Subadditive families of hypergraphs*

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Abstract

I analyse a natural class of proper forcings associated with actions of countable groups on Polish spaces, providing a practical and informative characterization as to when these forcings add no independent reals.

1 Introduction

Many σ-ideals on Polish spaces and their associated quotient posets of Borel sets are naturally associated with an action of a countable group on the underlying Polish space. In a narrow c.c.c. context it is possible to classify such situations, as Kunen showed in [4]. However, there are many examples in which the ideal is not c.c.c. A broad class of such examples has been introduced in [7, Section 2]. In this paper, I provide a practical and informative answer to the question which of the resulting quotient partial orders add independent reals. This is typically a difficult question to resolve for specific partial orders, requiring notationally demanding and repetitive fusion arguments.

First, I must define the class of σ-ideals and quotient posets in question. All of them are generated from hypergraphs. The common hypergraph nomenclature is captured in the following definition.

Definition 1.1. A *finitary hypergraph* G on a set X is a set of finite subsets of X. The elements of G are referred to as its *hyperedges*. A subset $A \subset X$ is a G-*anticlique* if its contains no subset belonging to G. A finitary hypergraph G on X is *Borel* if G is a Borel subset of the hyperspace $K(X)$ of X with the Vietoris topology.

The posets are generated from the hypergraphs in the following flexible way.

Definition 1.2. Let X be a Polish space and $\mathcal{G}$ be a countable family of analytic finitary hypergraphs on X. Then $I_{\mathcal{G}}$ denotes the σ-ideal on X generated by Borel sets $A \subset X$ which are G-anticliques for some $G \in \mathcal{G}$. In addition, $P_{\mathcal{G}}$ denotes the poset of Borel $I_{\mathcal{G}}$-positive subsets of X ordered by inclusion.

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The paper [7] provides a general method for analyzing the forcing properties of the posets of the form $P_{\mathcal{G}}$ and connecting them to simple combinatorial properties of the hypergraphs in the collection $\mathcal{G}$. Here, we deal with a large class of hypergraph posets associated with actions of countable groups as in the following definition.

Definition 1.3. [7, Definition 2.1] A countable collection $\mathcal{G}$ of analytic finitary hypergraphs on a Polish space X is *actionable* if there is a countable group Γ acting on X in a Borel way so that each hyperedge of each hypergraph in $\mathcal{G}$ consists of pairwise orbit equivalent elements and for each hypergraph $G \in \mathcal{G}$ and every $\gamma \in \Gamma$, $\gamma \cdot G \in \mathcal{G}$. A σ-ideal I on a Polish space X is *actionable* if it is equal to $I_{\mathcal{G}}$ for some actionable family $\mathcal{G}$ of analytic finitary hypergraphs. A poset P is *actionable* if it is in the forcing sense equivalent to the poset of Borel I-positive sets ordered by inclusion for some actionable σ-ideal I

It is proved in [7, Theorem 2.2, Corollary 3.18] that actionable posets are proper and bounding. Not adding independent reals is an important forcing property which escaped the theorems of [7]. Recall:

Definition 1.4. Let M be a transitive model of set theory and $x \subset \omega$ be an infinite binary sequence. The set x is an *independent real* over M if neither it nor its complement contain an infinite subset in M. A poset P *adds no independent reals* if P forces that there are no reals in the extension which are independent over the ground model.

Identifying subsets of ω with their chairacteristic functions, an independent real can be viewed as a point in 2^ω without an infinite subset in the model M. It turns out that there a simple property of actionable families of hypergraphs which is sufficient and in a suitable sense necessary to conclude that the associated poset adds no independent reals.

Definition 1.5. A family $\mathcal{G}$ of hypergraphs on a set X is *subadditive* if for every finite subset $\mathcal{H} \subset \mathcal{G}$ and a number $n \in \omega$ there is a hypergraph $G \in \mathcal{G}$ such that no G-hyperedge can be covered by a union of n-many sets, each of which is an H-anticlique for some hypergraph $H \in \mathcal{H}$.

The main theorem of this paper is the following.

Theorem 1.6. *Suppose that $\mathcal{G}$ is a subadditive, actionable countable collection of finitary Borel hypergraphs on a Polish space X. Then the poset $P_{\mathcal{G}}$ does not add independent reals.*

In fact, the criterion the theorem provides is optimal in a precise sense.

