Projection Theorems Using Effective Dimension

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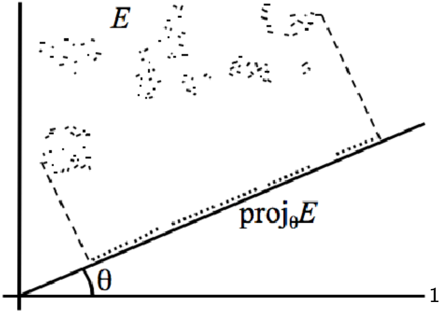
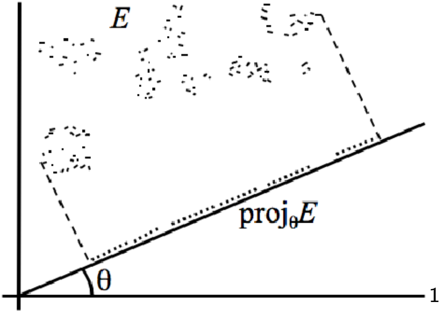
Orthogonal Projections



[1]Kenneth Falconer, Sixty years of fractal projections

Orthogonal Projections



If E is big, is it true that $\text{proj}_\theta E$ is big?

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Marstrands Projection Theorem

In the plane, we parameterize orthogonal projections with the angle the line makes with the x-axis, so

$$p_\theta : \mathbb{R}^2 \to \mathbb{R}$$
$$p_\theta(x, y) = x \cos\theta + y \sin theta.$$

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ be an analytic set with $\dim_H(E) = s$. Then for almost every $\theta \in (0, 2\pi)$,

$$\dim_H(p_\theta E) = \min\{s, 1\}.$$

Remark: Marstrand also showed that, if $\dim_H(E) > 1$, then $p_\theta E$ has positive (Lebesgue) measure.

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- Any subset of a line (or the real numbers) has Hausdorff dimension at most 1.
- Lipschitz functions (like p_θ) cannot increase the Hausdorff dimension of a set.
  - MPT shows that for a.e. angle, $\dim_H(p_\theta E)$ is maximal.
- Mattila generalized this to subspaces of dimension $1 \leq m < n$ in $\mathbb{R}^n$.
- Davies has shown that, assuming CH, there are sets for which this theorem does not hold.

Our Results

We use algorithmic information theory to reprove Marstrands theorem, and prove two new results on the fractal dimension of projections.

Theorem (N. Lutz and S. '17)

Let $E \subseteq \mathbb{R}^2$ *be any set with* $\dim_H(E) = \dim_P(E) = s$*. Then for almost every* $\theta \in (0, \pi)$,

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Kolmogorov Complexity

Definition

Fix a universal Turing machine U. Let u be a finite binary string. The *Kolmogorov complexity of u* is

$$K(u) = \min\{|\pi| \mid \pi \in \{0,1\}^*, \text{ and } U(\pi) = u\}.$$

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- The choice of universal TM is irrelevant.
- Can be extended to $\mathbb{N}$ and $\mathbb{Q}$ in a natural way.
- Can be relativized to an oracle $A \subseteq \mathbb{N}$, written as $K^A(u)$.

Kolmogorov Complexity in Euclidean Space

Definition

Let $n, r \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *Kolmogorov complexity of x at precision r* is

$$K_r(x) = \min\{K(q) \mid q \in B_{2^{-r}}(x) \cap \mathbb{Q}^n\},$$

where $B_{2^{-r}}(x)$ is the ball of radius 2^{-r} around x.

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For our purposes today, we may define

$$K_r(x) = K(d),$$

where $d = \frac{m}{2^r}$ is the closest dyadic rational at precision r to x.

Effective Dimensions of Points

Definition (Lutz '03, Mayordomo '03)

Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective Hausdorff) dimension of* x is

$$\dim(x) = \liminf_{r \to \infty} \frac{K_r(x)}{r}.$$

Definition (Athreya et al. '07, Lutz and Mayordomo '08)

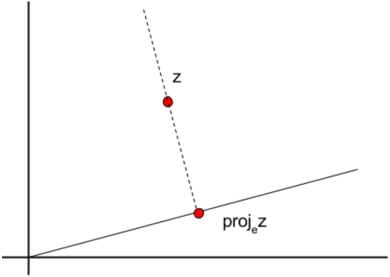
Let $n \in \mathbb{N}$, and $x \in \mathbb{R}^n$. The *(effective) strong dimension of* x is

$$\text{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}.$$

The effective dimensions of a point x measure the density of algorithmic information in x.

Projections of points

Let $z \in \mathbb{R}^2$ and $\theta \in [0, \pi)$. What can we say about $\dim^\theta(p_\theta(z))$?



Goal: If θ is *random* relative to z, then $\dim^\theta(p_\theta(z))$ is maximal, i.e.

$$\dim^\theta(p_\theta(z)) = \min\{\dim^\theta(z), 1\}. \tag{1}$$

Overview of Proof

For the remainder of the proof, fix z, θ and $r \in \mathbb{N}$. Want to show

$$K_r^\theta(p_\theta(z)) = \min\{K_r^\theta(z), 1\}$$

- Assume that $K_r(z)$ is "small".
- We want $K_r^\theta(p_\theta(z)) \geq K_r^\theta(z)$.
- Suffices to show that z is essentially the only low complexity point with this projection.
  - Enumerate all w such that $K_r(w)$ is small and $p_\theta(w) = p_\theta(z)$.

