Dynamics of Polish groups, submeasures, and a new concentration of measure

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Polish groups and their dynamics

A topological group G is **Polish** if its group topology completely metrizable and separable.

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Examples:

— the **unitary** group of the separable infinite dimensional Hilbert space;

— the group of all (classes of) **measure preserving transformations** of a Borel probability measure space;

— the **isometry** group of a Polish metric space;

— the **homeomorphism** group of a compact second countable space;

— the **automorphism** group of a countable (model theoretic) structure.

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Ellis: There exists a unique **universal minimal flow** $M(G)$, that is, $M(G)$ is unique such that each minimal G-flow is a continuous G-equivariant image of $M(G)$.

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G is **amenable** if each G-flow has a G-invariant, regular, Borel probability measure.

The first example of an extremely amenable group

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Herer–Christensen: If ϕ is a **pathological** submeasure, then $L_0(\phi, \mathbb{R})$ is extremely amenable.

Used methods of functional analysis.

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(A) Ramsey theory

Pestov: The automorphism group of $(\mathbb{Q}, <)$ is extremely amenable.

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Gromov–Milman: The unitary group of a separable, infinite dimensional Hilbert space is extremely amenable.

Glasner, Pestov: If ϕ is a **measure** and G is an **amenable** locally compact Polish group, then $L_0(\phi, G)$ is extremely amenable.

Submeasures and their classifications

Submeasures

$\mathcal{C}$ = an algebra of subsets of X

A function $\phi\colon \mathcal{C} \to \mathbb{R}$ is a **submeasure** if

- $\phi(\emptyset) = 0$,
- ϕ is *monotone*, that is, $\phi(A) \leq \phi(B)$ for all $A, B \in \mathcal{C}$ with $A \subseteq B$, and
- ϕ is *subadditive*, that is, $\phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \in \mathcal{C}$.

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All submeasures ϕ are assumed to be diffused, that is, for every $\epsilon > 0$, there exists a finite subset $\mathcal{B} \subseteq \mathcal{C}$ such that

$$X = \bigcup \mathcal{B} \quad \text{and} \quad \phi(B) \leq \epsilon \text{ for } B \in \mathcal{B}.$$

ϕ a submeasure on $\mathcal{C}$

ϕ is a **measure** if $\phi(A \cup B) = \phi(A) + \phi(B)$ for disjoint $A, B \in \mathcal{C}$.

ϕ is **pathological** if there does not exist a non-zero measure $\mu\colon \mathcal{C} \to \mathbb{R}$ with $\mu \leq \phi$.

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Herer–Christensen (1975), **Popov** (1976), **Erdős–Hajnal** (1967), **Davies–Rogers** (1969): There exists a pathological submeasure.

Talagrand: There exists an exhaustive pathological submeasure.

A submeasure ϕ on $\mathcal{C}$ induces a (pseudo-)metric on $\mathcal{C}$

$$\mathrm{dist}_\phi(A, B) = \phi(A \triangle B), \text{ for } A, B \in \mathcal{C}.$$

Classification of submeasures

Let $C_1, \ldots, C_m \subseteq X$. Define

$$t(C_1, \ldots, C_m)$$

to be the maximum of $k \in \mathbb{N}$ such that for each $x \in X$

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$\frac{t(C_1,\ldots,C_m)}{m}$ is the **covering number** of Kelley of the sequence $(C_1, \ldots, C_m)$.

$\phi\colon \mathcal{C} \to \mathbb{R}$ a submeasure

For $\xi > 0$, let

$$\mathcal{C}_{\phi,\xi} = \{A \in \mathcal{C} \mid \phi(A) \le \xi\}.$$

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Define $h_\phi\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ by

$$h_\phi(\xi) = \frac{1}{\xi} \sup\left\{ \tfrac{t(C_1,\ldots,C_m)}{m} \;\middle|\; m \in \mathbb{N},\ m > 0,\ C_1, \ldots, C_m \in \mathcal{C}_{\phi,\xi} \right\}.$$

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Theorem (Sch.–S.)

The limit $\lim_{\xi \to 0} h_\phi(\xi)$ *exists (possibly infinite).*

A submeasure ϕ is called

- **elliptic** if $h_\phi(\xi) = O(\xi)$ as $\xi \to 0$,
- **hyperbolic** if $\frac{1}{h_\phi(\xi)} = O(\xi)$ as $\xi \to 0$,
- **parabolic** if ϕ is neither elliptic, nor hyperbolic.

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There is a **rigidity** at the hyperbolic end.

Groups of the form $L_0(\phi, G)$ and their dynamics

Groups of the form $L_0(\phi, G)$

ϕ a submeasure on $\mathcal{C}$ and G a topological group

Let

$$L_0(\phi, G)$$

be the collection of all $f\colon X \to G$, for which there exists a finite partition $\mathcal{P}$ of X into elements of $\mathcal{C}$ with

$$f \text{ is constant on } B \text{ for } B \in \mathcal{P}.$$

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$1_G \in U \subseteq G$ open and $r > 0$ determine a neighborhood of $f \in L_0(\phi, G)$ as the set of all $g \in L_0(\phi, G)$ with

$$\phi(\{x \mid f(x)g(x)^{-1} \notin U\}) < r.$$

This is the **topology of convergence in** ϕ.

Dynamics of groups of the form $L_0(\phi, G)$

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Used methods of functional analysis. The proof does not generalize much beyond $G = \mathbb{R}$.

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Glasner, Pestov: If ϕ is a **measure** and G is an **amenable** locally compact Polish group, then $L_0(\phi, G)$ is extremely amenable.

Used concentration of measure.