Theorem 1.7. *Suppose that G is an actionable countable collection of finitary Borel hypergraphs on a Polish space X such that the poset $P_{\mathcal{G}}$ adds no independent reals. Then there is a subadditive, actionable countable collection $\mathcal{H}$ of finitary Borel hypergraphs on X such that $I_{\mathcal{H}} = I_{\mathcal{G}}$.*

The proof shows that there is a natural candidate for the family $\mathcal{H}$. Thus, if an actionable poset is at hand and the question is to be resolved whether it adds independent reals or not, the user only needs to check the subadditivity properties of the generating family of hypergraphs. This is typically a trivial procedure which includes zero fusion arguments, rejoice evermore.

It is now high time to consider several examples.

Example 1.8. (Silver forcing) Let Γ be the countable Cantor group of eventually zero binary sequences, acting on $X = 2^\omega$ by coordinatewise addition. Let G be the *Hamming graph* on X, connecting two infinite binary sequences if they differ in exactly one entry, and let $\mathcal{G} = \{G\}$. Since the graph G is invariant under the group action, it is clear that $\mathcal{G}$ is an actionable family. The quotient poset $P_\mathcal{G}$ is well-known to be equivalent to the Silver forcing [6, Section 4.7.4].

The family $\mathcal{G}$ is clearly not subadditive; in fact, the graph G is bipartite and therefore every finite set can be written as a union of two G-anticliques. Accordingly, the poset $P_\mathcal{G}$ adds an independent real. It is well-known that if $x \in 2^\omega$ is a $P_\mathcal{G}$-generic point, then the infinite binary sequence $y \in 2^\omega$ defined by $y(n) = 1$ if the cardinality of the set $\{m \in n \colon x(m) = 1\}$ is even is an independent real over the ground model.

Example 1.9. ($\mathbb{E}_0$-forcing) Let Γ be the countable Cantor group of eventually zero binary sequences, acting on $X = 2^\omega$ by coordinatewise addition. Let $\mathbb{E}_0$ be the modulo finite equality equivalence relation on X, and let $\mathcal{G} = \{\mathbb{E}_0\}$. The quotient poset $P_\mathcal{G}$ has been studied in [6, Section 4.7.1] or [2, Section 10.9], among other places.

Clearly, $\mathcal{G}$ is an actionable family of finitary Borel hypergraphs. It is not subadditive as is, but it can be naturally enlarged to a subadditive family which gives the same σ-ideal. Write G_n for the hypergraph of arity n consisting of sets of cardinality n consisting of pairwise $\mathbb{E}_0$-equivalent elements ($n \geq 2$), and write $\mathcal{H} = \{G_n \colon n \geq 2\}$. Clearly, $\mathcal{H}$ is an actionable subadditive family of Borel hypergraphs. Since $\mathbb{E}_0 = G_2$, it follows that $\mathcal{G} \subset \mathcal{H}$ and $I_\mathcal{G} \subseteq I_\mathcal{H}$. For the opposite inclusion, observe that every $I_\mathcal{G}$-positive Borel set must have an infinite intersection with some $\mathbb{E}_0$-classes, and therefore is not a G_n-anticlique for any number $n \geq 2$. It follows that $I_\mathcal{H} \subseteq I_\mathcal{G}$, so $I_\mathcal{G} = I_\mathcal{H}$. By Theorem 1.6, the poset $P_\mathcal{G}$ adds no independent reals.

Example 1.10. (The countable support product of $\mathbb{E}_0$-forcing) Let $X = \prod_i X_i$ be the product of countably many copies of the Cantor space. Let $\Gamma = \prod_i \Gamma_i$ be the finite support product of countably many copies of the countable Cantor group, acting on X coordinatewise. For each index $i \in \omega$, let G_i be the graph on X connecting points x_0, x_1 if they agree on all inputs except on i, and $x_0(i)$ and $x_1(i)$ differ in at most finitely many entries. Let $\mathcal{G} = \{G_i \colon i \in \omega\}$. A general theorem [7, Theorem 5.6] shows that the poset $P_\mathcal{G}$ is naturally equivalent to the product of countably many copies of $\mathbb{E}_0$-forcing in the sense that a Borel subset of X is $I_\mathcal{G}$-positive if and only if it contains a product of Borel $\mathbb{E}_0$-positive sets.

