Strong measure zero sets in Polish groups*

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Abstract

In the context of arbitrary Polish groups, we investigate the Galvin–Mycielski–Solovay characterization of strong measure zero sets as those sets for which a meager collection of right translates cannot cover the whole group.

1 Introduction

Emile Borel defined the collection of strong measure zero sets of reals:

Definition 1.1. A set $A \subset \mathbb{R}$ is a strong measure zero set if for every sequence $\langle \varepsilon_n : n \in \omega \rangle$ of positive real numbers there are intervals $I_n \subset \mathbb{R}$ of respective legths ε_n such that $A \subset \bigcup_n I_n$.

It is clear that every countable set of reals has strong measure zero. The failure of the effort to produce an uncountable strong measure zero set lead Borel to the following conjecture:

Conjecture 1.2. (Borel conjecture) *The strong measure zero sets are exactly the countable sets.*

Today we know [8] that under the Continuum Hypothesis there are uncountable strong measure zero sets, and in a certain model of ZFC (the Laver model) there are no uncountable strong measure zero sets. Thus, the Borel conjecture is not decidable in ZFC set theory. One interesting feature of strong measure zero sets is the following characterization, proved by Galvin, Mycielski, and Solovay:

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Fact 1.3. [3] *A set $A \subset \mathbb{R}$ is strong measure zero if and only if for every meager set $M \subset \mathbb{R}$, $A + M \neq \mathbb{R}$.*

The topic of this paper is a generalization of strong measure zero sets to arbitrary Polish groups and the verification of the above characterization in this more general context. One should remark right away that if Borel conjecture holds, then all strong measure zero sets in any given Polish group are countable and the GMS characterization holds automatically. Thus, one is interested only in the context of the Continuum Hypothesis, which is in a sense opposite of that of Borel conjecture.

Definition 1.4. (ZFC+CH) A Polish group G is GMS if for every set $A \subset G$, A is strong measure zero if and only if for every meager set $M \subset G$, $A \cdot M \neq G$.

The choice of the Continuum Hypothesis context is critical also because of the syntactical complexity of the GMS property. The statement "G is GMS" is $\Pi^2_1(G)$, and as such is decided in the Ω-logic by the Continuum Hypothesis. Also, if the decision is positive under CH, it is positive already in ZFC [7]; in other words, under CH there are as many non-GMS groups as possible.

An obvious question is whether there is a restatement of the above definition which would make an equivalent sense even in the context of ZFC. There is a natural conjecture which, if true, would identify the class of GMS groups with a classical class of groups:

Conjecture 1.5. (ZFC+CH) *A Polish group G is GMS if and only if it is locally compact.*

The right-to-left implication has been known [2, Section 534], [6, 4]. We investigate the problematic left-to-right implication and confirm it in two extensive classes of groups:

Theorem 1.6. *The conjecture holds in the class of groups with bi-invariant metric, and in the class of closed subgroups of S_∞.*

This greatly extends the result of [4], where it is proved that the Baer–Specker group $\mathbb{Z}^\omega$ is not GMS.

We use the set theoretic notational standard of [5]. Let G be a group. Unless $G = \mathbb{R}$, we denote the group operation by $\cdot$. If $g \in G$ and $A \subset G$ then gA denotes the set $\{gh \colon h \in A\}$; similar usage prevails for right translates and inverses of subsets of G.

2 Strong measure zero sets in arbitrary Polish groups

In this section, we define the strong measure zero sets in arbitrary Polish groups and verify their basic properties.

Definition 2.1. Let G be a Polish group. A set $A \subset G$ is *left strong measure zero, lsmz* if for every sequence $\langle U_n : n \in \omega \rangle$ of nonempty open subsets of the group there are elements $g_n \in G$ for $n \in \omega$ such that $A \subset \bigcup_n g_n U_n$.

