Forcing Borel reducibility invariants

Jindřich Zapletal
Academy of Sciences, Czech Republic
University of Florida

November 13, 2013

Preface

This book is a contribution to the classification theory of analytic equivalence relations in descriptive set theory. It shows that set theoretic techniques normally associated with the axiom of choice and combinatorics of uncountable cardinals can be efficiently used to prove new and difficult theorems about the structure of analytic equivalence relations. In many respects, this contradicts the conventional wisdom, which holds that the study of analytic equivalence relations is purely a matter of descriptive set theory and mathematical analysis and therefore impervious to efforts of combinatorial set theory. Thus, the resulting landscape is entirely unexpected and shows much promise for further investigation.

I developed the topic during my sabbatical year 2012-13. Before the summer 2012, almost no intuitions or results presented here existed. During the year, I gave a number of lectures (logic seminars at CalTech, UCLA, CTS Prague, as well as the set theory meeting in Luminy 2012) which documented the fast pace of development of the subject. In May 2013, I gave a minicourse at University of Münster that already contained many of the main themes present in this book.

I wish to thank the following individuals for insightful discussions on the subject of the book: Bohuslav Balcar, John Clemens, Vladimir Kanovei, Alexander Kechris, Andrew Marks, Benjamin Miller, Marcin Sabok, and Sławomir Solecki. I am also indebted to Alexander Kechris and Henry Macdonald for giving me permission to present their proof of Theorem 5.3.9 here.

I gratefully acknowledge the financial support from the following sources: DFG SFB grant 878, NSF DMS grant 1001623, Institutional Research Plan No. AV0Z10190503 and grant IAA100190902 of Grant Agency of the Academy of Sciences of the Czech Republic, NSF DMS grant 1161078, and the generous sabbatical funding from University of Florida. Last but not least, I would like to thank the Fields Institute for Research in Mathematical Sciences for its support and hospitality during the Thematic Program on Forcing and its Applications in the fall of 2012.

Contents

Chapter 1

Introduction

The theory of analytic equivalence relations has experienced a fast rate of growth in the last two decades, primarily due to its ability to connect many fields of mathematics in a substantial way [6, 12, 3, 13, 7]. It rates equivalence problems in mathematical analysis according to their intuitive complexity. Equivalence problems are understood to be analytic equivalence relations on Polish spaces. For such equivalence relations E, F on respective Polish spaces X, Y, write $E \leq_{\mathrm{B}} F$ if there is a Borel *reduction* of E to F, i.e. a Borel map $h : X \to Y$ such that for all points $x_0, x_1 \in X$, $x_0 \mathrel{E} x_1$ iff $h(x_0) \mathrel{F} h(x_1)$. The relation $\leq_{\mathrm{B}}$ is a quasiordering, and the main tasks of the theory of analytic equivalence relations are placing known equivalence relations into this quasiorder and finding other informative features of $\leq_{\mathrm{B}}$. Proving that a given analytic equivalence relation is Borel reducible to another one may be difficult, but the methodology of such a task is typically straightforward. The negative results (showing that a given equivalence relation is *not* reducible to another one) are typically much more challenging, and more often than not tools from mathematical logic are used.

In this book, I discuss several approaches for proving nonreducibility results using the method of forcing. In their majority, they go against the conventional wisdom in that they introduce reducibility invariants that are evaluated with the help of Axiom of Choice, and their valuation is not absolute among forcing extensions of the universe. Despite that, they are natural and useful; in the traditional descriptive set theoretic context, they seem to carry little content. In a good number of cases, an absolute and descriptive set theoretic result, such as nonreducibility of one analytic equivalence relation to another, is naturally obtained via comparison between sophisticated forcing extensions concerning issues high in the cumulative hierarchy.

Chapter 2 contains some useful preliminary generalities on forcing. In Chapters 3, 4 and 5 I analyze the concept of unpinned equivalence relations that first appeared in the work of Greg Hjorth. The current general definition is due to Kanovei:

Definition 1.0.1. [12, Chapter 17] Let E be an analytic equivalence relation on a Polish space X. Let P be a poset and τ a P-name for an element of $\dot{X}$. The name τ is *(E-)pinned* if $P \times P \Vdash \tau_{\text{left}}\ \dot{E}\ \tau_{\text{right}}$. The name is *($E$-)trivial* if $P \Vdash \tau\ \dot{E}\ \check{x}$ for some ground model element $x \in X$. The equivalence relation E is *pinned* if all E-pinned names on all posets are E-trivial. Otherwise, E is *unpinned.*