Decreasing Complexity

Conditional complexity gives a really nice way to decrease the complexity of a point.

Let $\eta \in (0, \min\{d, 1\}) \cap \mathbb{Q}$. Let $s \in \mathbb{N}$ such that $K_s(z) \approx \eta r$.

Let D be the oracle encoding the program testifying to $K_{r,s}(z \mid z)$.

- $K^D_r(z) \approx \eta r$.
- $K^D_t(z) \approx K^D_t(z)$ for all $t < s$.

Overview of Proof

Let $\eta \in (0, \min\{d, 1\}) \cap \mathbb{Q}$. Let D be oracle as in previous slide. Want to show

$$K_r^{D,\theta}(p_\theta(z)) = K_r^{D,\theta}(z)$$

- Know that $K_r^D(z)$ is ηr
- Suffices to show that z is essentially the only low complexity point with this projection.
  - Enumerate all w such that $K_r^D(w) \leq \eta r$ and $p_\theta(w) = p_\theta(z)$.

Lower Bounds on Complexity of Projections

Lemma

Suppose $z, w \in \mathbb{R}^2$ such that $p_\theta(z) = p_\theta(w)$. Given 2^{-r} approximations of z and w, it is possible to compute an approximation of θ to within 2^{-r+t}, where $t = -\log \|w - z\|$.

Corollary

Suppose $z, w \in \mathbb{R}^2$ such that $p_\theta(z) = p_\theta(w)$. Then $K_{r-t,r}(\theta \mid z) \leq K_r(w \mid z)$.

Lower Bounds on Complexity of Projections

Suppose that θ is random relative to z. Then for any w such that $p_\theta(w) = p_\theta(z)$,

$$\begin{aligned} r - t &\leq K^D_{r-t,r}(\theta \mid z) \\ &\leq K^D_r(w \mid z), \end{aligned}$$

Since w and z share about t bits, and $K^D_r(z) \approx \eta r$,

$$\begin{aligned} r - t &\leq K^D_r(w \mid z) \\ &\leq K^D_{r,t}(w \mid z) \\ &\leq K^D_r(w) - K^D_t(z) \\ &\leq K^D_r(z) - K^D_t(z) \\ &\leq \eta r - \eta t. \end{aligned}$$

So any w must be very close ($t \approx r - o(r)$) to z! Therefore $K^{D,\theta}_r(p_\theta(z)) = \eta r$

New Proof of Marstrands Theorem

Theorem (Marstrand '54)

Let $E \subseteq \mathbb{R}^2$ *be an analytic set with* $\dim_H(E) = d$*. Then for almost every* $\theta \in (0, 2\pi)$*,*

$$\dim_H(\text{proj}_\theta E) = \min\{d, 1\}.$$

By the point to set principle, it suffices to show that, for almost every θ, for every oracle $A \subseteq \mathbb{N}$, and every $\epsilon > 0$, there is a point $z \in E$ such that

$$\dim^A(\text{proj}_\theta z) \geq \min\{d, 1\} - \epsilon.$$

Using Restricted Point-to-Set Principle

Theorem (Hitchcock '03)

Let $E \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{N}$ be such that E is a Σ^0_2 set relative to B. Then

$$\dim_H(E) = \sup_{x \in E} \dim^B(x) .$$

- This restricted version gives us greater control on the oracle.
- Standard arguments show that if E is analytic, then there is a subset $F \subseteq E$ such that
  - $\dim_H(F) = \dim_H(E)$, and
  - For some oracle B, F is Σ^0_2 relative to B.
- For any such F, and any θ, $p_\theta F$ is Σ^0_2 relative to (B, θ).

New Proof of Marstrands Theorem

Let $F \subseteq E$ as in the previous slide, and $B \subseteq \mathbb{N}$ such that F is Σ^2_0 relative to B. It suffices to show that, for almost every θ and every $\epsilon > 0$, there is a point $z \in E$ such that

$$\dim^{B,\theta}(p_\theta z) \geq \min\{d, 1\} - \epsilon.$$

- *First* pick $z_1, z_2, \ldots$: Using the point to set principle, choose z_n such that $\dim^B(z_n) \geq d - 1/n$.
- For almost every θ,
    - For every n, $\dim^{B,z_n}(\theta) = 1$ (standard argument), and
    - For every n and r, $K_r^{B,\theta}(z_n) \approx K_r^B(z_n)$.
- Then we can simply relativize our result with oracle B, and therefore

$$\begin{aligned}\dim^{B,\theta}(p_\theta(z_n)) &= \liminf_{r\to\infty} \frac{K_r^{B,\theta}(p_\theta(z_n))}{r} \\ &\geq \liminf_{r\to\infty} \frac{K_r^{B,\theta}(z_n)}{r} \\ &\geq \min\{d, 1\} - 1/n.\end{aligned}$$

The End

Thank you!