Complexity of embeddings between bi-embeddable structures

Joint work with Nikolay Bazhenov and Maxim Zubkov

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Bi-embeddable structures

Two structures $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable if there are embeddings of either in the other.

($f : \mathcal{A} \to \mathcal{B}$ is an embedding if f is $1-1$ and for all $\bar{a} \in A^n$ and relation symbols R $\bar{a} \in R^{\mathcal{A}}$ iff $f(\bar{a}) \in R^{\mathcal{B}}$.)

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What is the complexity of embeddings between bi-embeddable structures?

Definition (Bazhenov, Fokina, R., San Mauro)

A countable (not necessarily computable) structure $\mathcal{A}$ is relatively Δ^0_α b.e. categorical if for any bi-embeddable copy $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable by $\Delta^{\mathcal{A}\oplus\mathcal{B}}_\alpha$ embeddings.

If $\mathcal{A}$ is computable and there is a least degree **d** that can compute embeddings between bi-embeddable computable copies of $\mathcal{A}$, then we say that **d** is the degree of b.e. categoricity of $\mathcal{A}$.

Hyperarithmetic is recursive

Theorem (Montalbán)

Every hyperarithmetic linear ordering is bi-embeddable with a computable one.

Theorem (Montalbán, Greenberg)

Every hyperarithmetic Boolean algebra, compact metric space, Abelian p-group is bi-embeddable with a computable one.

Theorem (Fokina, R., San Mauro)

Every equivalence structure is bi-embeddable with a computable one.

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The set of degrees of bi-embeddable copies of a structure is upwards closed.

$\Rightarrow$ If $\mathcal{A}$ is in one of the above classes and hyperarithmetic, then it has bi-embeddable copies of all Turing degrees.

This is in contrast to the set of isomorphic copies of a structure.

What is the complexity of embeddings between bi-embeddable structures in these classes?

Theorem (Bazhenov, Fokina, R., San Mauro)
The degree of b.e. categoricity of a computable equivalence structure is either $\mathbf{0}, \mathbf{0}'$*, or* $\mathbf{0}''$*.*

Theorem (BFRS)
Every equivalence structure is relatively Δ^0_3 *b.e. categorical.*

In this talk we will give partial answers for linear orders and Boolean algebras.

Definition

A linear order $\mathcal{L}$ is scattered if $\eta \not\hookrightarrow \mathcal{L}$.

Every countable linear order is embeddable into η, so if $\mathcal{L}$ is scattered, then it is bi-embeddable with η.

Proposition

If $\mathcal{L}$ is non-scattered, then it is not relatively Δ^0_α b.e. categorical for any computable α and does not have a degree of b.e. categoricity.

Proof sketch.

As $\mathcal{L}$ is non-scattered it is bi-embeddable with η. Take a standard copy of η and a copy $\mathcal{H}$ of $\omega_1^{\mathrm{C}K} \cdot (1+\eta)$ without hyperarithmetic decreasing sequences. Then any embedding of η into $\mathcal{H}$ computes a hyperarithmetic decreasing sequence, so it can not be hyperarithmetic. Every degree of b.e. categoricity is hyperarithmetic and thus $\mathcal{L}$ does not have such a degree. (BFRS) □

Definition

The class **VD** of linear orderings is defined by

1. $\mathsf{VD}_0 = \{0, 1\}$,
2. $\mathsf{VD}_\alpha = \{\sum_{i \in \tau} \mathcal{L}_i : \mathcal{L}_i \in \bigcup_{\beta < \alpha} \mathsf{VD}_\beta,\ \tau \in \{\omega, \omega^*, \zeta\}\}$,
3. and $\mathsf{VD} = \bigcup_\alpha \mathsf{VD}_\alpha$.

The *VD*-rank of a linear order $\mathcal{L}$ is the least α such that $\mathcal{L} \in \mathsf{VD}_\alpha$ and the Hausdorff-rank of $\mathcal{L}$ is the least α such that $\mathcal{L}$ is a finite sum of linear orders in VD_α.

Theorem (Hausdorff)

A countable linear order is scattered iff it has countable Hausdorff rank.

Indecomposable and h-indecomposable linear orders

Definition

A linear order $\mathcal{L}$ is indecomposable if whenever $\mathcal{L} = A + B$, either $\mathcal{L} \hookrightarrow A$ or $\mathcal{L} \hookrightarrow B$.

Theorem (Jullien)

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Definition (Montalbán)

A signed tree T is a tree in $\omega^{<\omega}$ with each node $\sigma \in T$ labelled by $s_T(\sigma) = \{+, -\}$.

Given a signed tree T define a linear order $lin(T)$. Let $\sigma \in T$,

1. if $rk(\sigma) = 0$ and $s_T(\sigma) = +/= -$, then $lin(T_\sigma) = \omega/= \omega^*$,
2. if $rk(\sigma) > 0$ and $s_T(\sigma) = +/= -$, then

$$lin(T_\sigma) = \sum_{i \in \omega} \sum_{k<i} T_{\sigma^\frown k} / = \sum_{i \in \omega^*} \sum_{k<i} T_{\sigma^\frown k}.$$

and $lin(T) = lin(T_\emptyset)$.

$\mathcal{L}$ is h-indecomposable if $\mathcal{L} = lin(T)$ for some signed tree T.

