Continuous Measures, Randomness, and Domination

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joint work with Mingyang Li

Randomness with respect to μ

There exists a representation R_μ of μ s.t. the real passes all μ-ML-tests that have access to R_μ

R & Slaman (2015) A real x is random for some μ with $\mu(\{x\}) = 0$ iff x is not computable

Things get more interesting if we pass to

- stricter randomness notions (Haken, 2014)
- smaller families of measures

this talk continuous measures

R & Slaman (2015) If x is not Δ^1_1, then it is random for a continuous measure

NCR = reals never random for a continuous measure

What is the structure of NCR inside the Δ^1_1 Turing degrees?

Kjos-Hanssen & Montalbán (2005)

If x is a member of a countable Π^0_1 class,

then $x \in NCR$

$\longrightarrow$ NCR is cofinal in the Turing degrees of Δ^1_1

(Cenzer, Clote, Smith, Soare, Wainer)

(Each degree $\underset{\sim}{0}^{(\alpha)}$ contains a member of a countable Π^0_1 class.)

Let μ be a probability measure on $2^{\mathbb{N}}$.

granularity function

$$g_\mu(n) = \min \{ \ell \ \forall |\sigma| = \ell \quad \mu[\![\sigma]\!] \leq 2^{-n} \}$$

In the following, all measures are assumed to be continuous,

so $g_\mu(n)$ is total

and $g_\mu(n) \longrightarrow \infty$

x Δ^0_2 with recursive approximation $\gamma(n, s)$

settling function

$$C_\gamma(n) = \min\{s \ \forall t \geq s \ \ \gamma(n,t) = \gamma(n,s)\}$$

R & Slaman If x is Δ^0_2 and μ-random, then

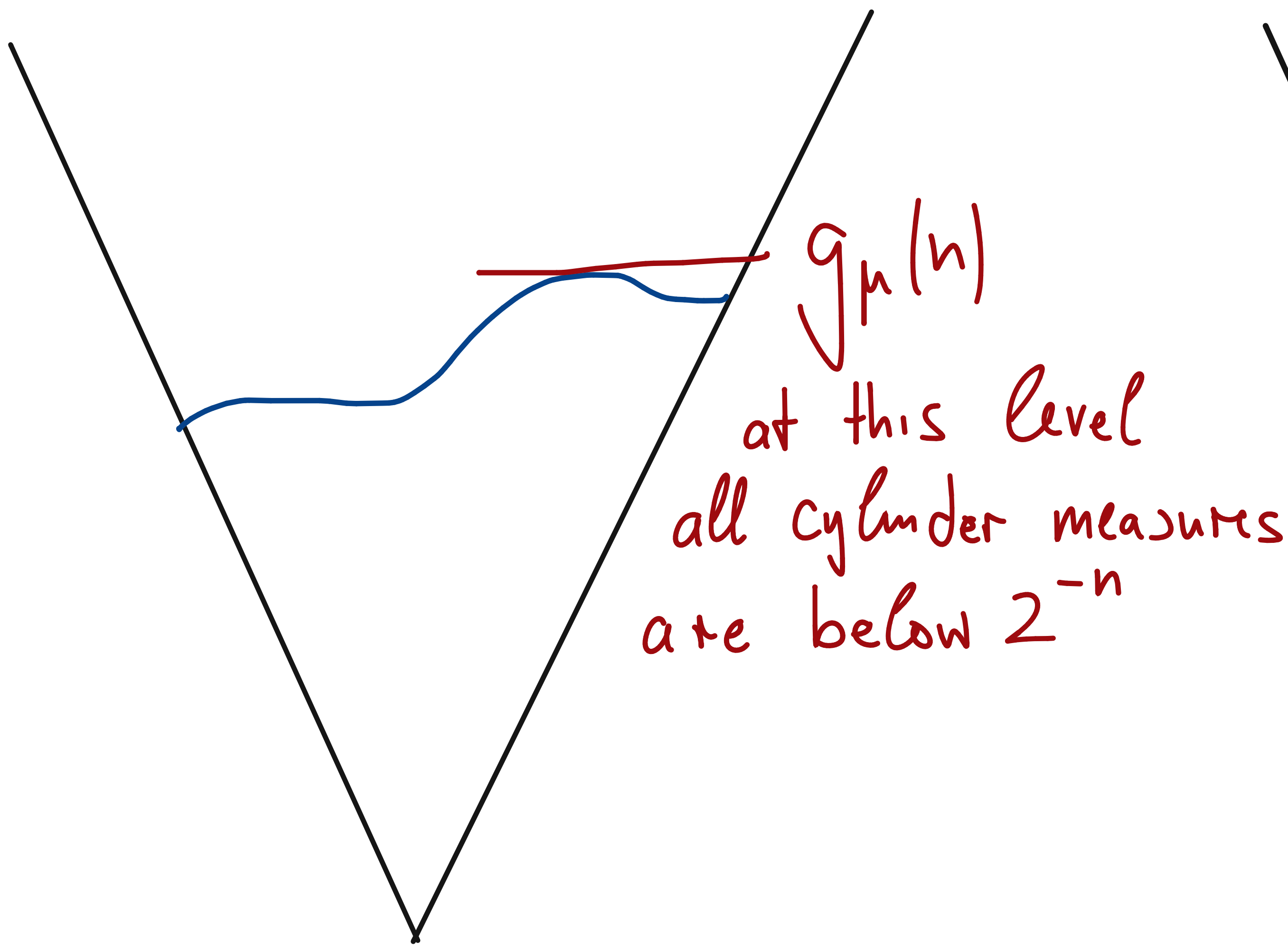
$C_\gamma(n) > g_\mu(n)$ for all but finitely many n

