In this chapter, we will discuss the structure of the real line from the set theoretic point of view. We will show that the ordering and algebraic structure of the real line are uniquely given by some natural demands. The hierarchy of Borel sets is constructed as well.

1 The order structure of the real numbers

Definition 1.1. Let $\langle P, \leq \rangle$ be a linearly ordered set. The ordering is *dense* if for every $p < q$ in P there is $r \in P$ such that $p < r < q$.

For example, the usual ordering of natural numbers is dense, while the usual ordering of integers is not. The following theorem shows that there is really only one countable dense linear ordering up to isomorphism. Clearly, this unique representative then must be isomorphic to the usual ordering of rational numbers.

Theorem 1.2. *Any two dense countable linear orders without endpoints are isomorphic.*

Proof. The trick used is known as a "back-and-forth argument". Suppose that $\langle P, \leq_P \rangle$ and $\langle R, \leq_R \rangle$ are two dense countable linear orders without endpoints. We must construct an isomorphism. Let $\langle p_n \colon n \in \omega \rangle$ and $\langle r_n \colon n \in \omega \rangle$ are enumerations of P and Q respectively. By recursion on $n \in \omega$, build partial functions $h_n \colon P \to R$ such that

- $0 = h_0 \subset h_1 \subset h_2 \subset \dots$;
- all maps h_n are finite injections;
- $p_n \in \mathrm{dom}(h_{2n+1})$ and $r_n \in \mathrm{rng}(h_{2n+2})$ for every $n \in \omega$;
- the maps h_n preserve the ordering: whenever $x <_P y$ are elements of $\mathrm{dom}(h_n)$ then $h_n(x) <_R h_n(y)$.

Once the recursion is performed, let $h = \bigcup_n h_n$. This is a function from P to Q which preserves the ordering, and $\mathrm{dom}(h) = P$ and $\mathrm{rng}(h) = Q$. That is, h is the requested isomorphism of the orderings P and Q.

To perform the construction, suppose that h_{2n} has been found. In the construction of h_{2n+1}, it is just necessary to include p_n in the domain of h_{2n+1}. If $p_n \in \mathrm{dom}(h_{2n})$ then let $h_{2n+1} = h_{2n}$ and proceed with the next stage of the recursion. If $p_n \notin \mathrm{dom}(h_{2n})$, then the construction of h_{2n+1} divides into several cases according to how p_n relates to the finite set $\mathrm{dom}(h_{2n}) \subset P$:

- if p_n is $\leq_P$-greater than all elements of $\mathrm{dom}(p_{2n})$, then pick a point $r \in R$ $\leq_R$-larger than all elements of $\mathrm{rng}(h_{2n})$(possible as the ordering R does not have a largest point) and let $h_{2n+1}(p_n) = r$;
- if p_n is $\leq_P$-smaller than all elements of $\mathrm{dom}(p_{2n})$, then pick a point $q \in Q$ $\leq_R$-smaller than all elements of $\mathrm{rng}(h_{2n})$ (possible as the ordering R does not have a smallest point) and let $h_{2n+1}(p_n) = r$;

- if neither of the above two items holds, then there must be a $\leq_P$-largest point $p' \in \mathrm{dom}(h_{2n})$ smaller than p_n, and a $\leq_P$-smallest point $p'' \in \mathrm{dom}(h_{2n})$ which is larger than p_n. Let $r \in R$ be any point strictly between $h_{2n}(p')$ (it exists as the ordering R is dense) and $h_{2n}(p'')$ and let $h_{2n+1}(p_n) = r$.

The induction step from h_{2n+1} to h_{2n+2} is performed similarly. □

Definition 1.3. A linear ordering $\langle P, \leq\rangle$ is *complete* if every bounded subset of P has a *supremum*. That is, whenever $A \subset P$ is a set such that the set $B = \{p \in P \colon \forall q \in A\ q \leq p\}$ is nonempty, then the set B has a $\leq$-smallest element.

Definition 1.4. Let $\langle P, \leq_P\rangle$ be a linear ordering. A *completion* of P is a order-preserving map $c : P \to R$ to a complete linear ordering $\langle R, \leq_R\rangle$ such that $c''P \subset R$ is dense.

