In this topic, we explore basic Ramsey theory and its applications. For a set a and a natural number r, the symbol $[a]^r$ denotes the set of all subsets of a of cardinality r.

Theorem 0.1. *For natural numbers r, k, for every function $f\colon [\omega]^r \to k$, there is an infinite set $a \subset \omega$ such that $f \restriction [a]^r$ is constant.*

The function f is often called a *coloring* or a *partition* of $[\omega]^r$. The set a such that $f \restriction [a]^r$ is constant is called *homogeneous*, and the unique $i \in k$ such that $f(b) = i$ for all $b \in [a]^r$ is called the *homogeneous color* or *homogeneous value*. The *arrow notation* is often used for Ramsey theorems of this kind. A typical arrow notation expression is of the form $\chi \to (\lambda)^\kappa_\mu$ where $\chi, \lambda, \kappa, \mu$ are cardinals (often natural numbers). This expression means the following: whenever A is a set of cardinality χ and $f\colon [a]^\kappa \to \mu$ is a function then there is a homogeneous set of size λ. So for example, the above theorem can be restated as $\omega \to (\omega)^r_k$ for all $r, k \in \omega$.

Proof. Induce on r. The base case $r = 1$ is immediate: the number k is finite and so there must be $i \in k$ such that the set $a = \{n \in \omega\colon f(\{n\}) = i\}$ is infinite. The set a is then homogeneous, with homogeneous value i.

Suppose that we have proved the theorem for some r. To argue for $r + 1$, suppose that $f\colon [\omega]^{r+1} \to k$ is a coloring. A piece of terminology will be useful: a set $a \subset \omega$ is *end-homogeneous* for f if for $b \in [\omega]^{r+1}$, the value $f(b)$ does not depend on the largest number in the set b.

Claim 0.2. *There is an infinite end-homogeneous set for f.*

Proof. By induction on $n \in \omega$, construct numbers i_n and infinite sets $c_n \subset \omega$ such that

- $i_0 < i_1 < i_2 < \dots$ and $c_0 \supset c_1 \supset c_2 \supset \dots$;
- $i_n < \min(c_n)$ and $i_{n+1} \in c_n$;
- for every set $b \subset \{i_m\colon m \leq n\}$ of size r and numbers $k, l \in c_{n+1}$, $f(b \cup \{k\}) = f(b \cup \{l\})$.

Suppose that the induction has been performed; then the set $\{i_n\colon n \in \omega\}$ is end-homogeneous by the last induction demand.

To perform the induction, let $i_0 = 0$ and $c_0 = \omega \setminus \{0\}$. This satisfies the induction hypotheses since the last item is void. Now suppose that the numbers i_m for $m \leq n$, as well as the set c_n have been constructed. We will first thin out c_n to c_{n+1} to satisfy the last item. List all subsets of $\{i_m\colon m \leq n\}$ of size r in some finite list $b_j\colon j \in J$. Use a counting argument repeatedly to find infinite sets $d_j \subset \omega$ so that $c_n = d_0 \supset d_1 \supset \dots$ such that for all $j \in J$ and all numbers $k, l \in d_j$, $f(b_{j+1} \cup \{k\}) = f(b_{j+1} \cup \{l\})$. Clearly, the set $c_{n+1} = b_J$ will satisfy the last item. Then, let $i_{n+1} \in c_{n+1}$ be some number larger than i_n. The induction step is complete! $\square$

Now, use the claim to find an infinite end-homogeneous set $c \subset \omega$ for the partition f. Let $g\colon [c]^r \to k$ be the function defined by $g(b)$ =the unique value of $f(b \cup \{k\})$ for $k \in a$, $k > \max(b)$. This is a well-defined function since the set c is end-homogeneous. By the induction hypothesis, there is an infinite homogeneous set $a \subset c$ for the partition g. An inspection of the definitions shows that the set a is homogeneous for the partition f as well. This concludes the induction step and the whole argument. □

The infinite Ramsey theorem allows for finite "miniaturizations".