The family $\mathcal{G}$ is not subadditive. However, it is immediately possible to replace it with a subadditive family which generates the same ideal. For each finite set $a \subset \omega$ and every number $n \geq 2$, let $H_{an} = \{b \subset X : b = \prod_i c_i$ where for $i \notin a$ the set $c_i \subset X_i$ is a singleton and for $i \in a$ the set $c_i \subset X_i$ is of cardinality n consisting of pairwise modulo finite equal sequences$\}$. Let $\mathcal{H} = \{H_{an} : a \in [\omega]^{<\aleph_0}, n \geq 2\}$. It is clear that $\mathcal{H}$ is an actionable family. It is also subadditive by a straightforward application of the finite rectangular Ramsey theorem. Moreover, the families $\mathcal{G}$ and $\mathcal{H}$ yield the same σ-ideal, since $\mathcal{G} \subset \mathcal{H}$ and every $I_{\mathcal{G}}$-positive Borel set contains a product of $\mathbb{E}_0$-positive Borel sets and therefore a hyperedge in each hypergraph in $\mathcal{H}$.

In consequence, Theorem 1.6 shows that the countable support product of the $\mathbb{E}_0$-forcing adds no independent reals.

Simple finitary Ramsey style theorems can be used to build actionable partial orders which do not add independent reals.

Example 1.11. (van der Waerden forcing) Let $\mathbb{Z}$ act on $X = 2^{\mathbb{Z}}$ by shift. For $n \geq 3$, let G_n be the hypergraph of arity n consisting of all sets of type $a \cdot x$ where $x \in X$ is arbitrary and $a \subset \mathbb{Z}$ is an arithmetic progression of length n. It is immediately clear that $\mathcal{G} = \{G_n : n \in \omega\}$ is an actionable family of Borel hypergraphs. In addition, the usual van der Waerden theorem shows that $\mathcal{G}$ is subadditive. By Theorem 1.6, the quotient poset $P_{\mathcal{G}}$ does not add independent reals.

The paper uses set theoretic notation standard of [1]. The idealized forcing background can be found in [6, Chapter 2]. If Γ is a group acting on a set X, $a \subset \Gamma$ is a set, and $x \in X$ is a point, then I write $a \cdot x = \{\gamma \cdot x : \gamma \in a\}$. Similarly, if $\gamma \in \Gamma$ and $B \subset X$ is a set, I write $\gamma \cdot B = \{\gamma \cdot x : x \in B\}$. For a finite sequence $\langle \gamma_m : m \in n\rangle$ of elements of Γ write $\prod_m \gamma_m$ for their product taken in increasing order of indices; I set $\prod 0 = 1_\Gamma$. For a finite sequence $\langle a_m : m \in n\rangle$ of nonempty finite subsets of Γ write $\prod_m a_m$ for the set of all group elements of the form $\prod_m \gamma_m$ where $\gamma_m \in a_m$ holds for all $m \in n$.

2 Proof of Theorem 1.6

Fix a Polish space X and a subadditive countable family $\mathcal{G}$ of Borel finitary hypergraphs on X which is actionable, as witnessed by a Borel action of some countable group Γ on X. Fix an arbitrary complete compatible metric on the underlying Polish space X. Fix also an enumeration $\mathcal{G} = \{G_m : m \in \omega\}$.

As proved in [7, Theorem 2.2, Corollary 3.18], the poset $P_{\mathcal{G}}$ is proper and bounding. This has the following well-known consequence [6, Theorem 3.3.2]:

Fact 2.1. *Whenever $B \subset X$ is a Borel $I_{\mathcal{G}}$-positive set and $f : B \to Y$ is a Borel function to another Polish space, then there is a compact $I_{\mathcal{G}}$-positive set $C \subset B$ such that $f \restriction C$ is continuous.*

In order to prove that the poset $P_{\mathcal{G}}$ does not add independent reals, I will use the following abstract partition fact.