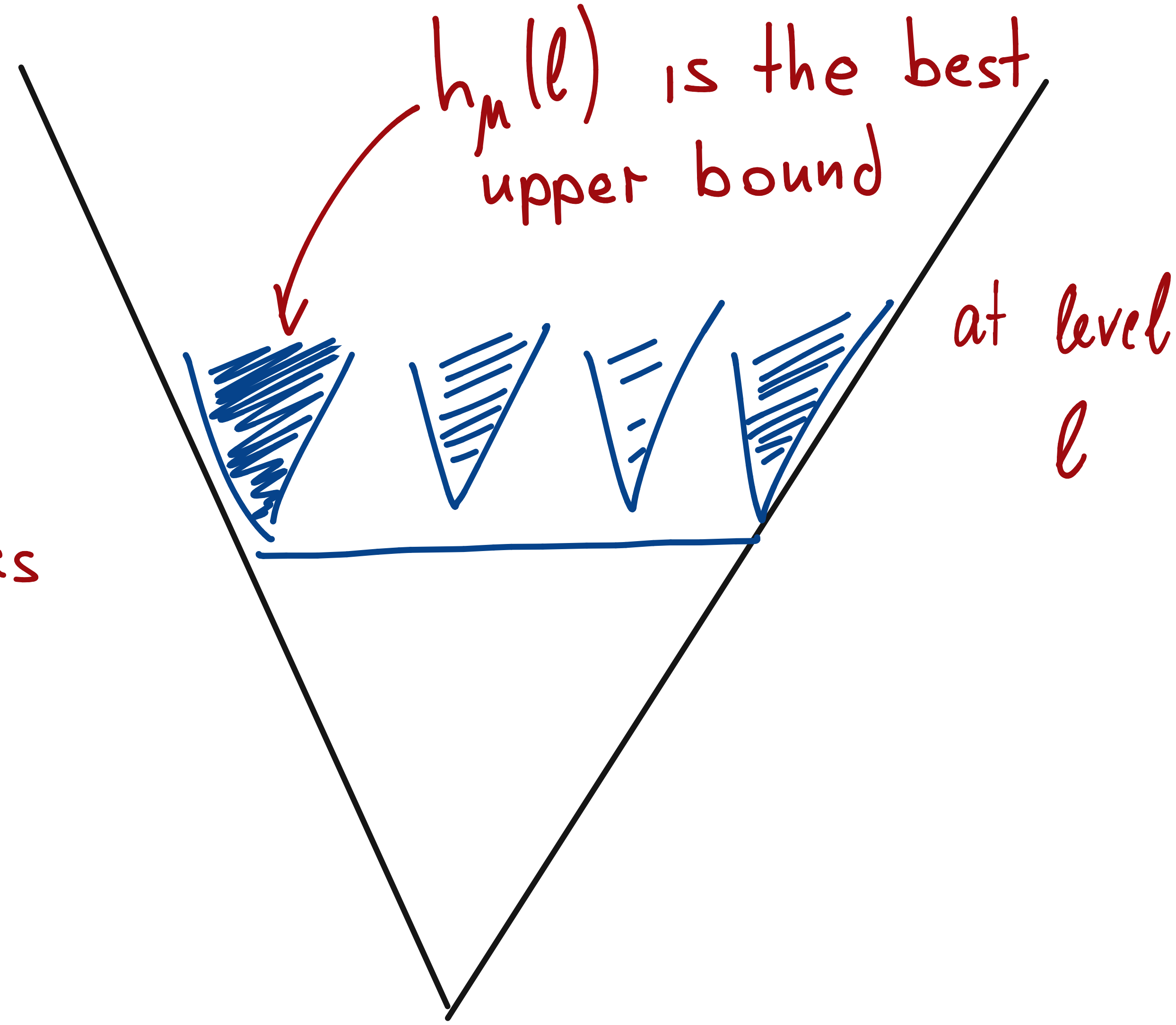
Barmpalias, Greenberg, Montalbán & Slaman used this to show

