# On the existence of a $\sigma$-closed dense subset*

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**Abstract**

It is consistent with the axioms of set theory that there are two co-dense partial orders, one of them $\sigma$-closed and the other one without a $\sigma$-closed dense subset.

## 1 Itroduction

One of the oldest properties of partial orders occurring in forcing arguments is $\sigma$-closedness. A partial ordering $\langle P, \leq\rangle$ is $\sigma$-closed if every countable decreasing sequence of elements of $P$ has a lower bound. This property easily implies that forcing with $P$ adds no new reals, preserves stationary subsets of $\omega_1$ and so on. In this note, partially answering a question of Bohuslav Balcar, I will prove that having a $\sigma$-closed dense subset is not a forcing property of partial orders–it is not invariant under the co-density equivalence. The story is somewhat parallel to the Axiom A case. While Axiom A is a property of posets that was used with great success in the early years of forcing and still occurs in many textbooks, it is not really a forcing property of posets in this sense. I will prove

**Theorem 1.1.** *It is consistent with ZFC set theory that there is a partial order $\langle P \cup Q, \leq\rangle$ such that both $P$ and $Q$ are dense parts in it, $P$ is $\sigma$-closed, while $Q$ has no $\sigma$-closed dense subset.*

The method of proof closely follows the argument of [3]. The result is perhaps not entirely satisfactory in the sense that the existence of such partial orders may be a theorem of ZFC, and it is even not excluded that the $\sigma$-closed part $P$ may be isomorphic to one of the standard $\sigma$-closed partial orders such as adding $\aleph_2$ many subsets of $\omega_1$ with countable approximations. In the model for

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the theorem, the continuum hypothesis holds and the posets have size $\aleph_2$, which is minimal possible by the results of Foreman [1] and Vojtáš [5].

The notation of the paper follows the set theoretic standard of [2].

## 2 The proof

Work in the theory ZFC+CH. The partial orders $P$ and $Q$ are added by countable approximations. Define a partial order $R$ to consist of quintuples $r = \langle P_r, Q_r, \leq_r, C_r, F_r\rangle$ such that

1. $P_r, Q_r$ are disjoint countable subsets of $\omega_2$;
2. $\leq_r$ is a partial order on $P_r \cup Q_r$ such that both $P_r$ and $Q_r$ are dense in it, and moreover $\alpha \leq_r \beta \to \beta \in \alpha \vee \beta = \alpha$;
3. $C_r$ is a countable set of descending chains in the poset $\langle Q_r, \leq_r\rangle$ with no lower bound in $\leq_r$;
4. $F_r : P_r \times C_r \to Q_r$ is a function such that $F_r(p, c) \in c$ is an element of the chain $c$ such that every common lower bound $p' \leq p, F_r(p, c)$ in $P_r$ is incompatible with some element of the chain $c$ in $\leq_r$.

The ordering on $R$ is defined by $r_1 \leq r_0$ if each coordinate of $r_0$ is a subset of the corresponding coordinate of $r_1$ and moreover, if $p, q \in P_{r_0} \cup Q_{r_0}$ are incompatible (resp. incomparable) in $\leq_{r_0}$ then they are also incompatible (resp. incomparable) in $\leq_{r_1}$.

A bit of explanation is necessary here. Let $G \subset R$ be a generic filter and look into the model $V[G]$. The partial order $\langle P \cup Q, \leq\rangle$ from the main theorem is obtained from the generic filter $G$ as the unions of the first three coordinates of the conditions in the generic filter. The last requirement in the second item is necessary to avoid the possibility that the density of $P$ is $\aleph_1$, which would be impossible by Foreman's result. The $\sigma$-closedness of $P$ will be guaranteed by a density argument. The descending chains in the set $C = \bigcup_{r \in G} C_r$ will have no lower bounds and will be plentiful enough so that $Q$ will contain no $\sigma$-closed dense subset. The function $F_r$ is a technical tool that guarantees that adding a lower bound to a countable decreasing chain in $P$ does not necessitate adding a lower bound to one of the chains in $C$.

I will proceed with a series of more or less immediate lemmas.