Theorem 2.2. *Let G be a Polish group.*

1. *The left strong measure zero sets form a bi-invariant σ-ideal;*

2. *if the Borel conjecture holds, then the left strong measure zero sets are exactly the countable sets;*

Proof. For (1), it is easy to verify that the left strong measure zero sets form a σ-ideal. Suppose $A = \bigcup_n A_n$ and for every $n \in \omega$ the set $A_n \subset G$ is lsmz. To verify that A is lsmz, suppose that U_m for $m \in \omega$ are nonempty open subsets of the group. Let $\{b_n : n \in \omega\}$ be a partition of ω into countably many infinite sets, and use the lsmz property of each set A_n to find group elements g_m for $m \in \omega$ such that for every $n \in \omega$, $A_n \subset \bigcup_{m \in b_n} g_m U_m$. Then $A \subset \bigcup_{m \in \omega} g_m U_m$.

To verify the invariance, suppose that $A \subset G$ is a left strong measure zero set and $g \in G$ is an arbitrary element. To see that gA is lsmz, suppose that $\langle U_n : n \in \omega \rangle$ is a sequence of nonempty open subsets of the group. As A is lsmz, there are elements $g_n \in G$ such that $A \subset \bigcup_n g_n U_n$, and then $gA \subset \bigcup_n g g_n U_n$. This confirms that the set gA is lsmz. To verify that the set Ag is lsmz, suppose that $\langle U_n : n \in \omega \rangle$ is a sequence of nonempty open subsets of the group. As A is lsmz, there are elements $g_n \in G$ such that $A \subset \bigcup_n g_n U_n g^{-1}$. Then, $Ag \subset \bigcup_n g_n U_n$, and so the set Ag is lsmz as required.

For (2), recall the result of Carlson: the Borel conjecture implies that separable strong measure zero metric spaces must be countable. See [1, Theorem 3.2]. □

The following two theorems record what is known regarding the GMS characterization of left strong measure zero sets in general Polish groups.

Theorem 2.3. *Let G be a Polish group.*

1. *Whenever $A \subset G$ is a set and $A \cdot C \neq G$ for every nowhere dense set $C \subset G$, then A is lsmz;*

2. *if G is locally compact and $A \subset G$ is a lsmz set, then $A \cdot C \neq G$ for every meager set $C \subset G$.*

Proof. For (1), fix a set $A \subset G$. Let $\langle U_n : n \in \omega \rangle$ be a sequence of nonempty open subsets of G. Let $\langle g_n : n \in \omega \rangle$ be a sequence of elements of the group such that the set $\bigcup_n g_n U_n \subset G$ is open dense. Let $C = (G \setminus \bigcup_n g_n U_n)^{-1}$; as the inverse map is a self-homeomorphism of G, the set $C \subset G$ is closed nowhere dense. Now, since $A \cdot C \neq G$, there is a group element $g \in G$ such that $g \notin A \cdot C$. This means that for every $h \in A$, $g \notin h \cdot C$, so $h \notin g \cdot C^{-1}$, in other words $h \in \bigcup_n g \cdot g_n U_n$. Thus, $A \subset \bigcup_n g \cdot g_n U_n$ and the set $A \subset G$ is lsmz.

For (2), see [6] or [2, Section534] or [4]. □

The attempts to prove the converse to Theorem 2.3(1) for Polish groups which are not locally compact are the starting point of the current paper. To violate the converse, we introduce a special type of nowhere dense set.

Definition 2.4. [4] Let G be a Polish group. A nonempty set $C \subset G$ is *anti-GMS* if it is nowhere dense and for every sequence $\langle U_n : n \in \omega \rangle$ of open neighborhoods of 1 there is a sequence $\langle g_n : n \in \omega \rangle$ of elements of G such that for every $g \in G$, the set $g \cdot \bigcup_n g_n \cdot U_n$ is dense in C.

Theorem 2.5. *(ZFC+CH) Let G be a Polish group. If there is an anti-GMS set in G then the group G is not GMS.*

Proof. Let $C \subset G$ be an anti-GMS set; passing to a closure if necessary, we may assume that it is closed. We will construct a lsmz set $A \subset G$ such that $A \cdot C^{-1} = G$.