More results on groups of the form $L_0(\phi, G)$

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Farah–S.: If ϕ is a submeasure and G is **compact solvable** Polish group, then $L_0(\phi, G)$ is extremely amenable.

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Using Ramsey theoretic methods coming from algebraic topology, related to Lovász's calculation of the chromatic number of the Kneser graph.

Sabok: If ϕ is a submeasure and G is **locally compact abelian** Polish group, then $L_0(\phi, G)$ is extremely amenable.

Extending methods of Farah–S.

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Theorem (Sch.–S.)

If ϕ is **parabolic** *or* **hyperbolic** *and G is* **amenable**, *then $L_0(\phi, G)$ is* **extremely amenable**.

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Theorem (Sch.–S.)

If ϕ is **parabolic** *or* **hyperbolic** *and G is* **amenable**, *then $L_0(\phi, G)$ is* **extremely amenable**.

The theorem above generalizes results of Herer–Christensen, Glasner, Pestov, and, to a large degree, Farah–S. and Sabok.

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Proposition (Sch.–S.)

If ϕ is **elliptic** *or* **parabolic** *and G is* **not amenable**, *then $L_0(\phi, G)$ is* **not extremely amenable**.

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Proposition (Sch.–S.)

If ϕ is **elliptic** *or* **parabolic** *and G is* **not amenable**, *then $L_0(\phi, G)$ is* **not extremely amenable**.
In fact, $L_0(\phi, G)$ is not even amenable.

Nets of *mm*-spaces and covering concentration of submeasures

mm-spaces and their nets

$\mathcal{X} = (X, d, \mu)$ is a **metric measure space**, ***mm*-space** for short, if

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— X is a standard Borel space,

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For a Borel set $A \subseteq X$ and $r > 0$, we write

$$B_r(A) = \{x \in X \mid d(A, x) < r\}.$$

Let $(\mathcal{X}_i)_{i \in I}$ be a net of *mm*-spaces along a directed order I.

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$(\mathcal{X}_i)_{i \in I}$ has **concentration of measure** if, given Borel sets $A_i \subseteq X_i$ and $r > 0$,

$$\inf_{i \in I} \mu_i(A_i) > 0$$

implies

$$\lim_{i \in I} \mu_i(B_r(A_i)) = 1.$$

Nets of *mm*-spaces associated with a submeasure

ϕ a submeasure on $\mathcal{C}$

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For a partition $\mathcal{P}$ into elements of $\mathcal{C}$ and a set Ω, define a pseudo-metric $\delta_{\mathcal{P},\phi}$ on $\Omega^{\mathcal{P}}$ by

$$\delta_{\mathcal{P},\phi}(x,y) = \phi\Big(\bigcup\{P \in \mathcal{P} \mid x_P \neq y_P\}\Big).$$

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Given a standard Borel probability space (Ω, μ), let

$$\mathcal{X}(\mathcal{P}) = \left(\Omega^{\mathcal{P}}, \delta_{\mathcal{P},\phi}, \mu^{\otimes \mathcal{P}}\right).$$

$\mathcal{X}(\mathcal{P})$ is an *mm*-space.

Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ into elements of $\mathcal{C}$, we write

$$\mathcal{P} \preceq \mathcal{Q} \iff \forall Q \in \mathcal{Q}\, \exists P \in \mathcal{P}\;\; Q \subseteq P.$$

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$$\mathcal{P} \preceq \mathcal{Q} \iff \forall Q \in \mathcal{Q} \, \exists P \in \mathcal{P} \;\; Q \subseteq P.$$

$\preceq$ is a directed order. So

$$(\mathcal{X}(\mathcal{P}))_{\mathcal{P}}$$

is a **net of** ***mm*****-spaces**.

Covering concentration of submeasures

We say that a submeasure ϕ has **covering concentration** if the associated with it net $(\mathcal{X}(\mathcal{P}))_{\mathcal{P}}$ of *mm*-spaces has concentration of measure.

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The connection with extreme amenability is given by the following proposition.

Proposition (Sch.–S.)

If ϕ has covering concentration and G is amenable, then $L_0(\phi, G)$ is extremely amenable.

The following theorem is our main result on covering concentration. It implies extreme amenability of $L_0(\phi, G)$ for ϕ hyperbolic or parabolic and G amenable.

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Theorem (Sch.–S.)

Every **hyperbolic** *or* **parabolic** *submeasure has covering concentration.*

The previous theorem does not extend to elliptic submeasures.

Theorem (Sch.–S.)

There is a submeasure (necessarily elliptic) that does not have covering concentration.

Concentration of measure in products

N a finite non-empty set and $m > 0$

$\mathcal{C} = (C_i)_{1 \leq i \leq m}$ a cover of N, and $w = (w_i)_{1 \leq i \leq m}$ where $w_i > 0$

$(\Omega_j)_{j \in N}$ a family of non-empty sets

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Define the metric $d_{\mathcal{C},w}$ on $\prod_{j \in N} \Omega_j$ by

$$d_{\mathcal{C},w}(x, y) = \inf \left\{ \sum\nolimits_{i \in I} w_i \mid \{j \in N \mid x_j \neq y_j\} \subseteq \bigcup\nolimits_{i \in I} C_i \right\}.$$

The metric $d_{\mathcal{C},w}$ generalizes the Hamming metric on product spaces in a direction that seems "orthogonal" to an important generalization due to Talagrand.

Theorem (Sch.–S.)

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Then, for every $r > 0$,

$$\Big(\bigotimes_{j\in N} \mu_j\Big)\left(\{x \mid f(x) - \mathbb{E}(f) \geq r\}\right) \leq \exp\Big(-\tfrac{kr^2}{4(w_1^2+\cdots+w_m^2)}\Big).$$

The proof uses entropy building on work of Ledoux and involving the Herbst argument.