Theorem (Montalbán)

If $\mathcal{L}$ is indecomposable and scattered of Hausdorff rank computable α, then $\mathcal{L}$ is bi-embeddable with $lin(T)$ for some computable tree T.

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Theorem (Montalbán)

If $\mathcal{L}$ is indecomposable and scattered of Hausdorff rank computable α, then $\mathcal{L}$ is bi-embeddable with $lin(T)$ for some computable tree T.

Theorem (Bazhenov, R., Zubkov)

If $\mathcal{L}$ is a linear order of finite Hausdorff rank n, then it is relatively Δ^0_{2n+2} b.e. categorical.

Theorem (BRZ)

If $\mathcal{L}$ is indecomposable and of finite rank n, then there are bi-embeddable computable copies $\mathcal{G}$ and $\mathcal{B}$ such that Δ^0_{2n+1} does not compute an embedding $\mathcal{B} \hookrightarrow \mathcal{G}$.

There is no Δ^0_{2n+1} embedding

Proof sketch. Let T be the computable signed tree of $\mathcal{L}$ and let σ be a node of maximal height in the tree. Let P be the tree consisting of a single path ending in σ. Assume $s_T(\emptyset) = +$.

Then $\mathcal{G} = lin(T_{\langle 0\rangle}) + lin(P) + lin(T_{\langle 0\rangle}) + lin(T_{\langle 1\rangle}) + lin(P) + \dots$.

Construct $\mathcal{B}$ by first constructing a Δ^0_{2n+1} copy of ω and then jump inverting it to obtain a computable copy $\mathcal{B}$ with required properties.

$$
\begin{array}{lccccccccccccccc}
\mathcal{G}_\alpha = & t^0_0 & + & p_0 & + & t^1_0 & + & t^1_1 & + & p_1 & + & t^2_0 & + & t^2_1 & + & t^2_2 \\
\mathcal{B}_\alpha = & t^0_0 & + & [p^l_0, p^r_0] & + & t^0_0 & + & t^1_0 & + & [p^l_1, p^r_1] & + & t^0_0 & + & t^1_0 & + & t^2_0
\end{array}
$$

The intervals $[p^l_i, p^r_i]$ are used to prevent the partial Δ^0_{2n+1} computable functions from being an embedding $\mathcal{B}_\alpha \hookrightarrow \mathcal{G}_\alpha$.

We then replace all the t_i^j with the corresponding h-indecomposable linear orders in $\mathcal{B}_\alpha$ and then jump invert the intervals $[p_i^l, p_i^r]$ n-times using a theorem by Ash and Knight.

Theorem (Ash, Knight)

Let $\mathcal{L}$ be a linear order. Then $\mathcal{L}$ is Δ_3^0 iff $\omega \cdot \mathcal{L}$ and $\omega^* \cdot \mathcal{L}$ are computable.

We use this theorem on $[p_i^l, p_i^r]$ n times using ω and ω^* as given by σ.

After this process we obtain a computable linear order

$$\mathcal{B} = lin(T_{\langle 0 \rangle}) + lin(P)k_0 + lin(T_{\langle 0 \rangle}) + +lin(T_{\langle 1 \rangle}) + lin(P)k_1 + \dots$$

where $k_i = |[p_i^l, p_i^r]|$. $\mathcal{B}$ and $\mathcal{G}$ are bi-embeddable but there can not be a Δ_{2n+1}^0 embedding of $\mathcal{B}$ in $\mathcal{G}$ as otherwise we would get a Δ_{2n+1}^0 embedding of $\mathcal{B}_\alpha$ in $\mathcal{G}_\alpha$. $\square$

Corollary (BRZ)

Let $\mathcal{L}$ be a linear order of rank n, then it is not relatively Δ^0_{2n+1} b.e. categorical.

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Conjecture

If $\mathcal{L}$ has computable Hausdorff rank α, then $\mathcal{L}$ is $\Delta^0_{2\alpha+2}$ b.e. categorical but not $\Delta^0_{2\alpha+1}$ b.e. categorical.

Question

Does every computable linear order of Hausdorff rank α have degree of b.e. categoricity $\mathbf{0}^{(2\alpha+2)}$ ($\mathbf{0}^{(2\alpha+1)}$ if α finite)?

Boolean algebras

A Boolean Algebra is superatomic, if it does not contain a atomless subalgebra.

The structure theory of countable Boolean algebras is simpler:

- If $\mathcal{A}$ is not superatomic, then $\mathcal{A}$ is bi-embeddable with the countable atomless Boolean algebra.
- If $\mathcal{A}$ is superatomic, then $\mathcal{A}$ is isomorphic to *Int*(α) for some ordinal α.

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- If $\mathcal{A}$ is superatomic, then $\mathcal{A}$ is isomorphic to *Int*(α) for some ordinal α.

For α an ordinal, let α_0 be the largest exponent in its Cantor normal form. Then:

Theorem

1. *If $\mathcal{A}$ is not superatomic, then $\mathcal{A}$ does not have a degree of b.e. categoricity and is not hyperarithmetically b.e. categorical.*
2. *If $\mathcal{A}$ is computable, infinite, and superatomic, then $\mathcal{A} \cong$ Int(α) for some computable infinite α and $\mathcal{A}$ has degree of b.e. categoricity* $\mathbf{0}^{(2\alpha_0)}$ *(*$\mathbf{0}^{(2\alpha_0-1)}$ *if α_0 finite).*

Thank you!