Let $\{U_n^\alpha : n \in \omega, \alpha \in \omega_1\}$ be an enumeration of all ω-sequences of nonempty basic open subsets of G. Since C is an anti-GMS set, we can choose elements $\{g_n^\alpha : n \in \omega, \alpha \in \omega_1\}$ such that for every $\alpha \in \omega_1$ the set $\bigcup_n g_n^\alpha U_n^\alpha$ is dense in C. Let $\{h_\alpha : \alpha \in \omega_1\}$ be an enumeration of the group G. For each ordinal $\alpha \in \omega_1$ we will find elements $g_\alpha \in G$ and $k_\alpha \in C$ so that

- if $\beta < \alpha$ then $g_\alpha \in \bigcup_n g_n^\beta U_n^\beta$;
- $g_\alpha \cdot k_\alpha^{-1} = h_\alpha$.

Let $A = \{g_\alpha : \alpha \in \omega_1\}$. The first item ensures that A is a lsmz set. The second item shows that $G = A \cdot C^{-1}$, and this will complete the proof of the theorem.

To construct g_α, k_α, note that for every $\beta \in \alpha$, the sets $\bigcup_n h_\alpha^{-1} g_n^\beta U_n^\beta$ are open dense in C. Since there are only countably many such sets, there is some $k_\alpha \in C$ which belongs to all of them. Let $g_\alpha = h_\alpha \cdot k_\alpha$. By the choice of k_α, for every $\beta \in \alpha$ it is the case that $g_\alpha \in \bigcap_{\beta \in \alpha} \bigcup_n g_n^\beta U_n^\beta$ as desired. Finally, $g_\alpha k_\alpha^{-1} = h_\alpha k_\alpha k_\alpha^{-1} = h_\alpha$ as desired in the second item and the proof is complete. □

We do not know if the converse to Theorem 2.5 holds. Nevertheless, the failure of GMS property in Polish groups in this paper is always obtained through a construction of an anti-GMS set. Note that if a set $C \subset G$ is anti-GMS then its closure is anti-GMS as well. Thus, the existence of anti-GMS set is a projective statement.

As a final remark in this section, in Definition 2.1 one multiplies by the group elements from the left. It is also possible to consider a similar definition with multiplication from the right: a set $A \subset G$ is *right strong measure zero, rsmz* if for every sequence $\langle U_n : n \in \omega \rangle$ of nonempty open subsets of the group there are elements $g_n \in G$ for $n \in \omega$ such that $A \subset \bigcup_n U_n g_n$. We will not work on the difference between the left and right strong measure zero sets beyond the following basic theorem.

Theorem 2.6. *Let G be a Polish group.*

1. *For every set $A \subset G$, A is lsmz if and only if $A^{-1} \subset G$ is rsmz;*

2. *if G admits a bi-invariant metric or the Borel conjecture holds then the lsmz and rsmz sets coincide;*

3. *if $G = S_\infty$ and the continuum hypothesis holds, then the ideals of lsmz and rsmz sets are not the same.*

Proof. For the left-to-right directon of (1), suppose that A is lsmz. To verify the rsmz of A^{-1}, suppose that $\langle U_n : n \in \omega\rangle$ is a sequence of nonempty open subsets of the group. By the lsmz property of the set A, there is a sequence $\langle g_n : n \in \omega\rangle$ of elements of the groups such that $A \subset \bigcup_n g_n U_n^{-1}$. Then, $A^{-1} \subset \bigcup_n U_n g_n^{-1}$, verifying the rsmz of A^{-1}. The right-to-left implication is similar.

For (2), if d is a bi-invariant metric on the grup G, then both left and right shfts of d-balls of a fixed radius $\varepsilon > 0$ are exactly the d-balls of radius ε. In the definition of both lsmz and rsmz sets, shrinking the sets U_n if necessary we may assume that they are d-balls, and then the definitions of rsmz and lsmz coincide. If the Borel conjecture holds then both the lsmz and rsmz ideal coincide with the σ-ideal of countable sets.

For (3), in view of (1) it is enough to find a set $A \subset S_\infty$ such that A is left strong measure zero while A^{-1} is not. Let $\langle x_\alpha : \alpha \in \omega_1\rangle$ be an enumeration of increasing functions in ω^ω and let $\langle y_\alpha : \alpha \in \omega_1\rangle$ be an enumeration of S_∞^ω. By transfinite induction on α we will produce points $z_\alpha \in S_\infty^\omega$ and $g_\alpha \in S_\infty$ such that

1. for every $\beta \in \alpha$, $g_\alpha \in \bigcup_n [z_\beta(n) \restriction x_\beta(n)]$ and $g_\alpha^{-1} \notin \bigcup_n [y_\alpha^{-1}(n) \restriction n]$.