Theorem 0.3. *For all natural numbers n, k, r there is a natural number m such that $m \to (n)^k_r$ holds.*

Proof. Suppose towards contradiction that this fails for some n, k, r. For each $m \in \omega$ let $f_m\colon [m]^r \to k$ be a coloring with no homogeneous set of size n. Let U be a non-principal ultrafilter on ω. (There is a simple proof which uses no ultrafilters or the Axiom of Choice.) Define a function $g\colon [\omega]^r \to k$ be setting $g(b) = i$ if the set $d_b = \{m \in \omega\colon f_m(b) = i\}$ belongs to the ultrafilter U. By the infinite Ramsey theorem, there has to be an infinite set $a \subset \omega$ homogeneous for the partition g. Let $c \subset a$ be the finite set of the first n elements of the set c. Consider the intersection $\bigcap\{d_b\colon b \in [c]^r\} \subset \omega$. This is an intersection of finitely many sets in the ultrafilter U and so it is nonempty. Let m be a number in the intersection. By the definition of the function g, $f_m \restriction [c]^r = g \restriction [c]^r$ and so c is a homogeneous set for the partition f_m of size n. This contradicts the choice of the partition f_m! □

1 Negative results

Theorem 1.1. *For no set A, $A \to (\omega)^\omega_2$.*

In other words, for every set A there is a map f assigning each infinite countable subset of A color 0 or 1 such that there is no homogeneous infinite set for f.

Proof. Let E be the equivalence relation on $\mathcal{P}(A)$ defined by $b\ E\ c$ if $b \Delta c$ is finite. Use the Axiom of Choice to find a set S which visits each E-equivalence class in exactly one element. Define the function $f\colon \mathcal{P}(A) \to 2$ by $f(a)$ =parity of the number $|a \Delta b|$ where b is the unique element of S such that $a\ E\ b$.

Suppose that $c \subset A$ is an infinite set; we must show that it is not homogeneous for the partition f. To see this, let $a \subset c$ be an infinite countable set. Let $b \in S$ be the unique element of S which is E-equivalent to a. The set $a \cap b$ is infinite, so choose an element i in the intersection, and let $a' = a \setminus \{i\}$. Then the set $a' \Delta b$ differs from $a \Delta b$ exactly by this element i, so one of these sets has even size, while the other has an odd size. Thus $f(a) \neq f(a')$ and the set c is not homogeneous. □

Theorem 1.2. $\mathbb{R} \to (\textit{uncountable})^2_2$ *fails.*

In other words, there is a partition $f\colon [\mathbb{R}]^2 \to 2$ such that no uncountable subset of reals is homogeneous for f.

Proof. Use the Axiom of Choice to find a well-ordering $\prec$ on $\mathbb{R}$. Let $<$ be the usual ordering of $\mathbb{R}$. Define $f(x, y) = 0$ if $x < y \leftrightarrow x \prec y$ (meaning that $<$ and $\prec$ order the set $\{x, y\}$ in the same way) and $f(x, y) = 1$ otherwise.

Suppose that $c \subset \mathbb{R}$ is a homogeneous set in color 0; we will show that c is countable. To see this, note that c is well-ordered by $\prec$ and (since $\prec$ and $<$ agree on c) it is well-ordered by $<$. Thus, for every real $r \in c$ except the largest element of c, the set $\{s \in c\colon s > r\}$ has a smallest element; call it r^+. Let $g\colon c \to \mathbb{Q}$ be a function which, to each $r \in c$, assigns a rational between r and r^+. Then g is an injection from c to a countable set, showing that c is countable.

Suppose that $c \subset \mathbb{R}$ is a homogeneous set in color 1; we again must show that c is countable. The proof is similar as in the previous case observing that c is well-ordered by the reverse of $<$. □

2 Canonical Ramsey theorem

Given a natural number r and a set $a \subset r$, define the equivalence relation E_a on $[\omega]^r$ in the following way. Given sets $b, c \in [\omega]^r$, put $b \; E_a \; c$ if, fixing the increasing enumerations $b = \{n_i\colon i \in r\}$ and $c = \{m_i\colon i \in r\}$, it is the case that $\forall i \in a \; n_i = m_i$.