Every x Turing below an incomplete r.e. degree is in NCR

The dissipation function of a measure μ

$$h_\mu(\ell) = \max\{n \quad \forall |\sigma| = \ell \quad \mu(\sigma) < 2^{-n+1}\}$$





We have

$$n < g(n) < g(n+1) < g(g(n+1))$$

$$h(\ell) < h(\ell+1) \leq h(\ell)+1 \leq \ell+1$$

$$h(g(n)) = n+1$$

$$h(\ell) \longrightarrow \infty$$

$$g \equiv_T h$$

In general, g_μ and h_μ are only μ-c.e.

LEM: For any continuous μ, there are μ-computable, non-decreasing $h^*_\mu(n)$, $g^*_\mu(n)$ s.t. for all n,

$$h_\mu(n) \le h^*_\mu(n) \le \min\{n, h_\mu(n)+1\}$$

$$g_\mu(n) \le g^*_\mu(n) \le g_\mu(n+1)$$

Let μ be a continuous measure

A level-n Solovay test for μ is a μ-r.e. set of strings $\{\sigma_i \; i \in \mathbb{N}\}$ s.t. $\quad n \geq 1$

$$\sum_i \left(h^{(n)}(|\sigma_i|)\right)^{\log n} 2^{-h^{(n)}(|\sigma_i|)} < \infty$$

$h^{(n)}$ n-th iterate of h

$x \in 2^{\mathbb{N}}$ is non-μ-random of level n if it fails some level-n Solovay test, i.e. $x \in [\sigma_i]$ for infinitely many $i \in \mathbb{N}$

level ω non-random of level n f all $n \geq 1$

Observations

- If $x \leq_T \mu$, then x is non-μ-random of level ω
- If x is non-μ-random of level 1, then x is not μ-ML-random
- Every level-$(n+1)$ test is also a level-n test