Once this is done, it is clear that the set $A = \{g_\alpha : g \in \omega_1\}$ is strong left measure zero while the set A^{-1} is not. An additional inductive hypothesis on the points z_α will be the following:

2. Let s be a finite injection from ω to ω. Let $m < k$ be the first two numbers not in the range of s. We require that there will be a set $a \subset \omega$ of size at least $k+1$ such that the finite injections $z_\alpha(n) \restriction x_\alpha(n)$ for $n \in a$ extend s, they all have m in their range, neither has k in their range, and their preimages of m are pairwise distinct.

Now, suppose that z_β, x_β have been found for $\beta \in \alpha$. It is easy to find z_α satisfying (2); this does not use the inductive assumption at all. To construct the point g_α, choose an enumeration $\{\beta_i : i \in \omega\} = \alpha$. By induction on $i \in \omega$ find $n_i \in \omega$ such that the finite injections $s_i = z_{\beta_i}(n_i) \restriction x_{\beta_i}(n_i)$ form a chain with respect to inclusion, and writing m_i for the smallest number not in the range of s_i, $m_i \in \mathrm{rng}(s_{i+1})$ and $[s_i]^{-1} \cap \bigcup_{n \leq m_i} [y_\alpha^{-1}(n) \restriction n] = 0$. Once this is done, the point $x_\alpha = \bigcup_i s_i$ is as required.

To perform the induction on i, start with $s_{-1} = 0$. Suppose s_i has been constructed. Let $m < k$ be the first two numbers not in the range of s_i. Use the induction hypothesis (2) at β_i and s_i to find a number $n_i \in \omega$ such that the injection s_{i+1} contains m in its range, does not contain k in its range, and for every $m < n < k$, the preimages of m under $y_\alpha(n)$ and s_{i+1} are distinct. Then, s_{i+1} works as desired. □

3 Submeasures on groups

A useful tool for packaging the proof of Theorem 1.6 in the case of groups with bi-invariant metric is the concept of left invariant submeasure on a topological group. Recall that a submeasure μ on a Polish space X is outer regular if for every set $A \subset X$, $\mu(A) = \inf\{\mu(O) \colon A \subset O, O \text{ open}\}$. The submeasure μ is non-atomic if $\mu(\{x\}) = 0$ for every point $x \in X$.

Theorem 3.1. *Let G be an uncountable Polish group. There is a left invariant, outer regular, countably subadditive, nonatomic, nonzero submeasure on G.*

Proof. It will be enough to produce a local inclusion-decreasing basis $\langle U_n \colon n \in \omega \rangle$ at 1 such that whenever $m \in \omega$ is a number and $a_n \subset G$ are sets of respective size n for all $n > m$, then U_m is not a subset of $\bigcup_{n>m} a_n \cdot U_n$. Once this is done, define the submeasure μ by setting $\mu(A) = \inf\{\Sigma_i 1/n_i \colon \langle g_i, n_i \colon i \in \omega \rangle$ are such that $A \subset \bigcup_i g_i U_{n_i}\}$. The requested properties of μ follow from the definition. The only important point is that $\mu(U_n) = 1/n$, in particular open sets have nonzero μ-mass. This, however, follows from the initial properties of the local basis.

To construct the local basis, choose a left invariant metric d and a complete metric e for the group G. By induction find the open sets $U_n \subset G$ together with sets $b_n \subset U_n$ of size $n+1$ such that

- writing ε_n for the d-diameter of U_{n+1}, the points in b_n are $d - 3\varepsilon_n$ distant from each other and also from the complement of U_n;
- for every $m < n$ and every choice $g_i \in b_i$ for $m < i < n$ the sets $\prod_{m<i<n} g_i \cdot U_n$ have e-diameter $< 1/n$.