Theorem 2.1. *For every $r \in \omega$ and every equivalence relation E on $[\omega]^r$ there is an infinite set $c \subset \omega$ and a set $a \subset r$ such that the equivalence relations E and E_a coincide on $[c]^r$.*

Proof. We will prove the theorem in the case $r = 2$. Fix the equivalence relation E on $[\omega]^2$; we will find the set c by a multiple application of the basic Ramsey theorem. For a set $b \in [\omega]^4$, write $b = \{b(0), b(1), b(2), b(3)\}$ for the increasing enumeration of the set b.

The first partition $f_0\colon [\omega]^4 \to 2$ is defined by $f_0(b) = 0$ if $\{b(0), b(1)\} \; E \; \{b(2), b(3)\}$. Let $c_0 \subset \omega$ be an infinite homogeneous set for f_0.

Claim 2.2. *If the homogeneous color is 0 then any two pairs $d_0, d_1 \in [c_0]^2$ are E-equivalent.*

Proof. Let e be a pair of natural numbers in c_0 which are bigger than all numbers in d_0, d_1. Then, by the homogeneity of the set c_0, it must be the case that $d_0 \; E \; e$ and $d_1 \; E \; e$ holds. By the transitivity of the equivalence relation E, $d_0 \; E \; d_1$ holds. □

So, if the homogeneous color is 0 then we are done–E on $[c_0]$ is equal to E_0. Now assume that the homogeneous color for f_0 is 1. The next couple of partitions are designed to produce an an infinite set in which disjoint pairs are necessarily E-unrelated.

The second partition $f_1 \colon [c_0]^4 \to 2$ is defined by $f_1(b) = 0$ if $\{b(0), b(3)\}\ E\ \{b_1, b_2\}$. Let $c_1 \subset c_0$ be an infinite homogeneous set guaranteed by the Ramsey theorem.

Claim 2.3. *The homogeneous color for f_1 is 1.*

Proof. If the homogeneous color is 0, then choose numbers $n_0 < n_1 < n_2 < n_3 < n_4 < n_5$ in c_1. By the homogeneity for the partition f_1, $\{n_0, n_5\}$ is E-related to both $\{n_1, n_2\}$ and $\{n_3, n_4\}$. By the transitivity of the relation E, $\{n_1, n_2\}\ E\ \{n_3, n_4\}$ holds. This contradicts the homogeneity for partition f_0. $\square$

The third partition $f_2 \colon [c_1]^4 \to 2$ is defined by $f_2(b) = 0$ if $\{b(0), b(2)\}\ E\ \{b(1), b(3)\}$. Let $c_2 \subset c_1$ be an infinite homogeneous set guaranteed by the Ramsey theorem.

Claim 2.4. *The homogeneous color for f_2 is 1.*

Proof. If the homogeneous color is 0, then choose numbers $n_0 < n_1 < n_2 < n_3 < n_4 < n_5$ in c_2. By the homogeneity for the partition f_2, $\{n_0, n_3\}$ is E-related to both $\{n_2, n_4\}$ and $\{n_1, n_5\}$. By the transitivity of the relation E, $\{n_2, n_4\}\ E\ \{n_1, n_5\}$ holds. This contradicts the homogeneity for partition f_1. $\square$

A brief analysis of all possible configurations of disjoint pairs of natural numbers shows now that any two disjoint pairs of natural numbers in the set c_2 are E-unrelated. Now we start dealing with pairs that have nonempty intersection. Let $f_3 \colon [c_2]^3 \to 2$ is defined by $f_3(b) = 0$ if $\{b(0), b(1)\}\ E\ \{b(1), b(2)\}$. Let $c_3 \subset c_2$ be an infinite homogeneous set guaranteed by the Ramsey theorem.