Theorem 1.5. *Every linear ordering has a completion. The completion is unique up to isomorphism.*

Proof. For simplicity of notation, we will consider only the case of dense linear ordering $\langle P, \leq_P\rangle$. First, construct some completion of P. Call a pair $\langle A, B\rangle$ a *Dedekind cut* if $A \cup B = P$, $A \cap B = 0$, for every $p \in A$ and every $q \in B$ $p <_P q$, and A does not have a largest element. Let R be the set of all Dedekind cuts, and define $\langle A_0, B_0\rangle \leq_R \langle A_1, B_1\rangle$ if $A_0 \subseteq A_1$.

Claim 1.6. *$\langle R, \leq_R\rangle$ is a complete linear ordering.*

Proof. It is immediate that $\leq_R$ is an ordering. The first challenge is its linearity. Suppose that $\langle A_0, B_0\rangle$ and $\langle A_1, B_1\rangle$ are Dedekind cuts. We must show that either $A_0 \subseteq A_1$ or $A_1 \subseteq A_0$ holds. If $A_0 = A_1$ then this is clear. Otherwise, one of the sets $A_1 \setminus A_0$ or the set $A_0 \setminus A_1$ must be nonempty. Suppose for definiteness it is the set $A_1 \setminus A_0$, and choose an element $q \in A_1$ which is not in A_0. As $\langle A_0, B_0\rangle$ is a Dedekind cut, it must be the case that $q \in B_0$ and all elements of A_0 are $<_P$-smaller than q. As $\langle A_1, B_1$ is a Dedekind cut, every element $p <_P q$ must belong to A_1. Therefore, $A_0 \subseteq A_1$. This confirms the linearity of $\leq_R$.

Now, we have to prove that $\leq_R$ is complete. Suppose that $S \subset R$ is a bounded set. Its supremum is defined as the pair $\langle A, B\rangle$ where $A = \bigcup\{A' : \exists B'\ \langle A', B'\rangle \in S\}$ and $B = \bigcap\{B' : \exists A'\ \langle A', B'\rangle \in S\}$. □

Now, we have to produce an order-preserving map $c \colon P \to R$ such that $c''P \subset R$ is dense. Just let $c(p) = \langle A, B\rangle$ where $A = \{q \in P : q <_P p\}$ and $B = \{q \in P \colon p \leq_P q\}$. ???

Thus, the map $c : P \to R$ is a completion of the ordering P. The final task is to show that any other completion of P is isomorphic to R. ??? □

Now it makes sense to define $\langle \mathbb{R}, \leq\rangle$ as the completion of $\langle \mathbb{Q}, \leq\rangle$, which is unique up to isomorphism. This is again a linear ordering which has some uniqueness features.

Theorem 1.7. *Every linear ordering which is dense with no endpoints, complete, and has a countable dense subset, is isomorphic to $\langle \mathbb{R}, \leq \rangle$.*

At this point, it is possible to introduce a problem which, together with the Continuum Hypothesis, shaped modern set theory. Note that the real line has the following remarkable property: if A is a collection of pairwise disjoint open intervals, then A is countable. To see this, for every interval $I \in A$ pick a rational number $f(I) \in I$. The function f is then an injection from A to the rationals, showing that A is countable. One can now ask whether this property can be used to characterize the real line similarly to Theorem 1.7.

Question 1.8. (Suslin's problem) Suppose that a linear ordering is dense with no endpoints, complete, and any collection of its pairwise disjoint open intervals is countable. Is it necessarily isomorphic to $\langle R, \leq \rangle$?

It turns out that the answer to the Suslin's problem cannot be decided within ZFC set theory.

2 The algebraic structure of the real numbers

I n this section we will show that the algebraic structure of the real line is in some way uniquely determined by the natural demands on the operations and the ordering.

Definition 2.1. A *field* is a tuple $\langle F, 0, 1, +, \cdot \rangle$ such that F is a set, $0, 1 \in F$ and $+, \cdot$ are binary operations on F such that the following hold:

1. $+, \cdot$ are associative and commutative;
2. $0 + a = a$, $0 \cdot a = 0$ and $1 \cdot a = a$ holds for all $a \in F$;
3. for every $a \in F$ there is b such that $a + b = 0$, and if $a \neq 0$ then there is b such that $a \cdot b = 1$;
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ holds for all $a, b, c \in F$.