**Lemma 2.1.** *The forcing $R$ is $\sigma$-closed.*

*Proof.* If $\langle r_n : n \in \omega\rangle$ is a descending chain of conditions in $R$ then its coordinatewise union is still a condition in $R$ and is the lower bound. □

**Lemma 2.2.** *The forcing $R$ has $\aleph_2$-c.c.*

In fact, I will prove that $R$ has the $\aleph_2$-p.i.c. introduced by Shelah [4, Chapter VIII]. Strictly speaking, the lemma is not neccessary for the proof of the main theorem. Its proof is embedded in the proof of Lemma 2.4 and should be viewed as a warm-up for that more complicated argument. Note that the poset $P \cup Q$ must have size greater than $\aleph_1$ in order to satisfy the desired properties, and therefore I must at least show that the forcing $R$ preserves $\aleph_2$.

*Proof.* Suppose that $\langle r_\alpha : \alpha \in \omega_2 \rangle$ is a collection of conditions in $R$. I must produce $\alpha \neq \beta$ such that the conditions $r_\alpha$ and $r_\beta$ are compatible in $R$. Choose a large enough cardinal $\theta$ and countable elementary submodels $M_\alpha$ of $H_\theta$ containing the collection of conditions as well as the ordinal $\alpha$. By standard $\Delta$-system and counting arguments, using the continuum hypothesis assumption, I will be able to find ordinals $\alpha \in \beta$ such that the corresponding models are isomorphic via a function $\pi : M_\alpha \to M_\beta$ which is the identity on their intersection (the *root*) and satisfies $\pi(\alpha) = \beta$. I can also require that all ordinals in $\omega_2$ and the root are smaller than all ordinals in $\omega_2 \cap M_\alpha$ and not the root, which are in turn smaller than all the ordinals in $\omega_2 \cap M_\beta$ and not the root. I will prove that the conditions $r_\alpha$ and $r_\beta$ have a lower bound. Write $r_\alpha = \langle P_\alpha, Q_\alpha, \leq_\alpha, C_\alpha, F_\alpha \rangle$ and similarly for $\beta$ and note that $\pi(r_\alpha) = r_\beta$.

The common lower bound $r$ is defined as the coordinatewise union on the first three coordinates of $r_\alpha$ and $r_\beta$. The function $F_r$ must extend $F_\alpha \cup F_\beta$. It is necessary to define $F_r(p, c)$ where $p \in P_\alpha \setminus P_\beta$ and $c \in C_\beta \setminus C_\alpha$, or vice versa, where $p \in P_\beta \setminus P_\alpha$ and $c \in C_\alpha \setminus C_\beta$. The latter case is just a mirror image of the former case. In the former case, note that $c$ must contain some condition $q \notin M_\alpha$ (otherwise $c = \pi^{-1} c \in C_\alpha$) and let $F_r(p, c)$ be one such condition in the chain $c$.

It is not difficult to verify that indeed $r \in R$. Consider for example the condition (4) in the case where $p \in P_\alpha$ and the root and $c \in C_\beta$ and not the root. Then, $F_r(p, c)$ is a condition in $Q_\beta$ and not the root. All conditions $\leq_r p$ are in $P_\alpha \cup Q_\alpha$ and not the root, all conditions $\leq_r F_r(p, c)$ are in $P_\beta \cup Q_\beta$ and not the root, these two sets are disjoint, therefore $p, F_r(p, c)$ are $\leq_r$-incompatible and (4) holds.

In order to verify that $r \leq r_\alpha, r_\beta$, I need to show that the incompatibility relation on $\leq_r$ extends that of $\leq_\alpha$ and $\leq_\beta$. For this, note that if a condition $p \in P_\alpha \cup Q_\alpha$ does not belong to the root, it has no elements of $P_\beta \cup Q_\beta$ below it. $\square$

**Lemma 2.3.** *$R \Vdash \dot{P}$ is $\sigma$-closed.*

*Proof.* Suppose that $r \in R$ forces $\dot{a} = \langle \dot{p}_n : n \in \omega \rangle$ is a descending chain of elements of $\dot{P}$. I must find a stronger condition forcing a lower bound to this chain. A reference to genericity will then complete the argument.

Use the $\sigma$-closedness of $R$ to strengthen $r$ if necessary to decide the names $\dot{p}_n$ to be certain specific elements $p_n \in P_r$. Define a condition $r' \leq r$ by extending the poset $P_r \cup Q_r$ by adding an element $p \in P_{r'}$ such that for every $q \in P_r \cup Q_r$, $p \leq_{r'} q$ if and only if there is $n \in \omega$ with $p_n \leq_r q$; and adding a countable chain $b$ below $p$ which contains alternately elements of $P_{r'}$ and $Q_{r'}$. Define $C_{r'} = C_r$

and $F_{r'}$ to be a certain extension of $F_r$. I must define the values $F_{r'}(p, c)$ for every chain $c \in C_r$. The values $F_{r'}(q, c)$ for $q \in b$ will be defined in the same way.