To check the desired properties of the basis, suppose that $m \in \omega$ is a number and $a_n \subset G$ are sets of respective size n for all $n > m$. By induction on $n > m$ choose elements $g_n \in b_n$ so that the set $\prod_{m<i<n} g_i U_n$ is disjoint from all sets gU_n for $g \in a_n$. This is possible since the metric d is left-invariant and so each set gU_n for $g \in a_n$ intersects at most one of the sets $\prod_{m<i<n} g_i \cdot h \cdot U_n$ for $h \in b_n$. The closures of the sets $\prod_{m<i<n} g_i U_n$ for $n < m$ form an inclusion-decreasing sequence with e-diameter tending to zero and so their intersection is nonempty by the completeness of the metric e. The point in their intersection belongs to U_m but not to the set $\bigcup_{n>m} a_n \cdot U_n$. This completes the proof. □

It is entirely natural to consider left invariant submeasures in conjunction with left strong measure zero sets, as the following theorem, due to Jan Grebík, shows. Lsmz sets are in a sense exactly the universally left invariant submeasure zero sets:

Theorem 3.2. *Let G be a Polish group, and $A \subset G$ be a set. The following are equivalent:*

1. *A is lsmz;*

2. *for every left invariant, outer regular, countably subadditive, nonatomic submeasure μ on G, $\mu(A) = 0$.*

Proof. To see why (1) implies (2), suppose that A is lsmz, μ is a left invariant submeasure on G, and $\varepsilon > 0$ is a real number. To show that $\mu(A) < \varepsilon$, just use the assumption that μ is nonatomic to find open neighborhoods $U_n \subset G$ of the unit for $n \in \omega$ such that $\Sigma_n \mu(U_n) < \varepsilon$. Use the assumption that A is lsmz to find group elements $g_n \in G$ for $n \in \omega$ such that $A \subset \bigcup_n g_n \cdot U_n$. Finally use the countable subadditivity and invariance of μ to conclude that $\mu(A) \leq \Sigma_n \mu(g_n \cdot U_n) < \varepsilon$ as desired.

To see why (2) implies (1), suppose that (2) is satisfied and $U_n \subset G$ for $n \in \omega$ are nonempty open sets; we must find group elements $g_n \in G$ for $n \in \omega$ such that $A \subset \bigcup_n g_n \cdot U_n$. Thinning out the sets U_n and shifting them on the left, we may assume that they are all neighborhoods of the unit and form a basis at the unit. Find numbers $n_i \in \omega$ for $i \in \omega$ such that $n_{i+1} > 2n_i$, write $\mathcal{O} = \{A_{n_i} : i \in \omega\}$ and define a submeasure μ on the group G by setting $w(A_{n_0}) = 1$ and $w(A_{n_{i+1}} = 1/n_i$ and $\mu(B) = \inf\{\Sigma_j w(V_j) : V_j \in \mathcal{O}$ and there are group elements $g_j \in G$ such that $B \subset \bigcup_j g_j \cdot V_j\}$. It is immediate that this is a left invariant, countably subadditive etc. submeasure on G. By (2), $\mu(A) < 1$ holds, and so there are sets $V_j \in \mathcal{O}$ and group elements g_j such that $A \subset \bigcup_j g_j \cdot V_j$ and moreover, $\Sigma_j w(V_j) < 1$. The latter inequality shows that for every $i \in \omega$, the set $U_{n_{i+1}}$ occurs fewer than n_i-many times among the V_j's, and the set U_{n_0} does not occur at all. The indexes n_{i+1} increase so fast that it is possible to find sets $W_j \in \{U_n : n \in \omega\}$ for $j \in \omega$ such that $W_j \supset V_j$ and each U_n occurs at most once among the W_j's. But then, $A \subset \bigcup_j g_j \cdot W_j$ and the set A is lsmz as desired. □

4 Groups with bi-invariant metric

Theorem 4.1. (ZFC+CH) *Suppose that G is a non-locally compact Polish group with a bi-invariant metric. Then G is not GMS.*

Proof. By Theorem 2.5 it is enough to construct an anti-GMS set in the group G. Let μ be a left invariant countably subadditive diffuse submeasure on G. We will build a closed nowhere dense set $C \subset G$ such that whenever $O \subset G$ is an open set with nonempty intersection with C, there are nonempty open sets $\{Q_m : m \in \omega\}$, all subsets of O, all left translates of each other, such that the numbers $\mu(Q_m \setminus C)$ tend to zero as m tends to infinity. Such a set $C \subset G$ will be anti-GMS.