Fact 2.2. *Let $\langle a_n, \mu_n : n \in \omega\rangle$ be a sequence of nonempty finite sets with a submeasure on each such that $\limsup_n \mu_n(a_n) = \infty$. Suppose that $f : \prod_n a_n \times \omega \to 2$ is a Borel function. Then there are nonempty finite sets $b_n \subset a_n$ for $n \in \omega$ and an infinite set $c \subset \omega$ such that $\limsup_n \mu_n(b_n) = \infty$ and the function f is constant on $\prod_n b_n \times c$.*

This is an immediate consequence of [5, Theorem 1.4]. To see why, just find an infinite subset $d \subset \omega$ such that the numbers $\mu_n(a_n)$ for $n \in d$ increase fast enough for that theorem to apply, for each $n \notin d$ replace a_n with any of its singleton subsets, naturally identify $\prod_{n\in\omega} a_n \times \omega$ with $\prod_{n\in d} a_n \times \omega$ and apply the theorem.

I need a certain notion of measured fusion sequence of conditions inside the poset $P_{\mathcal{G}}$. This is codified in the following definition.

Definition 2.3. Let $B \subset X$ be an $I_{\mathcal{G}}$-positive Borel set. A *measured fusion sequence* below B is a sequence $\langle C_n, a_n, \mu_n : n \in \omega\rangle$ such that for every $n \in \omega$, the following holds:

1. the sets C_n are compact I-positive subsets of B decreasing with respect to inclusion, with respective metric diameters smaller than 2^{-n};
2. $a_n \subset \Gamma$ are nonempty finite sets and μ_n are submeasures on each such that $\mu_n(a_n) \geq n$;
3. for all $\gamma \in a_n$, $\gamma \cdot C_{n+1} \subset C_n$ and the action of γ on C_{n+1} is continuous;
4. for all $\gamma \in \prod_{m\leq n} a_m$, the set $\gamma \cdot C_{n+1}$ has metric diameter smaller than 2^{-n};
5. for every $\gamma \in \prod_{m\in n} a_m$, for every $m \in n$ and every set $b \subset a_n$ such that $\mu_n(b) \geq 1$, and for every point $x \in C_{n+1}$, there is a set $c \subset b$ such that $c \cdot x \in \gamma^{-1} G_m$.

Proposition 2.4. *Below every Borel I-positive set $B \subset X$ there is a measured fusion sequence.*

Proof. The fusion sequence is constructed by recursion. The recursion starts with any compact $I_{\mathcal{G}}$-positive set $C_0 \subset B$ of metric diameter smaller than 1. Now, suppose that C_m for $m \leq n$ and a_m for $m < n$ have been constructed.

Let $G \in \mathcal{G}$ be a hypergraph such that no hyperedge $e \in G$ can be covered by n many sets each of which is an anticlique in one of the hypergraphs $\gamma^{-1} G_k$ where $k < n$ and $\gamma \in \prod_{m\in n} a_m$. To see the key point behind the choice of the hypergraph G, suppose that $e \in G$ is a hyperedge in C_n, $x \in e$ is a point, and $a \subset \Gamma$ is a set such that $e = a \cdot x$. Then consider the pavement submeasure μ on a in which the sets in the following collection are pavers with weight one: $\{b \subset a : \exists\gamma \in \prod_{m\in n} a_m\ \exists k < n\ b \cdot x$ is a $\gamma^{-1}G_k$-anticlique$\}$. It will be the case that $\mu(a) > n$.

Now, argue that there must be a finite set $a \subset \Gamma$ and a submeasure μ on a such that $\mu(a) \geq n$, such that the Borel set $D_{a\mu} = \{x \in C_n :$ for all

$\gamma \in \prod_{m\in n} a_m$, for all $m \in n$ and for all $b \subset a$ such that $\mu(b) \geq 1$ there is $c \subset b$ such that $c \cdot x \in \gamma^{-1}G_m\}$ is $I_{\mathcal{G}}$-positive. To see this, consider the Borel set $A = C_n \setminus \bigcup_{a\mu} D_{a\mu}$. The previous paragraph shows that A is a Borel G-anticlique and therefore belongs to $I_{\mathcal{G}}$. If each set $D_{a\mu}$ were $I_{\mathcal{G}}$-small, then $C_n = A \cup \bigcup_{a,\mu} D_{a\mu}$ would be a union of countably many sets in $I_{\mathcal{G}}$, contradicting the assumption that the set C_n is $I_{\mathcal{G}}$-positive.