For the definition, write $d = \{q \in c : \exists n \in \omega\ p_n \leq q\}$. Nnote that $d \neq c$: either $F_r(p_0, c) \notin d$, or else, if $F_r(p_0, c) \in d$ as witnessed by $p_n$, then $p_n$ is incompatible with some element of $c$ by the properties of the function $F_r$, and this element then must fall out of $d$. In any case, let $F_{r'}(p, c)$ be any element of $c \setminus d$. It is immediate that $F_{r'}(p, c)$ is incompatible with $p$ in $\leq_{r'}$ and therefore the condition (4) is satisfied in this case.

It is now not difficult to check that $r' \in R$, $r' \leq r$ and $r' \Vdash \check{p}$ is a lower bound of $\dot{a}$ as desired. $\square$

**Lemma 2.4.** *$R \Vdash \dot{Q}$ does not have a dense $\sigma$-closed subset.*

*Proof.* Suppose that $r \Vdash \dot{D} \subset \dot{Q}$ is dense. I will find a condition $r' \leq r$ such that there is a chain $d \in C_{r'}$ such that for every $q \in d$, $r' \Vdash \check{q} \in \dot{D}$. Such condition of course forces that $\check{d}$ is a descending chain in $\dot{D}$ with no lower bound.

For every ordinal $\alpha \in \omega_2$ find a condition $r_\alpha \leq r$ such that there is a condition $q_\alpha \in Q_{r_\alpha}$ which is as an ordinal larger than $\alpha$ and $r_\alpha \Vdash \check{q}_\alpha \in \dot{D}$. This is possible by the second requirement in (2), $R \Vdash \dot{Q} \cap \alpha$ is not dense in $\dot{Q}$. Thinning out if necessary, I may assume that $\langle q_\alpha : \alpha \in \omega_2 \rangle$ in fact form an increasing sequence as ordinals. Now, let $\theta$ be a large enough cardinal number and for every ordinal $\alpha \in \omega_2$ choose a countable elementary submodel $M_\alpha \prec H_\theta$ containing $\dot{D}$ as well as $r_\alpha, q_\alpha$. By a standard $\Delta$-system and counting arguments using the continuum hypothesis assumptions, find ordinals $\alpha_n : n \in \omega$ such that the models $\langle M_{\alpha_n} : n \in \omega \rangle$ form a $\Delta$-system, they are pairwise isomorphic via functions $\pi_{mn} : M_{\alpha_m} \to M_{\alpha_n}$ which form a commuting system and are equal to the identity on the root of the $\Delta$-system, $\pi_{mn}(r_{\alpha_m}) = r_{\alpha_n}$, $\pi_{mn}(q_{\alpha_m}) = q_{\alpha_n}$, and moreover, whenever $m \in n$ then all ordinals in $\omega_2 \cap M_{\alpha_m} \setminus M_{\alpha_n}$ are smaller than all ordinals in $\omega_2 \cap M_{\alpha_n} \setminus M_{\alpha_m}$, but greater than all ordinals in the root and $\omega_2$. I will produce a lower bound $r'$ of the conditions $\{r_{\alpha_n} : n \in \omega\}$ such that $d = \{q_{\alpha_n} : n \in \omega\} \in C_{r'}$. This will complete the proof.

In fact, there is a canonical such condition $r'$. In order to facilitate the notation during the construction, write $r_{\alpha_n} = \langle P_n, Q_n, \leq_n, C_n, F_n \rangle$ and $q_{\alpha_n} = q_n \in Q_n$ for every number $n \in \omega$. We are going to have $P_{r'} = \bigcup_n P_n$, $Q_{r'} = \bigcup_n Q_n$. The ordering $\leq_{r'}$ is the inclusion-minimal one which extends all $\leq_n$: $n \in \omega$ and contains $d$ as a chain. Since I want to make sure to get a condition $\leq r_n$ for all $n$, I must verify that the incompatibility relation of $\leq_{r'}$ extends the incompatibility relations of all $\leq_n$: $n \in \omega$. Well, suppose that $n \neq m \in \omega$ and $p, p' \in P_n \cup Q_n$ are conditions and $q \in P_m \cup Q_m$ is their lower bound in $\leq_{r'}$; I must find their lower bound in $\leq_n$. There are two cases. Either $q$ belongs to the root, in which case it is enough to observe that $\leq_n = \leq_{r'}$ on the root and therefore $q$ is the required lower bound in $P_n$ as well. Or $q$ does not belong to the root. In such a case, the minimality condition on $\leq_{r'}$ implies that either $q \leq_m p$, or $n \in m$ and $q_n \leq_n p$ and $q \leq_m q_m$ (and the same condition on $p'$). In any case, this means that $\pi_{mn}(q)$ is the required lower bound of $p, p'$ in