To see this, choose a countable collection of open neighborhoods $\{U_n : n \in \omega\}$ of the unit; shrinking if necessary, we may assume that their diameters tend to zero. Choose a pairwise disjoint collection $\{a_O : O \subset G$ basic open$\}$ of infinite subsets of ω. For each O such that $O \cap C \neq 0$ find an open neighborhood $W_O \subset G$ of the unit such that O contains infinitely many left translates of W_O such that the μ-masses of their intersections with the complement of C tend to 0. Removing finitely many elements from a_O if necessary, we may find an open neighborhood V_O of the unit such that for each $n \in a_O$ $V_O \cdot U_n \subset W_O$. Now, pick elements $g_n, h_n \in G$ for $n \in a_O$ such that $h_n W_O \subset O$ and $\mu(h_n W_O \setminus C) < \mu(U_n)$, and $G = \bigcup_{n \in a_O} h_n V_O g_n^{-1}$. This is possible since there is a bi-invariant compatible metric. We claim that the group elements $\{g_n : n \in \bigcup_O a_O\}$ work as desired.

Indeed, if $g \in G$ is an arbitrary element and $O \subset G$ is a basic open set with nonempty intersection with C, there must be a number $n \in a_O$ such that $g \in h_n V_O g_n^{-1}$, or in other words $g \cdot g_n \in h_n V_O$. Then $g \cdot g_n U_n \subset h_n V_O U_n \subset h_n W_O \subset O$. Now, $\mu(h_n W_O \setminus C) < \mu(U_n)$ and so $C \cap O \cap g \cdot g_n U_n \neq 0$ as desired.

The construction of the nowhere dense closed set $C \subset G$ is performed by a routine induction. Let $\langle O_k : k \in \omega \rangle$ be an enumeration of a basis for the group G. By induction on $k \in \omega$ build sets $P_k \subset G$ and countable sets A_k such that

1. P_k is an increasing sequence of open subsets of G and $O_k \cap P_{k+1} \neq 0$;
2. A_k is a countable set of pairs of the form $\langle Q, \varepsilon \rangle$ such that $Q \subset G$ is open, $\varepsilon > 0$. No point of G belongs to more than k many sets mentioned in A_k;
3. whenever $\langle Q, \varepsilon \rangle \in A_k$ then $\mu(\bar{P}_{k+1} \cap Q \setminus \bar{P}_k) < 2^{-k}\varepsilon$. Moreover, if k is the smallest such that $\langle Q, \varepsilon \rangle \in A_k$ then $\bar{P}_k \cap Q = 0$.
4. either (a) $O_k \subset P_{k+1}$, or (b) the set A_{k+1} contains an infinite collection of open sets, all left translates of each other, all subsets of O_k, paired with real numbers tending to zero.

In the end, let $C = G \setminus \bigcup_k P_k$. By the first item, this is a closed nowhere dense set. By the third item, whenever $\langle Q, \varepsilon \rangle \in \bigcup_k A_k$ then $\mu(Q \setminus C) < \varepsilon$. Now, if O_k is a basic open set with nonempty intersection with C, then (b) must have occurred at k in the last item, and so the set O_k contains an infinite collection of open subsets with the required properties.

To perform the induction, start with $P_0 = A_0 = 0$. If P_k has been constructed, the next step divides into two cases, corresponding to (a) and (b) of the last item. Either $O_k \subset \bar{P}_k$, in which case let $A_{k+1} = A_k$ and $P_{k+1} = P_k \cup O_k$, and proceed to $k+1$. If $O_k \not\subset \bar{P}_k$, find disjoint open sets $R, S \subset O_k \setminus \bar{P}_k$. Use the second item to find a nonempty open subset $R' \subset R$ which is either disjoint from or a subset of every open set mentioned in A_k. Since the submeasure μ is diffuse, there is a nonempty open set $R'' \subset R'$ with $\mu(\bar{R}'')$ so small that $P_{k+1} = P_k \cup R''$ is going to satisfy the demands of the third item for all pairs $\langle Q, \varepsilon \rangle \in A_k$ with $R' \subset Q$. Finally, use the fact that the set S is not compact to find infinitely many balls of equal diameter with pairwise disjoint closures inside S, and include them all in A_{k+1}, paired with some real numbers tending to zero. □