Let $a_n \subset \Gamma$ be a finite set and μ_n a submeasure as in the previous paragraph. Use Fact 2.1 to find a compact $I_{\mathcal{G}}$-positive subset $C_{n+1} \subset C_n$ such that for every $g \in \prod_{m\leq n} a_m$ the action by g is continuous on C_{n+1}, and the metric diameter of the set $g \cdot C_{n+1}$ is smaller than 2^{-n}. The recursion step has been performed. □

Definition 2.5. Let $\langle C_n, a_n, \mu_n : n \in \omega\rangle$ be a measured fusion sequence below B. The *associated map* is the map $\pi\colon \prod_n a_n \to B$ defined by $\pi(y) = \lim_n \prod_{m\in n} y(m) \cdot x$ where $x \in X$ is the unique point in the intersection $\bigcap_n C_n$.

Proposition 2.6. *The associated map π is well-defined and continuous. In addition, if $\langle b_n : n \in \omega\rangle$ is a sequence of nonempty subsets of a_n such that $\limsup_n \mu_n(b_n) = \infty$, then $\pi'' \prod_n b_n \notin I_{\mathcal{G}}$.*

Proof. For the first sentence, the sets $C_n \subset X$ for $n \in \omega$ are compact, nested, and of decreasing metric diameter, therefore $\bigcap_n C_n$ is a singleton, containing a unique point $x \in X$. By induction on $n \in \omega$ from Definition 2.3(3), one can easily show that for every point $y \in \prod_n a_n$ and every number $n \in \omega$, $\prod_{m\in n} y(m) \cdot C_n$ is a superset of $\prod_{m\in n+1} y(m) \cdot C_{n+1}$. Thus, the sets $\prod_{m\in n} y(m) \cdot C_n$ for $n \in \omega$ are compact, nested, and of decreasing metric diameter, and their intersection is a singleton containing the unique point $\pi(y)$. It also follows that the function π is continuous.

The second sentence is more demanding. I will need the following observation. For every point $y \in \prod_n a_n$ and every number $k \in \omega$ write $\pi_k(y) = \lim_n \prod_{k\in m\in n} y(m) \cdot x$.

Claim 2.7. *The point $\pi_k(y) \in X$ is well defined exists and belongs to the set C_{k+1}. In addition, $\prod_{k\leq m} y(m) \cdot \pi_k(y) = \pi(y)$.*

Proof. The first sentence of the claim is proved just like the first sentence of the proposition. The second sentence follows from the fact that the action by the group element $\prod_{k\leq m} y(m)$ is continuous on the set C_k by item (3) of Definition 2.3. □

Let $Z = \prod_n b_n$, let $G \in \mathcal{G}$ be an arbitrary hypergraph, $D \subset X$ be a Borel G-anticlique, and work to show that $\pi^{-1}D \cap Z$ is a set meager in Z; this will prove the second sentence by a Baire category argument with the space Z. Suppose towards a contradiction that the set $\pi^{-1}D \cap Z$ is nonmeager in Z. Since this set is Borel, it has the Baire property and it has to be comeager in some basic open set determined by a finite tuple t of length some k such that for every $m \in k$, $t(m) \in b_m$. Extending the sequence t if necessary, I may assume that k is greater than the index of G in the enumeration of $\mathcal{G}$ and $\mu_k(b_k) \geq 1$ holds. By a standard procedure, find a point $y \in \prod_n b_n$ such that $t \subset y$, and every point in

$\prod_n b_n$ which agrees with y at every entry except possibly the k-th entry belongs to $\pi^{-1}D$. Consider the point $\pi_k(y) \in X$, which belongs to C_{k+1} by Claim 2.7. By Definition 2.3(5), there is a set $c \subset b_k$ such that $c \cdot \pi_k(y) \in (\prod_{m<k} y(m))^{-1}G$. Thus, $\prod_{m\in k} t(m) \cdot c \cdot \pi_k(y) \in G$ holds. The latter set is a subset of $\pi^{-1}D$ by the choice of the point y and the second sentence of Claim 2.7. This completes the proof. $\square$