Example 2.2. The rational numbers, the real numbers, and the complex numbers with their usual operationas are all fields.

Example 2.3. If p is a prime, then Z_p, the set of all natural numbers smaller than p with addition and multiplication modulo Z is a (finite) field.

Definition 2.4. An *ordered field* is a tuple $\langle F, 0, 1, +, \cdot, \leq \rangle$ where F is a field and $\leq$ is a linear ordering on F such that $b \leq c$ implies $a + b \leq a + c$, and if $a > 0$ then $ab \leq ac$. A field $\langle F, 0, 1, +, \cdot \rangle$ is *orderable* if there is an ordering on F which makes it inot ordered field.

Example 2.5. $\mathbb{R}$ with the usual operations and ordering is an ordered field.

Example 2.6. The field of rational functions on $\mathbb{R}$, with the usual operations, is orderable. To see one of the possible orderings, let $f \leq g$ if for some number r, for all real numbers $s > r$ $f(s) \leq g(s)$ holds. One has to verify that this in fact is a linear field ordering.

Basic rules about field orders are the following:

- if $a \neq 0$ then either $a > 0$ or $-a > 0$ and not both;
- if $a \neq 0$ then $a^2 > 0$;
- if $a, b > 0$ then $a + b > 0$.

To see the first item, if $a < 0$ then we can add $-a$ to both sides of this inequality to yield $0 < -a$. To see why $a > 0$ implies $0 > -a$ just add $-a$ to both sides of the inequality, To see the second item, the first item shows that it is enough to consider two cases: $a > 0$ and $-a > 0$. In the first case, multiply both sides of the inequality by a to get $a^2 > 0$; in the second case, multiply both sides of the inequality by $-a$ to get $a^2 > 0$ as well. To see the third item, look at the inequality $a > 0$, add b to both sides, and use the transitivity of the ordering to see that $a + b > b > 0$.

These rules show that some fields are not orderable:

Example 2.7. No finite field is orderable. To see this, note that the unit is a square and therefore has to be greater than 0 in every ordered field. Thus $0 < 1 < 1 + 1 < 1 + 1 + 1 < \dots$ must hold, where each successive identity is obtained from the previous one by adding 1 to both sides. This produces an infinite collection of distinct elements of the ordered field.

Example 2.8. The field of complex numbers is not orderable. To see this, note that in this field both 1 and -1 are squares, so in the ordering they would have to be both greater than 0 by the second item, but this would contradict the first item.

Theorem 2.9. *(Artin–Schreier) A field F is orderable just in case* 0 *is not a sum of nonzero squares in F.*

Proof. For the left-to-right direction, observe that in a field ordering, nonzero squares are always bigger than zero, and so is their sum. The right-to-left direction is harder and uses the axiom of choice.

Define a cone to be a subset of F which is closed under addition and multiplication and does not contain 0. We will use Zorn's lemma to show that there is a cone $C \subset F$ such that for every element $a \in F$, either $a \in C$ or $-a \in C$. With such a cone C, define the relation $<$ on F by $a < b$ if $b - a \in C$. The following claim shows that this relation will witness the statement of the theorem.

Claim 2.10. *$<$ is a linear order compatible with the field structure of F.*

Proof. The linearity of $<$ is the main issue. Given $a \neq b \in F$, by the maximality of C it is the case that $a - b \in C$ or $b - a \in C$; consequently, either $a < b$ or $b < a$ must hold. The relation $<$ is transitive, since the cone C is closed under addition: if $a < b$ and $b < c$, then $a < c$ holds because $c - a = (c - b) + (b - a)$ and the two parenthesised differences belong to the set C. $\square$

For the construction of a cone with the desired property, first argue that the set C_0 of all sums of nonzero squares is a cone; it is clearly closed under addition and multiplication, and it does not contain 0 by the assumption on the field F. Now let $\mathcal{C}$ be the collection of all cones on F extending C_0, ordered by inclusion. It is not difficult to verify that every linearly ordered subset $\mathcal{D} \subset \mathcal{C}$ has an upper bound in $\mathcal{C}$–the union $\bigcup \mathcal{D}$ is a cone and therefore an upper bound of $\mathcal{D}$. An application of Zorn's lemma yields an inclusion maximal cone $C \in \mathcal{C}$.