5 Closed subgroups of S_∞

Theorem 5.1. (ZFC+CH) *Let G be a non-locally-compact, closed subgroup of S_∞. Then G is not GMS.*

Proof. By Theorem 2.5, it is enough to construct an anti-GMS set $C \subset G$. By the lack of local compactness, there is an infinite set $a \subset \omega$ such that for every $n \in a$ there are infinitely many m such that there is $g \in G$ for which $g \restriction n$ is the identity and $g(n) = m$. For numbers $n < m$ both in a define $R_{n,m}$ to be the relation on G connecting g, h if $g(n) \notin \mathrm{rng}(h \restriction m)$, and $h(n) \notin \mathrm{rng}(g \restriction m)$. Note that the relation $R_{n,m}$ is left invariant.

Let B be the set of all finite injections u from ω to ω such that $\mathrm{dom}(u) \in a$ and there is $g \in G$ with $u \subset g$. For $u \in B$ write $[u] = \{g \in G : u \subset g\}$.

Claim 5.2. *Let $n < m$ be numbers in a and $u \in B$ is of size n. There are $g, h \in G$ which both extend u and such that $g\ R_{n,m}\ h$ holds.*

Proof. Since the relation $R_{n,m}$ is left invariant, it is enough to prove this for u equal to the identity function on n. Let $g_k : k \in \omega$ be elements of G which up to n are equal to the identity, and $g_k(n)$ for $k \in \omega$ are pairwise distinct numbers. Thinning the collection if necessary, we may assume for every number $l \in [n, m)$ the numbers $g_k(l)$ are either pairwise distinct or all equal. Since the g_k's are injections, this means that for each $k_0 \in \omega$ the set of all k_1 such that $g_{k_0}(n) \in \mathrm{rng}(g_{k_1} \restriction [n, m))$, is finite. Thus, for every number k large enough the pair $g = g_0$ and $h = g_k$ will work as required. $\square$

Let $A \subset B$ be a dense set such that

- the ranges of injections in A are linearly ordered by inclusion;
- whenever $u \in B$ has no initial segment in A then for every m it has an extension $v \in B$ with no initial segment in A and such that $m \in \mathrm{rng}(v)$.

It is easy to construct such a set A by a bookkeeping argument. Let $C = G \setminus \bigcup_k [s_k]$; this is a closed nowhere dense subset of C. Note that if $u \in B$ then $C \cap [u] = 0 \leftrightarrow \exists s \in A\ s \subseteq u$. We claim this set C works.

Suppose that $b \subset a$ is an infinite set. We must find group elements $g_n \in G$ for $n \in b$ such that for every $g \in S_\infty$ $g \cdot \bigcup_n [g_n \restriction n] \cap C$ is dense in C. Just use the claim to find the group elements so that for every $u \in B$ of length some $k \in a$ there are numbers $n_0 < n_1 \in b$ such that both g_{n_0}, g_{n_1} extend u and $g_{n_0}\ R_{k,n_1}\ g_{n_1}$ holds. We claim that $\{g_n : n \in b\}$ works.

Suppose that $g \in G$ is an element and $v \in B$ is such that $C \cap [v] \neq 0$. Let $k = \mathrm{dom}(v)$ and $u = g^{-1} \cdot v$, and find numbers $n_0 < n_1$ in b such that g_{n_0}, g_{n_1} extend u and are R_{k,n_1}-related. We claim that either $[g \cdot g_{n_0} \restriction n_0]$ or $[g \cdot g_{n_1} \restriction n_1]$ must have nonempty intersection with C–this will suffice as both of these sets are subsets of $[v]$. If this is not the case, there must be injections $s_0, s_1 \in A$ such that $s_0 \subseteq g \cdot g_{n_0} \restriction n_0$ and $s_1 \subseteq g \cdot g_{n_1} \restriction n_1$. However, neither s_0, s_1 can be a subset of v since $C \cap [v] \neq 0$, so their respective ranges must contain the numbers $g \cdot g_{n_0}(k)$ and $g \cdot g_{n_1}(k)$ which do not belong to the range of the other sequence. This contradicts the assumption that the ranges of s_0, s_1 are linearly ordered by inclusion. $\square$

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