For the proof of Theorem 1.6, let $B \subset X$ be a Borel $I_{\mathcal{G}}$-positive set and τ be a $P_{\mathcal{G}}$-name for a subset of ω. I must find an infinite subset $c \subset \omega$ and a condition stronger than B which forces c either to be disjoint from τ or to be a subset of τ. To do this, use the Borel reading of names to thin out B if necessary and find a Borel function $h\colon B \to \mathcal{P}(\omega)$ such that $B \Vdash \tau = h(\dot{x}_{gen})$. Let $\langle C_n, a_n, \mu_n \colon n \in \omega\rangle$ be a fusion sequence below B, and $\pi\colon \prod_n a_n \to B$ the associated map. Let $f\colon \prod_n a_n \times \omega \to 2$ be the Borel function defined by $f(y, n) = 1$ if $n \in h(\pi(y))$. By Fact 2.2, there are nonempty sets $b_n \subset a_n$ for $n \in \omega$ and an infinite set $c \subset \omega$ such that $\limsup_n \mu_n(b_n) = \infty$ and f is constant on $\prod_n b_n \times c$.

By Proposition 2.6, the set $C = \pi'' \prod_n b_n \subset B$ is compact and $I_{\mathcal{G}}$-positive. A standard Mostowski absoluteness argument shows that either $C \Vdash \check{c} \subset \tau$ or $\check{c} \cap \tau = 0$ depending on the constant value the function f takes on $\prod_n b_n \times c$. This completes the proof of Theorem 1.6.

3 Proof of Theorem 1.7

Suppose that X is a Polish space and $\mathcal{G}$ is a countable collecton of Borel finitary hypergraphs which is actionable, as witnessed by a Borel action of a countable group Γ on the space X. Suppose that the quotient poset $P_{\mathcal{G}}$ adds no independent reals. For every finite collection $a \subset \mathcal{G}$ and every number $n \in \omega$ let H_{an} be the hypergraph of all finite subsets $b \subset X$ which consist of pairwise orbit equivalent elements which cannot be covered by n-many sets each of which is an anticlique for some hypergraph in the set a. (It may occur that $H_{an} = 0$.) Let $\mathcal{H} = \mathcal{G} \cup \{H_{an} \colon a \subset \mathcal{G}$ is finite and $n \in \omega\}$.

Proposition 3.1. *The set $\mathcal{H}$ is a countable actionable subadditive collection of finitary Borel hypergraphs on X.*

Proof. It is immediate from the definition that each hypergraph H_{an} is Borel. By definitions, each hyperedge in H_{an} is finite and consists of pairwise orbit equivalent elements of X. If $\gamma \in \Gamma$ is any element, then $\gamma \cdot H_{an} = H_{\gamma \cdot a, n}$, so the family $\mathcal{H}$ is actionable since $\mathcal{G}$ is. Finally, I must check that the family $\mathcal{H}$ is subadditive.

This is again immediate from the definitions. Let $b \subset \mathcal{H}$ be a finite set and let $m \in \omega$ be a number. Consider the hypergraph H_{an} where $a \subset \mathcal{G}$ is the finite set of all hypergraphs in $\mathcal{G}$ mentioned in the set b, either elements of b or in the subscript of an element of b. Moreover, n is a number larger than the sum of all numbers mentioned in the subscripts of elements of b plus $|b|$. It is not difficult to see that the hypergraph H_{an} works as required in the definition of subadditivity. $\square$

In view of the proposition, to prove Theorem 1.7 it is enough to show that $I_{\mathcal{G}} = I_{\mathcal{H}}$. Suppose towards a contradiction that this fails. Since $\mathcal{G} \subset \mathcal{H}$, it must be the case that there is a Borel set $B \subset X$ which belongs to $I_{\mathcal{H}}$ but not to $I_{\mathcal{G}}$. A countable additivity of the σ-ideals shows that thinning down the set B if necessary, one may find a finite set $a \subset \mathcal{G}$ and a number $n \in \omega$ such that B is an H_{an}-anticlique. The following general proposition is a key to the proof. Recall that a countable Borel equivalence relation is *hyperfinite* if it is an increasing union of a countable sequence of Borel equivalence relations all of whose equivalence classes are finite.

Proposition 3.2. *There is a Borel $I_{\mathcal{G}}$-positive set $C \subset B$ such that the Γ-orbit equivalence relation on C is hyperfinite.*

Proof. Let M be a countable elementary submodel of a large enough structure containing among others $X, \mathcal{G}, \Gamma$, and B. Let Y be the Polish space of all M-generic filters on $P_{\mathcal{G}} \cap M$, equipped with the usual topology in which the basic open sets are the sets of filters containing a given condition. Let $h\colon Y \to X$ be the function assigning to a filter $y \in Y$ the unique element of X contained in all conditions in y. It is easy to check that the function h is injective and continuous. Consider the countable Borel equivalence relation F on Y connecting y_0, y_1 if $h(y_0)$ and $h(y_1) \in X$ are Γ-orbit equivalent.

