Uniform Computable Categoricity and Scott Families

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Flavors of computable categoricity

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Definitions

Let $\mathcal{A}$ be a structure with domain ω.

- $\mathcal{A}$ is *computably categorical* if, for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is a computable isomorphism $f : \mathcal{A} \to \mathcal{B}$.

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- $\mathcal{A}$ is *relatively computably categorical* if, for all structures $\mathcal{B}$, $\mathcal{C}$ (on domain ω) isomorphic to $\mathcal{A}$, there is an isomorphism $f : \mathcal{B} \to \mathcal{C}$ with $f \leq_T \mathcal{B} \oplus \mathcal{C}$.

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- $\mathcal{A}$ is *uniformly computably categorical* if there is a Turing functional Φ such that, for all structures $\mathcal{B}$, $\mathcal{C}$ (on domain ω) isomorphic to $\mathcal{A}$, $\Phi^{\mathcal{B} \oplus \mathcal{C}}$ is an isomorphism from $\mathcal{B}$ onto $\mathcal{C}$.

(Also cf. Downey-Hirschfeldt-Khoussainov for a uniform version for computable structures only.)

Example of uniform computable categoricity

Consider the ground field $E = \mathbb{Q}(\sqrt{p_n}\ :\ n \in \omega)$ generated by all square roots of prime numbers; and then the field

$$K = E(\sqrt[4]{p_n}\ :\ n \in \emptyset').$$

So when $n \in \emptyset'$, one of $\pm\sqrt{p_n}$ has a square root of its own, and the other doesn't. K is not computably categorical.

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$$F = E(\sqrt[4]{p_n} \ : \ n \notin \emptyset').$$

This F is uniformly computably categorical. Given two copies F_0, F_1 of F, for each n, we wait until either some $\sqrt[4]{p_n}$ appears in F_0 or else $n \searrow \emptyset'$. Once either of these occurs, we know what to do: if $n \in \emptyset'$, we can map $\sqrt{p_n}$ in F_0 to either of $\pm\sqrt{p_n}$ in F_1; while if $\sqrt[4]{p_n} \in F_0$, we map it to $\pm\sqrt[4]{p_n} \in F_1$ and use this to define the map on $\pm\sqrt{p_n}$.

Scott families

Defn. (folklore)

A *Scott family* for a structure $\mathcal{A}$ is a set S of wff's $\varphi(\vec{x})$ such that:

- For every $\vec{a} \in \mathcal{A}^{<\omega}$, some wff $\varphi(x_1, \ldots, x_n) \in S$ has $\models_{\mathcal{A}} \varphi(\vec{a})$.
- If $\varphi \in S$ and $\models_{\mathcal{A}} (\varphi(\vec{a})\ \&\ \varphi(\vec{b}))$, then $(\exists \alpha \in \text{Aut}(\mathcal{A}))\ \alpha(\vec{a}) = \vec{b}$.
- For every $\varphi \in S$, some $\vec{a} \in \mathcal{A}^{<\omega}$ makes $\models_{\mathcal{A}} \varphi(\vec{a})$.

So each $\varphi(x_1, \ldots, x_n) \in S$ defines some automorphism orbit in $\mathcal{A}^n$.

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Thm. (Ventsov; cf. Downey-Hirschfeldt-Khoussainov)

A computable structure $\mathcal{A}$ is uniformly computably categorical iff $\mathcal{A}$ has a c.e. Scott family of Σ_1^c formulas.

Thm. (Ash-Knight-Manasse-Slaman; Chisholm)

A computable structure $\mathcal{A}$ is relatively computably categorical iff, for some $\vec{a} \in \mathcal{A}^{<\omega}$, $(\mathcal{A}, \vec{a})$ has a c.e. Scott family of Σ_1^c formulas.