Now, we have to argue that the maximal cone C has the desired property: if $a \in F$ then either $a \in C$ or $-a$ in C. Suppose that both of these two fail for some $a \in F$. Let $D_a = D \cup \{ba+c \colon b, c \in D\}$ and $D_{-a} = D \cup \{-ba+c \colon b, c \in D\}$.

Claim 2.11. *Both sets D_a and D_{-a} are closed under addition and multiplication.*

Proof. To show for example that D_a is closed under multiplication, suppose that $b_0, c_0, b_1, c_1 \in C$ are arbitrary elements and consider the product $(b_0 a + c_0)(b_1 a + c_1)$, This is equal to $b_0 b_1 a^2 + (c_0 b_1 + b_0 c_1)a + c_0 c_1$. In this expression, the first term belongs to C because $b_0, b_1 \in D$ and $a^2 \in C_0 \subset C$. Thus, the product is again of the form $b_2 a + c_2$ for suitable coefficients $b_2, c_2 \in C$ and therefore belongs to D_a. $\square$

The sets D_a, D_{-a} are both larger than C and so they cannot be cones by the maximality of C. This means that both of them must contain 0. So, choose $b_0, b_1, c_0, c_1 \in C$ such that $b_0 a + c_0 = 0$ and $-b_1 a + c_1 = 0$. The first equation, multiplied by b_1, yields $b_0 b_1 a + b_0 c_0 = 0$. The second equaltion, multiplied by b_0, yields $-b_0 b_1 a + b_0 c_1 = 0$. Adding these two, we get $b_0 c_0 + b_0 c_1 = 0$. However, this contradicts the assumption that C is a cone and so multiplying and adding some of its elements cannot yield a zero result. $\square$

Theorem 2.12. *$\mathbb{R}$ is, up to isomorphism, the unique ordered field whose ordering is complete.*

Proof. Let $\langle F, 0, 1, +, \cdot, \leq \rangle$ be an ordered field and assume that $\leq$ is a complete ordering. We will show that there is a unique isomorphism between $\mathbb{R}$ and F. For each natural number n, let n_F denote the field element obtained by adding 1_F to itself n many times. The following is the key observation:

Claim 2.13. *For every $a \in F$ there is $n \in \omega$ such that $a \leq n_F$ holds.*

The fields whose ordering satisfies the statement of the claim are called *archimedean.*

Proof. Suppose this fails for some $a \in F$; so $a > n_F$ for all $n \in \omega$. Thus, the set $\{n_F : n \in \omega\}$ is bounded from above by a and so by the completeness of the ordering it has the least upper bound, call it b. Then $b - 1 < b$, $b - 1$ is not an upper bound of the set $\{n_F : n \in \omega\}$, and so there must be a number $n \in \omega$ such that $b - 1 < n_F$. But then, $b < n_F + 1 = (n+1)_F$, contradicting the assumption that b is an upper bound of the set $\{n_F : n \in \omega\}$. $\square$

For every rational number $q = n/m$, define $q_F \in F$ to be the field element equal to n_F/m_F.

Claim 2.14. *Whenever $a < b$ are field elements then there is a rational number q such that $a < q_F < b$.*

Proof. For simplicity assume that $0 < a < b$. Let $m \in \omega$ be a natural number such that $m_F > \frac{1}{b-a}$; such a number exists by the previous claim. We will show that there is a natural number n such that n_F/m_F is between a and b. ??? $\square$

Now let $\pi \colon \mathbb{Q} \to F$ be the function mapping each rational q to q_F. This is a bijection between a dense subset of $\mathbb{R}$ and a dense subset of F by the previous claim. By the completion theorem ??? it has a unique extension to a bijection between $\mathbb{R}$ and F. It is not difficult to see that this extension must transport the algebraic structure of $\mathbb{R}$ to the algebraic structure of F. $\square$

3 The Borel hierarchy

In this section, we will define the Borel hierarchy on the space of real numbers. The construction transfers without change to other similar topological spaces, such as the Euclidean spaces $\mathbb{R}^n$ for $n \in \omega$.

Definition 3.1. A set $O \subset \mathbb{R}$ is *open* if it is union of some collection of intervals (p, q) where p, q are rational numbers.

