Definition. A theory Γ is *complete* if for every formula ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

Lemma. Every consistent theory can be extended to a complete consistent theory.

Proof. Let $\langle \phi_n : n \in \omega \rangle$ enumerate all formulas. By induction on $n \in \omega$ build theories Γ_n such that

- $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$
- Γ_n is consistent;
- $\phi_n \in \Gamma_{n+1}$ or $\neg \phi_n \in \Gamma_{n+1}$.

The induction step is performed by lemma on "proof by cases". If both Γ_n, ϕ_n and $\Gamma_n, \neg\phi_n$ are inconsistent then so is Γ_n, which contradicts the induction hypothesis.

Let $\Delta = \bigcup_n \Gamma_n$. This theory is consistent; any (finite) proof of contradiction from Δ would have appeared in some Γ_n, which is impossible.

Definition. A truth assignment V is a *model* of Γ if $V(\phi) = 1$ for every $\phi \in \Gamma$.

Lemma. Γ is consistent if and only if it has a model.

So, the following are equivalent:

- $\Gamma \models \phi$
- $\Gamma, \neg\phi$ has no model;
- $\Gamma, \neg\phi$ is inconsistent;
- $\Gamma \vdash \phi$.

Suppose first that Γ has a model V. Argue that formulas appearing in every formal proof from Γ have truth value 1 in V; this prevents reaching a contradiction from Γ.

Suppose now that Γ is consistent. Extend it if necessary to a complete consistent theory. Define a function V by setting $V(\phi) = 1$ if $\phi \in \Gamma$. It will be enough to show that this is a truth assignment–then, it is a model for Γ.

Verification of truth assignment properties at negation:

- if $V(\phi) = 1$ then we should have $V(\neg\phi) = 0$. Indeed, if $\phi \in \Gamma$ then $\neg\phi \notin \Gamma$ by the consistency of Γ;

- if $V(\phi) = 0$ then we should have $V(\neg\phi) = 1$. Indeed, if $\phi \notin \Gamma$ then $\neg\phi \in \Gamma$ by the completeness of Γ.

Verification of truth assignment properties at implication $\phi \to \psi$:

- if $V(\psi) = 1$ then we should have $V(\phi \to \psi) = 1$. Indeed, the following formulas are in Γ: ψ, $\psi \to (\phi \to \psi)$, $\phi \to \psi$ and so $V(\phi \to \psi) = 1$.

- if $V(\phi) = 0$ then we should have $V(\phi \to \psi) = 1$. The following formulas are in Γ: $\neg\phi$, $\neg\phi \to (\neg\psi \to \neg\phi)$, $\neg\psi \to \neg\phi$, $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$, $\phi \to \psi$.

- if $V(\psi) = 0$ and $V(\phi) = 1$, then we should have $V(\phi \to \psi) = 0$. Indeed $\phi, \neg\psi \in \Gamma$ and so $\phi \to \psi$ cannot be in Γ by the consistency of Γ.

First order logic: language

- logical connectives, parentheses;
- variables (infinitely many of them);
- quantifiers $\forall$ (for every) and possibly $\exists$ (there is);
- equality symbol;
- special functional and relational symbols, each with assigned arity.

0-ary functional symbols are *constants*.

Example. $\in$ is a special binary relational symbol for ZFC, $+, \cdot, 0, 1$ are special functional symbols of arithmetic

First order logic: terms

- every variable is a term;
- if F is a n-ary functional symbol and $t_0, t_1, \ldots t_{n-1}$ are terms then $f(t_0, t_1, \ldots t_{n-1})$ is a term;
- all terms are obtained by repeated applications of the previous items.

Example. $(x^2+0)\cdot 1$ is a term of the language of arithmetic.

First order logic: formulas

- if t_0, t_1 are terms then $t_0 = t_1$ is a formula;
- if R is a n-ary relational symbol and $t_0, t_1, \ldots t_{n-1}$ are terms then $R(t_0, t_1, \ldots t_{n-1})$ is a formula;
- if ϕ, ψ are formulas then $\neg(\phi)$ and $(\phi) \to (\psi)$ are formulas;
- if x is a variable then $\forall x(\phi)$ is a formula;
- all formulas are obtained by repeated application of previous items.

Example. $\forall x \neg \forall y\ (x = y \to x \in z)$ is a formula of the language of set theory.

Free variables, substitution

$\phi = \ldots \forall x (\psi) \ldots$: ψ is the *range* of the quantifier, every occurence of x inside ψ is *bounded*. An occurence of x in ϕ which is not bounded is *free*. Formula without free variables is a *sentence*.

Example. $\forall x \ (x \in y \vee x = y)$: x is not free, y is.

If t is a term and x is free in ϕ then $\phi(t/x)$ results from replacing all free occurences of x in ϕ by t. Similarly for a sequence of terms $\vec{t}$ and a sequence of free variables $\vec{x}$ of same length: $\phi(\vec{t}/\vec{x})$. The substitution is *proper* if the variables in the terms do not become bounded.

Example. $x^2 + y^2$ cannot be properly substituted for z in $\forall x \ (y + x = z)$.