Back to our example

Our field $F = \mathbb{Q}(\sqrt[4]{p_n}\ :\ n \notin \emptyset')(\sqrt{p_n}\ :\ n \in \emptyset')$ is u.c.c., and does have a Scott family of Σ_1 formulas. The important formulas are:

$$\begin{array}{ll} (\text{for } n \in \emptyset') & x^2 = p_n (= 1 + \cdots + 1,\ \ p_n \text{ times}). \\ (\text{for } n \notin \emptyset') & x^2 = p_n \ \&\ (\exists y)\ y^2 = x. \\ (\text{for } n \notin \emptyset') & x^2 = p_n \ \&\ (\exists y)\ y^2 + x = 0. \\ (\text{for } n \notin \emptyset') & x^4 = p_n. \end{array}$$

Enumerating this Scott family requires a $\emptyset'$-oracle.
But when our categoricity functional Φ is given $(F_0 \oplus F_1)$ as an oracle, it can compute $\emptyset'$ uniformly from those diagrams: just wait until either $n \searrow \emptyset'$ or a fourth root of p_n appears in F_0.

General theorem for UCC

Theorem

A countable structure $\mathcal{A}$ is uniformly computably categorical iff $\mathcal{A}$ has a Scott family S of Σ_1 formulas such that $S \leq_e \Sigma_1$-$Th(\mathcal{A})$.

An oracle for the atomic diagram of a copy of $\mathcal{A}$ always allows the categoricity operator to enumerate Σ_1-$Th(\mathcal{A})$, and it then enumerates S using the e-reduction.

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But this theorem will not be the final word . . . .

Work of Csima & Harrison-Trainor

Theorem (Csima & Harrison-Trainor, 2017)

For every countable structure $\mathcal{A}$, there is a Turing degree $\boldsymbol{d} \in \text{Spec}(\mathcal{A})$ and a countable $\boldsymbol{d}$-computable ordinal α such that:

- all $\boldsymbol{d}$-computable copies of $\mathcal{A}$ are $\boldsymbol{d}^{(\alpha)}$-computably isomorphic;
- there exist $\boldsymbol{d}$-computable $\mathcal{A}_0 \cong \mathcal{A}_1 \cong \mathcal{A}$ for which every isomorphism $f : \mathcal{A}_0 \to \mathcal{A}_1$ has $\boldsymbol{d}^{(\alpha)} \leq_T \deg(f) \cup \boldsymbol{d}$;
- and $(\forall \boldsymbol{c} \geq \boldsymbol{d})$ these both hold with $\boldsymbol{c}$ in place of $\boldsymbol{d}$.

(The idea: $\boldsymbol{d}$ can enumerate a Scott family of Σ^{in}_{α} formulas for $\mathcal{A}$.)

So, relative to $\boldsymbol{d}$, $\boldsymbol{d}^{(\alpha)}$ is the strong degree of categoricity of $\mathcal{A}$. Moreover, while many degrees $\boldsymbol{d}$ witness this theorem, the ordinal α is the same for all of them. I will call α the *categoricity ordinal* of $\mathcal{A}$.

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It is possible for α to decrease if $\mathcal{A}$ is replaced by $(\mathcal{A}, \vec{a})$, but there is a finite tuple $\vec{a}$ that yields the least possible α across all finite tuples.

Applying this theorem

The C-HT theorem allows us to parse out two necessities for computing isomorphisms between copies $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$.

- We may actually need α jumps of the atomic diagrams: a functional Φ can compute isomorphisms $\Phi^{\mathcal{B}^{(\alpha)} \oplus \mathcal{C}^{(\alpha)}}$;
- or we may only need a fixed oracle D, using $\Phi^{\mathcal{B} \oplus \mathcal{C} \oplus D}$.

For example, consider the field $E(\sqrt[4]{p_n} \ : \ n \in D)$. Categoricity is generally harder here, but only the atomic diagram and a fixed D-oracle are required: this field is UCC relative to D.

Similarly, every algebraic field K is *uniformly D-computably categorical* for some fixed set D. Indeed, it is sufficient to take D to be the jump of $\{f \in \mathbb{Q}[X] : f \text{ has a root in } K\}$.

In contrast, $(\omega^2, <)$ has categoricity ordinal 3, and $(\omega^n, <)$ has $(2n - 1)$.