Definition 3.2. Let X be a set. A *σ-algebra* of subsets of X is a set $A \subset \mathcal{P}(X)$ such that $0 \in A$ and A is closed under countable union, countable intersection, and complement.

Theorem 3.3. *Let X be a set and $B \subset \mathcal{P}(X)$ be a set. Among all σ-algebras of subsets of X containing B as a subset, there is an inclusion-smallest one.*

Proof. Let $\mathcal{C} = \{C \subset \mathcal{P}(X) \colon C$ is a σ-algebra of subsets of X and $B \subset C\}$, and let $A = \bigcap \mathcal{C}$. We will show that A is a σ-algebra of subsets of X and $B \subset A$. By the definition of A then, it has to be the inclusion-smallest σ-algebra containing B.

It is clear that $0 \in A$ since $0 \in C$ for all $C \in \mathcal{C}$. Similarly, $B \subset A$ since for every $C \in \mathcal{C}$, $B \subset C$. Now for the closure under countable unions, suppose that $a \subset A$ is a countable set; I need to show that $\bigcup a \in A$ holds. To see this, note that for all $C \in \mathcal{C}$, $a \subset C$ must hold. Since every $C \in \mathcal{C}$ is a σ-algebra, it must be the case that $\bigcup a \in C$. Thus, $\forall C \in \mathcal{C}\ \bigcup a \in C$ and so $\bigcup a \in A = \bigcap \mathcal{C}$. The closure on the other operations is proved in the same way. $\square$

Definition 3.4. The *σ-algebra of Borel sets* is the inclusion-smallest σ-algebra of subsets of $\mathbb{R}$ which contains all open sets.

There is a different way of presenting the σ-algebra of Borel sets, which also stratifies it into levels indexed by countable ordinals.

Definition 3.5. By transfinite recursion on a countable ordinal $\alpha > 0$ define the sets $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ in the following way:

1. $\mathbf{\Sigma}^0_1$ is the collection of all open subsets of $\mathbb{R}$;
2. $\mathbf{\Pi}^0_\alpha$ is the set of complements of all sets in $\mathbf{\Sigma}^0_\alpha$;
3. $\mathbf{\Sigma}^0_\alpha$ is the set of all countable unions of sets in $\bigcup_{\beta\in\alpha}\mathbf{\Pi}^0_\alpha$, if $\alpha > 1$.

$\mathbf{\Sigma}^0_2$ sets are often referred to as the F_σ-sets, while $\mathbf{\Pi}^0_2$ sets are referred to as the G_δ-sets.

Thus, all sets $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ are collections of sets of reals. The sets in $\mathbf{\Sigma}^0_2$ are commonly called F_σ-sets, and the sets in $\mathbf{\Pi}^0_2$ are commonly called G_δ sets.

Theorem 3.6. *For every ordinal $\alpha > 0$, both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ contain $\mathbf{\Sigma}^0_\beta$ and $\mathbf{\Pi}^0_\beta$ as subsets for all $\beta < \alpha$.*

Since the sets $\mathbf{\Sigma}^0_\alpha$ form a transfinite inclusion-increasing sequence of subsets of $\mathcal{P}(\mathbb{R})$, by Theorem ??? there must be a an ordinal α at which they stop growing. The next theorem identifies the fixed point.

Theorem 3.7. $\mathbf{\Sigma}^0_{\omega_1} = \mathbf{\Sigma}^0_{\omega_1+1} = \mathbf{\Pi}^0_{\omega_1}$ *is the σ-algebra of Borel sets.*

4 Examples of non-Borel sets

By far not every set of reals is Borel. In this section, we will provide several ways of producing a non-Borel set of reals.

Definition 4.1. Let X be a set and A a σ-algebra of subsets of X. A *σ-additive measure on A* is a function μ defined on A such that

1. the functional values of μ are non-negative real numbers or possibly infinity;
2. $\mu(0) = 0$;
3. if $B_0, B_1 \in A$ are sets such that $B_0 \subset B_1$, then $\mu(B_0) \leq \mu(B_1)$;
4. if $a \subset A$ is a countable set then $\mu(\bigcup a) \leq \Sigma_{B\in a}\mu(B)$;
5. if $a \subset A$ is a countable set consisting of pairwise disjoint sets, then $\mu(\bigcup a) = \Sigma_{B\in a}\mu(B)$.