First order logic: axioms

- axioms of propositional logic;
- $(\forall x\ \phi) \to \phi(t/x)$ if the substitution is proper;
- $\forall x\ (\phi \to \psi) \to (\forall x \phi \to \forall x\ \psi)$;
- $\phi \to \forall x \phi$ if x is not free in ϕ.

Also add universal quantifiers in front of these.
Inference rule: modus ponens.

First order logic: models

Suppose $\mathfrak{L} = \{R_i : i \in I, F_j : j \in J\}$ is a language of first order logic, with arities n_i, n_j respectively. An $\mathfrak{L}$-*model* is a tuple $\mathfrak{M} = \langle M, R_i^{\mathfrak{M}} : i \in I, F_j^{\mathfrak{M}} : j \in J \rangle$ where

- M is a nonempty set–universe of the model;
- $R_i^{\mathfrak{M}} \subset M^{n_i}$ is a relation for each $i \in I$– realization of the symbol R_i;
- $F_j^{\mathfrak{M}} : M^{n_j} \to M$ is a function for $j \in J$– realization of the functional symbol F_j.

Example. $\langle \mathbb{N}, 0, 1, +, \cdot \rangle$ is a model for the language of arithmetic.

Models: plugging in

Let $\mathfrak{L}$ be a language of first order logic and $\mathfrak{M}$ an $\mathfrak{L}$-model. If t is an $\mathfrak{L}$ term with variables $\vec{x}$, and $\vec{m}$ is a tuple of elements of M, define $t^{\mathfrak{M}}(\vec{m}/\vec{x})$:

- if $t = x$ for a variable x then $t^{\mathfrak{M}}(m/x) = m$;
- if $t = F_j(t_0, \ldots t_{n_j - 1})$ then $t^{\mathfrak{M}}(\vec{m}/\vec{x})$ equals to $F_j^{\mathfrak{M}}(t_0^{\mathfrak{M}}(\vec{m}/\vec{x}), \ldots)$.

Models: satisfaction

Let $\mathfrak{L}$ be a language of first order logic and $\mathfrak{M}$ an $\mathfrak{L}$-model. For every formula $\phi(\vec{x})$ and a tuple $\vec{m}$ of elements of M, define $\mathfrak{M} \models \phi(\vec{m}/\vec{x})$:

- if ϕ is $t_0 = t_1$ then $\mathfrak{M} \models \phi(\vec{m}/\vec{x})$ if $t_0^{\mathfrak{M}}(\vec{m}/\vec{x}) = t_1^{\mathfrak{M}}(\vec{m}/\vec{x})$;

- if ϕ is $R_i(t_0, t_1, \dots)$ then $\mathfrak{M} \models \phi(\vec{m}/\vec{x})$ if $(t_0^{\mathfrak{M}}(\vec{m}/\vec{x}), \dots) \in R_i^{\mathfrak{M}}$;

- if $\phi = \neg\psi$ then $\mathfrak{M} \models \phi$ if $\mathfrak{M} \models \psi$ fails;

- if $\phi = \forall y \psi$ then $\mathfrak{M} \models \phi(\vec{m}/\vec{x})$ if for every $n \in M$, $\mathfrak{M} \models \psi(\vec{m}/\vec{x}, n/y)$.

Example. Theory of dense linear order without endpoints.

- language: $\leq$

- axioms: $\forall x, y\ x \leq y \vee y \leq x$, $x \leq y \wedge y \leq x \rightarrow x = y$, $\ldots x < y \rightarrow \exists z\ x < z < y$, $\exists z\ z < x$, $\exists z\ x < z$.

- models: the rational numbers.

The theory has only one countable model. It is complete, and there is an algorithm for identifying its theorems.

Example. Theory of groups.

- language: $\cdot$, inverse, 1;

- axioms: $\forall x \forall y \forall z \;\; x(yz) = x(yz)$, $xx^{-1} = x^{-1}x = 1$, $xy = 1 \to x = y^{-1}$.

- models: every group is a model of the theory of groups.

Question A. Is there an algorithm identifying theorems of theory of groups?

Question B. Given a group G, is there an algorithm for deciding which sentences G satisfies?

Example. Peano Arithmetic.

- language: $0, <$, S, $+$, $\cdot$, exponentiation

- axioms: some statements such as $\forall x \forall y\ x + Sy = S(x+y)$, plus the induction scheme: if $\phi(x)$ is a formula, then $\phi(0) \wedge \forall x\ (\phi(x) \rightarrow \phi(Sx))$ implies $\forall x \phi$.

- models: $\langle \mathbb{N}, 0, S, +, \cdot,$ exponentiation$\rangle$.

Question A. Is there an algorithm identifying theorems of Peano Arithmetic?

Question B. Is there any other model?

Question C. Is Peano Arithmetic complete?

Completeness theorem for first order logic

For a set Γ of formulas, define $\Gamma \vdash \phi$ if there is a formal proof of ϕ from Γ. For a set Γ of sentences, define $\Gamma \models \phi$ if every model $\mathfrak{M} \models \Gamma$ also satisfies ϕ.

Theorem. $\Gamma \vdash \phi$ if and only if $\Gamma \models \phi$.

Restatement. A theory is consistent if and only if it has a model.