$\leq_n$. A similar break into cases also proves the following implications for every $p \in P_n \cup Q_n$ and $q \in P_m \cup Q_m$: if $p \geq_{r'} q$ then $p \geq_n \pi_{mn}(q)$, and if $p$ and $q$ are compatible in $\leq_{r'}$ then $p$ and $\pi_{mn}(q)$ are compatible in $\leq_n$.

Let $C_{r'} = \{d\} \cup \bigcup_n C_n$. Note that $d$ has no lower bound in $P_{r'} \cup Q_{r'}$ since it is cofinal in this set with the ordinal ordering. Finally, the function $F_{r'}$ will extend $\bigcup_n F_n$. Note that $\bigcup_n F_n$ is indeed a function: if $p \in P_n$ and $c \in C_n$ for some $n \in \omega$ then either $c$ is not in the root and then $\langle p, c \rangle$ is not in the domain of the functions $F_m : m \neq n$, and if $p, c$ both belong to the root then so does $F_n(p, c)$ and for every $m \in \omega$, $F_n(p, c) = \pi_{nm}(F_n(p, c)) = (\pi_{nm} F_n)(\pi_{nm} p, \pi_{nm} c) = F_m(p, c)$. To verify that (4) holds, suppose that $p \in P_n, c \in C_n$, and $q \leq_{r'} p, F(p, c)$. I must show that $q$ is not compatible with all elements of the chain $c$. Indeed, if $q \in P_m \cup Q_m$ were compatible with all elements of the chain $c$ (which are all in $P_n \cup Q_n$), by the last sentence of the previous paragraph $\pi_{mn} q$ would be $\leq_n$ compatible with all elements of $c$, contradicting the property (4) of the function $F_n$.

I must define the values $F_{r'}(p, c)$ where $p \in P_n$ and not in the root, and $c \in C_m$ not in the root, for some $n \neq m \in \omega$, Here, observe that all but finitely many elements of $c$ fall out of the root of the $\Delta$-system: the $\pi$ embeddings move countable sequences pointwise and if they fixed all elements of $c$, they would all fix $c$ and put $c$ in the root. Then note that all but finitely many elements of $c$ are not above $q_m$ in $\leq_m$ because $q_m$ is not a lower bound of $c$ in that ordering. The definition of $F_{r'}(p, c)$ divides into two possibilities, $m \in n$ and $n \in m$. If $m \in n$, let $F_{r'}(p, c) = q$ be an element of $c$ which is not above $q_m$ and not in the root. The minimality of the ordering $\leq_{r'}$ then implies that $p$ and $q$ are incompatible and therefore (4) is satisfied. If $n \in m$ then let $F_{r'}(p, c) = q$ be an element of $c$ which is not in the root, not above $q_m$, and below $F_m(q_m, c)$. The verification of (4) is more complicated here. If $p \not\geq_n q_n$ then $p$ is incompatible with $q$ and therefore (4) holds. If $p \geq_n q_n$ then indeed there may be a lower bound $p'$ of $p$ and $q$. By the minimality of $\leq_{r'}$ it must be the case that $p' \in P_m \cup Q_m$ but not in the root, and $p' \leq q_m$. Then $p'$ is incompatible with one of the elements of $c$ by (4) applied to $F_m(q_m, c)$ and the minimality of $\leq_{r'}$.

Finally, I have to define the values of $F_{r'}(p, d)$ for $p \in P_{r'}$. Just let $F_{r'}(p, d) = q_0$. To see that (4) holds, let $p' \leq p$ be an arbitrary element of $P_{r'}$ below $q_0$. $p'$ does not belong to the root, and must belong to $P_n$ for some $n \in \omega$. The minimality of $\leq_{r'}$ implies that $p' \leq_{r'} q_n$. However, $p' \neq q_n$ since $q_n \notin P_n$, and the minimality of $\leq_{r'}$ implies that $p'$ is incompatible with $q_{n+1}$. (4) follows.

$\square$

Together, the lemmas show that $V[G]$ has the same cardinals and reals as $V$, and $P, Q$ are codense partial orders, one of them $\sigma$-closed and the other without a $\sigma$-closed dense subset, proving the theorem.

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