By a well-known and several times rediscovered result [3, Theorem 12.1], there is a co-meager Borel set $D \subset Y$ such that $F \restriction D$ is hyperfinite. The set $C = \{h(y)\colon y \in D, B \in y\}$ is a continuous injective image of a relatively open subset of the Borel set D and as such is Borel. The Γ-orbit equivalence relation on D is hyperfinite. Thus, it is enough to show that C is $I_{\mathcal{G}}$-positive.

For this, by the Baire category theorem applied with the space Y it is enough to show that for every hypergraph $G \in \mathcal{G}$ the h-preimage of any Borel G-anticlique is meager in Y. Suppose towards a contradiction that $A \subset X$ is a Borel G-anticlique whose h-preimage is not meager, and therefore co-meager in some nonempty open set $\{y \in Y\colon B_0 \in y\} \subset Y$ for some condition B_0. There must be a finite set $a \subset \Gamma$ such that the Borel set $B_a = \{x \in B_0\colon a \cdot x \subset B_0 \text{ and } a \cdot x \in G\}$ is $I_{\mathcal{G}}$-positive: otherwise the set B_0 would be a union of countably many $I_{\mathcal{G}}$-small sets and a Borel G-anticlique, which would contradict the positivity of B_0.

Pick a finite set $a \subset \Gamma$ as in the previous paragraph. Note that for every $\gamma \in a$, the action of γ on X naturally extends to an action on Borel subsets of X to a permutation of the poset $P_{\mathcal{G}} \cap M$, and finally to a self-homeomorphism of the space Y. Thus, by a standard argument it is possible to find a filter $y \in Y$ containing B_a such that for each $\gamma \in a$, the filter $\gamma \cdot y$ belongs to the relatively co-meager set $h^{-1}A$. But then, $h'' a \cdot y$ is a G-hyperedge in the set A, contradicting the initial assumptions on the set A. $\square$

Let $C \subset B$ be a Borel $I_{\mathcal{G}}$-positive Borel set as in the proposition, and let $\langle E_m\colon m \in \omega\rangle$ be an increasing sequence of Borel equivalence relations with finite classes on the set C such that $\bigcup_m E_m$ is the Γ-orbit equivalence relation restricted to the set C. Since the set C is a H_{an}-anticlique, for each $m \in \omega$, each E_m-class $c \subset C$ can be covered by n many sets each of which is an anticlique in one of the hypergraphs in the set a. By an application of the Lusin–Novikov theorem, for each $m \in \omega$ there is a Borel function f_m with domain $[C]^{<\aleph_0}$ which to each E_m-class c assigns such a cover. More specifically, the value $f_m(c)$ is a function from c to $n \times a$ such that for each $i \in n$ and $G \in a$, the set $\{x \in c\colon f_m(c)(x) = \langle i, G\rangle\}$ is a G-anticlique.

Define a Borel function $h\colon C \to (n \times a)^\omega$ by setting $h(x)(m) = f_m([x]_{E_m})(x)$. Since the poset $P_{\mathcal{G}}$ does not add independent reals, there must be a Borel $I_{\mathcal{G}}$-positive set $D \subset C$, an infinite set $d \subset \omega$, and a pair $\langle i, G\rangle \in n \times a$ such that for each $m \in \mathrm{dom}(f)$ and each $x \in D$, $h(x)(m) = \langle i, G\rangle$. Since the set D is $I_{\mathcal{G}}$-positive, there must be a G-hyperedge $e \subset D$. Since all elements of e are pairwise orbit equivalent and the equivalence relations E_m for $m \in d$ exhaust the orbit equivalence relation, there must be a number $m \in d$ and an E_m-class c such that $e \subset c$. Then, for each $x \in e$, it is the case that $f_m(c)(x) = \langle i, G\rangle$. The choice of the function f_m implies that e is a G-anticlique, which it is not. This contradiction completes the proof of Theorem 1.7.

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