Example 4.2. Let X be a set and A a σ-algebra on it. The following are σ-additive measures on A:

1. $\mu(B) = 0$ for all $B \in A$;
2. $\mu(0) = 0$ and $\mu(B) = \infty$ for every set $B \in A$ distinct from 0;
3. $\mu(B) = |B|$ if B is finite, and $\mu(B) = \infty$ if B is infinite.

Theorem 4.3. (Caratheodory) *The function $(p, q) \mapsto q - p$ defined on basic open intervals extends in a unique way to a σ-additive measure on the σ-algebra of Borel sets.*

The resulting σ-additive measure on Borel subsets of reals is called the Lebesgue measure. Observe that the Lebesgue measure is translation invariant: if $B \subset \mathbb{R}$ is a Borel set and $r \in \mathbb{R}$ is a number then $\mu(B) = \mu(r + B)$. To see this, consider the function λ on Borel sets defined by $\lambda(B) = \mu(r + B)$. It is not difficult to see that this is a σ-additive measure. Moreover, it extends the function $(p, q) \mapsto q - p$. By the uniqueness part of the Caratheodory theorem, it must be the case that $\mu = \lambda$, which says precisely that μ is translation-invariant.

5 A set which is F_σ but not G_δ

It turns out that for each countable ordinal α, there is a set in $\mathbf{\Sigma}^0_\alpha$ which is not in $\mathbf{\Pi}^0_\alpha$; in other words, the Borel hierarchy keeps growing at all countable stages. We will prove only the most basic theorem of this kind. The argument uses a basic mathematical tool, the Baire category theorem. We state it only for the real numbers; it can be immediately generalized to much wider classes of spaces, including all Euclidean spaces.

Theorem 5.1. (*Baire category theorem) Let $O_n \subset \mathbb{R}$ be dense open sets for each $n \in \omega$. Then $\bigcap_n O_n \neq 0$.*

Proof. By induction on $n \in \omega$ build rational numbers p_n, q_n such that $p_n < p_{n+1} < q_{n+1} < q_n$ and $(p_n, q_n) \subset O_n$ holds for every number $n \in \omega$. If the induction succeeds, use the completeness of the real numbers to let $r = \sup_n p_n$. By the first part of the induction hypothesis, $r \in (p_n, q_n)$ and by the second part of the induction hypothesis $r \in O_n$ holds. In other words, $r \in \bigcap_n O_n$ and so $\bigcap_n O_n \neq 0$.

To perform the induction, start with any open interval $(p_0, q_0) \subset O_0$ with rational endpoints. For the induction step, if the interval (p_n, q_n) has been constructed, use the density of the set O_{n+1} to find a real number $s \in (p_n, q_n)$ in the set O_{n+1} and then use the openness of the set O_{n+1} to find an open interval (p_{n+1}, q_{n+1}) around s which is a subset of O. Shrinking the interval if necessary, one can achieve $p_n < p_{n+1} < s < q_{n+1} < q_n$ and make the endpoints rational. This completes the induction step and the proof. $\square$

Theorem 5.2. *Let $A \subset \mathbb{R}$ be a countable dense set. The set A is F_σ and not G_δ.*

Proof. The set A, as any countable set, is a countable union of sets each of which contain just one element. One element sets are closed, and so A is F_σ. The harder part of the theorem is showing that A is not G_δ.

Suppose towards contradiction that $A = \bigcap_n O_n$ is an intersection of dense sets. Since A is dense, each set O_n is dense as well. Now, use the countability of the set A to list its elements, $A = \{r_n : n \in \omega\}$. Let $P_n = O_n \setminus \{r_n\}$; the set P_n is still both open and dense, since it results from a removal of a single point from the open dense set O_n. Now, look at the intersection $\bigcap_n P_n$.

Since $P_n \subset O_n$, it must be the case that $\bigcap_n P_n \subseteq \bigcap_n O_n = A$. However, for each point $r \in A$, there is $n \in \omega$ such that $r = r_n$ and so $r = r_n \notin P_n$ and $r \notin \bigcap_n P_n$. It follows that $\bigcap_n P_n \neq 0$. This contradicts the Baire category theorem applied to the intersection $\bigcap_n P_n$. $\square$