For $x \in 2^{\mathbb{N}}$ and $f : \mathbb{N} \to \mathbb{N}$ non-decreasing, define

$$y = x + f$$

by

$$y_0 = 1^{f(0)} {}^\frown 0 {}^\frown x_0$$

$$y_{n+1} = y_n {}^\frown 1^{f(|y_n|)} {}^\frown 0 {}^\frown x_{n+1}$$

$$y = \lim_n y_n$$

Let $x \in 2^{\mathbb{N}}$, $k \geq 1$, f non-decreasing, μ continuous, $y = x + f$

LEM If f is not dominated by any μ-computable function,

then $\exists^\infty n \quad g_\mu^{*(k)}(2|y_n|+1) < f(|y_n|)$

$f: \mathbb{N} \to \mathbb{N}$ is a modulus for $x \in 2^{\mathbb{N}}$

if every function that dominates f

computes x

self-modulus $\quad f \equiv_T x$

Every $x \leq_T 0'$ has a self-modulus

THM If a real x has a self-modulus f, then there exists a real $y \equiv_T x$ s.t. for any continuous measure μ y is non-μ-random of level ω.

Solovay test of level k

$$T_k = \{ \sigma ^\frown 1^{g_\mu^{*(k)}(2|\sigma|)} \quad \sigma \in 2^{<\mathbb{N}} \}$$

THM

Suppose y is r.e.a. X. There exists $z \equiv_T y$ s.t. for any continuous measure μ, if X is non-μ-random of level $2n$, then z is non-μ-random of level n.

<u>Construction</u> Given $x \in 2^{\mathbb{N}}$, y r.e. above x

Assume $y = W_e^x$

Let $m_i = \min\{j > i \quad \Phi_e^x(j)\downarrow\}$

$$f(i) = \begin{cases} \min\{s \quad \forall k \leq m_i \ (\Phi_e^x(k)\downarrow \implies \Phi_{e,s}^x(k)\downarrow)\} & \text{if } i \in y \\ 1 & \text{if } i \notin y \end{cases}$$

$f \leq_T y$

$$z = y_0^{f(0)} \frown 0 \frown y_1^{f(1)} \frown 0 \frown y_2^{f(2)} \frown 0 \frown$$

$z \equiv_T y$

NCR vs Self-moduli

Does every NCR degree have a self-modulus?

(tentative answer: No)

Application: Simultaneous randomness

A, B simultaneously cont random

$\exists Z, \mu$ cont, $Z \geq_T \mu$ s.t.

A, B μ-random rel to Z

Conjecture (Day & Marks)

A, B NSCR $\iff$ A or B NCR

f self-modulus for $0'$

$$S_0 = \emptyset, \quad S_{n+1} = \{\sigma^\frown 1^{f(|\sigma|)\frown} 0^\frown a \;\; \sigma \in S_n,\ a \in \{0,1\}\}$$

$$S = \{y \in 2^{\mathbb{N}} \;\; \forall n\, \exists \sigma \in S_n \;\; \sigma \sqsubset y\}$$

Then

- S is perfect
- If $y \in S$ is μ-random, then any representation of μ computes a function dominating f

Pick x_1 ML-random $\in \mathring{\Delta}_2$

Distribute unit mass uniformly along S

$\rightsquigarrow$ cont μ Pick x_2 μ-random

Then $x_1, x_2 \notin NCR$, but

$$(x_1, x_2) \in NSCR$$

The role of h

How much is NCR actually governed by the dissipation functions h_μ, instead of the measures themselves?

R (2008) Suppose $h : \mathbb{N} \to \mathbb{R}^{\geq 0}$ is computable, non-decreasing, unbounded, and

$$h(n+1) \leq h(n) + 1.$$

If x is such that for all n,

$$-\log \underset{\sim}{M}(x \upharpoonright n) \geq h(n) \qquad (*)$$

then x is random for a continuous measure μ with

$$g_\mu \in O\left(\frac{n}{s}\right).$$

(Effective Frostman Lemma)

THM: If X is level-1 random for some continuous measure μ, then X is not in NCR.

- For computable measures, this was also observed by Hölzl & Porter.
- For NCR, this also means that the level-n hierarchy collapses, while it can be proper for single measures.