Claim 2.5. *The homogeneous color for f_3 is 1.*

Proof. If the homogeneous color is 0, then choose numbers $n_0 < n_1 < n_2 < n_3$ in c_3. By the homogeneity for the partition f_3, $\{n_0, n_1\}$ is E-related to both $\{n_1, n_2\}$ which is in turn E-related to $\{n_2, n_3\}$. By the transitivity of the relation E, $\{n_0, n_1\}\ E\ \{n_2, n_3\}$ holds. This contradicts the homogeneity for partition f_0. $\square$

Finally, consider partitions f_4 and $f_5 \colon [c_3]^3 \to 2$ defined as follows: $f_4(b) = 0$ if $\{b(0), b(1)\}\ E\ \{b(0), b(2)\}$ and $f_5(b) = 0$ if $\{b(1), b(2)\}\ E\ \{b(0), b(2)\}$. Let $c_4 \subset c_3$ be an infinite set homogeneous for both of these partitions. The analysis of possible configurations of pairs of natural numbers gives the following:

- if the homogeneous colors for both f_4, f_5 are 1, then no two distinct pairs of numbers in the set c_4 are E-related. Therefore, $E = E_{\{0,1\}}$ on c_4;
- if the homogeneous color for f_4 is 0 while the homogeneous color for f_5 is 1, then $E = E_{\{}0\}$ on c_4;

- if the homogeneous color for f_4 is 1 while the homogeneous color for f_5 is 0, then $E = E_{\{}1\}$ on c_4;
- the homogeneous colors for both f_4 and f_5 cannot be both 0. To see this, choose numbers $n_0 < n_1 < n_2$, and argue that $\{n_0, n_1\}\ E\ \{n_0, n_2\}$ (homogeneity for f_4), $\{n_0, n_2\}\ E\ \{n_1, n_2\}$ (homogeneity for f_5) $\{n_0, n_1\}\ E\ \{n_1, n_2\}$ (transitivity of E), and now this contradicts the homogeneity for f_3.

The proof of the theorem for $r = 2$ is complete. □

3 Applications

Theorem 3.1. *Every infinite sequence of reals contains an infinite monotone subsequence.*

Proof. Let $\langle x_i : i \in \omega\rangle$ be an infinite sequence of real numbers. Let $f: [\omega]^2 \to 2$ be a function defined by $f(i,j) = 0$ if $i < j \leftrightarrow x_i < x_j$. Ramsey theorem provides a homogeneous infinite set c. It is clear that if the homogeneous color is 0 then the sequence $\langle x_i : i \in c\rangle$ is increasing, and if the homogeneous color is 1 then the sequence $\langle x_i : i \in c\rangle$ is nonincreasing. □

One interesting application is a simple proof of the Bolzano–Weierstrass theorem: a bounded infinite sequence of reals contains a convergent subsequence. To see this, use the theorem to find a monotone subsequence. Such a subsequence must be converging: if it is nonincreasing then its infimum is the limit, if it is nondecreasing then its supremum has a limit.

Theorem 3.2. *For every $n \in \omega$ there is $m \in \omega$ such that among any m many points in the plane, no three of which are colinear, there are vertices of convex n-gon.*

Proof. Then, assume without loss that $n \geq 5$ and let m be such that $m \to (n)^4_2$; we claim that m works. Suppose that $\{x_i : i \in m\}$ are points in the plane, no three of which are colinear. Let $f: [m]^4 \to 2$ be the functions defined by $f(a) = 0$ if the points x_i for $i \in a$ form a convex quadrilateral, and $f(a) = 1$ otherwise. Let $c \subset m$ be a homogeneous set of size n. We will show that the homogeneous color must be 0 and that the points $\{x_i : i \in c\}$ are vertices of a convex n-gon.

First of all, the homogeneous color cannot be 1, because a simple analysis of possible configurations show that among any 5 points, no three of which are colinear, there are vertices of convex quadrilateral. So, the homogeneous color is 0. To see that the points $\{x_i : i \in c\}$ form vertices of a convex n-gon, note that if they do not, then there must be a number $i \in c$ such that the point x_i is inside the triangle with vertices x_{j_0}, x_{j_1} and x_{j_2} for some numbers $j_0, j_1, j_2 \in c$. This would mean that $f(a) = 1$ where $a = \{i, j_0, j_1, j_2\}$. This contradicts the assumption that the set c is homogeneous for the partition f with homogeneous color 0. □