Back to our $\emptyset'$ example

Once again let $F = E(\sqrt[4]{p_n} \ : \ n \notin \emptyset')$. We already saw a $\emptyset'$-c.e. Scott family of Σ_1 formulas for F. But now consider the following family:

$$\begin{array}{rl} (\text{for } n \in \emptyset') & x^2 = p_n (= 1 + \cdots + 1, \ \ p_n \text{ times}). \\ (\text{for all } n) & x^2 = p_n \ \& \ (\exists y)\ y^2 = x. \\ (\text{for all } n) & x^2 = p_n \ \& \ (\exists y)\ y^2 + x = 0. \\ (\text{for all } n) & x^4 = p_n. \end{array}$$

This family is sufficient to accomplish the same purpose, and it is just plain c.e., with no oracle.

What do Scott families really want?

Defn., revised

A *pure Scott family* for a structure $\mathcal{A}$ is a set S of wff's $\varphi(\vec{x})$ such that:

- For every $\vec{a} \in \mathcal{A}^{<\omega}$, some wff $\varphi(x_1, \ldots, x_n) \in S$ has $\models_{\mathcal{A}} \varphi(\vec{a})$.
- If $\varphi \in S$ and $\models_{\mathcal{A}} (\varphi(\vec{a})\ \&\ \varphi(\vec{b}))$, then $(\exists \alpha \in \text{Aut}(\mathcal{A}))\ \alpha(\vec{a}) = \vec{b}$.
- For every $\varphi \in S$, some $\vec{a} \in \mathcal{A}^{<\omega}$ makes $\models_{\mathcal{A}} \varphi(\vec{a})$.

A *cluttered Scott family* for a structure $\mathcal{A}$ is a set S of wff's $\varphi(\vec{x})$ s.t.:

- For every $\vec{a} \in \mathcal{A}^{<\omega}$, some wff $\varphi(x_1, \ldots, x_n) \in S$ has $\models_{\mathcal{A}} \varphi(\vec{a})$.
- If $\varphi \in S$ and $\models_{\mathcal{A}} (\varphi(\vec{a})\ \&\ \varphi(\vec{b}))$, then $(\exists \alpha \in \text{Aut}(\mathcal{A}))\ \alpha(\vec{a}) = \vec{b}$.

In a cluttered Scott family S, every orbit is defined by some $\varphi \in S$, and each $\varphi \in S$ defines an orbit – but the empty set is considered an orbit. Extraneous wff's φ are allowed in, provided that they cause no harm.

Pure vs. cluttered

Theorem

For a countable structure $\mathcal{A}$ and $\alpha < \omega_1^{CK}$, the following are equivalent:

1. $\mathcal{A}$ has a pure Scott family S of Σ^c_α formulas, all $\leq_e \Sigma^c_{\alpha+1}$-$Th(\mathcal{A})$.

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1. $\mathcal{A}$ has a pure Scott family S of Σ^c_α formulas, all $\leq_e \Sigma^c_{\alpha+1}$-$Th(\mathcal{A})$.
2. $\mathcal{A}$ is uniformly α-jump categorical (a.k.a. $\Delta^0_{\alpha+1}$-categorical).

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The same holds relative to an oracle D.

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(1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) is clear. For (2) $\Rightarrow$ (3), let Γ be an α-categoricity operator for $\mathcal{A}$. Then for each m and $\sigma \in 2^m$ such that $\Gamma^{\sigma\oplus\sigma}(x)$ halts for every $x < n$, we add to C the wff

$$(\exists y_{n+1} \cdots \exists y_m)\ \psi(x_1, \ldots, x_n, y_{n+1}, \ldots, y_m)$$

where ψ is the Π^c_α formula described by σ, using our fixed Gödel coding of the Π^c_α-formulas..

The conundrum

Corollary

For a countable structure $\mathcal{A}$ and $\alpha < \omega_1^{CK}$, the following are equivalent:

- $\mathcal{A}$ has a pure Scott family S of Σ^c_α formulas with $S \leq_e \Sigma^c_{\alpha+1}$-$Th(\mathcal{A})$.
- $\mathcal{A}$ has a c.e. cluttered Scott family C of $\Sigma^c_{\alpha+1}$ formulas.

The same holds relative to an oracle D.

There should be a direct proof of this equivalence, without the intermediate step of uniform D-computable α-jump categoricity.

For $\Leftarrow$, it simply involves using $\Sigma^c_{\alpha+1}$-$Th(\mathcal{A})$ to unclutter the Scott family. But what about $\Rightarrow$?