The main point behind this concept is the fact that the class of pinned equivalence relations is closed downwards under Borel reducibility. Thus for example proving that the equivalence relation E_{K_σ} is pinned while F_2 is not yielded a conceptual proof of Borel nonreducibility of the latter to the former [12, Theorem 17.1.4]. Kechris conjectured that for Borel equivalence relations E, $F_2 \leq_{\mathrm{B}} E$ is in fact equivalent to the statement that E is unpinned. While this is false in ZFC [27], I show that in fairly common choiceless contexts this is in fact true:

Theorem 1.0.2. (Corollary 3.4.2) *In the choiceless Solovay model derived from a measurable cardinal, whenever E is a Borel equivalence relation then E is unpinned if and only if $F_2 \leq_{\mathrm{B}} E$.*

I also show that the unpinned status of an analytic equivalence relation is suitably absolute and that it is detected by posets of size $\aleph_1$. I proceed to refine the pinned concept to obtain a number of new nonreducibility results. The underlying idea is the following extension of the equivalence relation E to the space of all pinned names:

Definition 1.0.3. Let E be an analytic equivalence relation on a Polish space X, and let τ, σ be E-pinned names on respective posets P, Q. Say that $\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$ holds if $P \times Q \Vdash \tau\ E\ \sigma$.

It turns out that $\bar{E}$ is an equivalence relation and it is interesting to count the number of its equivalence classes. One possible way of doing so uses the following central definition.

Definition 1.0.4. Let E be an analytic equivalence relation on a Polish space X. The *pinned cardinal* of E, $\kappa(E)$ is the smallest cardinal κ such that every E-pinned name is $\bar{E}$-equivalent to a name on a poset of size $< \kappa$ if such κ exists; otherwise $\kappa(E) = \infty$. If E is pinned then write $\kappa(E) = \aleph_1$.

The main point of this definition is again the fact that the pinned cardinal is a Borel reducibility invariant–$E \leq_{\mathrm{B}} F$ implies $\kappa(E) \leq \kappa(F)$. Thus, the evaluation of the pinned cardinal again leads to nonreducibility results. It turns out that the pinned cardinal can attain fairly exotic values for quite simple equivalence relations, obtaining some sort of parallel of Shelah's classification theory of models for analytic equivalence relations. For example,

Theorem 1.0.5. (Corollary 4.4.8 and 4.4.10) *For every countable ordinal $\alpha > 0$ there is a Borel equivalence relation E_α such that (ZFC provably) $\kappa(E_\alpha) = \aleph_\alpha$. There are Borel equivalence relations E and F such that (ZFC provably) $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$ and $\kappa(F) = \max(\aleph_{\omega+1}, \mathfrak{c})^+$.*

One curious feature of the pinned cardinal is that the comparison of its values may differ in different forcing extensions, which opens the gate towards methods that were heretofore considered irrelevant for the theory of analytic equivalence relations. For example, the natural proof of nonreducibility of E to F uses the independence of the Singular Cardinal Hypothesis at $\aleph_\omega$.

Chapter 5 looks at unpinned equivalence relations from a different direction. What is the nature of forcings that can carry nontrivial pinned names? It turns out that notrivial pinned names for orbit equivalence relations are essentially cardinal collapse names–Theorem 5.2.1. On the other hand, there are equivalence relations for which say Namba forcing can carry a nontrivial pinned name. This leads to an ergodicity result for such equivalence relations.

Definition 1.0.6. The *mutual domination* equivalence relation E on $X = (\omega^\omega)^\omega$ connects points $x, y \in X$ if for every $n \in \omega$ there are $m_0, m_1 \in \omega$ such that $x(m_0)$ modulo finite dominates $y(n)$ and $y(m_1)$ modulo finite dominates $x(n)$.

Theorem 1.0.7. (Theorem 5.3.3 simplified.) *The mutual domination equivalence relation is F-I-ergodic for every orbit equivalence relation F of a continuous action of a Polish group, where I is the mutual domination ideal on X.*

Chapters 6 and 7 deal with generalizations of another concept of Greg Hjorth–turbulence of Polish group actions. The main motivating result can be phrased as follows.

Theorem 1.0.8. (Theorem 6.1.2 simplified.) *If E is an orbit equivalence relation on a Polish space X obtained from a generically turbulent action, then in a Cohen forcing extension there are points $x_0, x_1 \in X$ such that*

1. *$x_0 \mathrel{E} x_1$;*
2. *there is no element of the ground model E-related to x_0, x_1;*
3. *$V[x_0] \cap V[x_1] = V$.*

This immediately motivates the definition of the class of *trim* equivalence relations:

Definition 1.0.9. An equivalence relation E on a Polish space X is *(proper)-trim* if for all (proper forcing) generic extensions $V[G]$ and $V[H]$ and all E-related points $x \in V[G]$ and $y \in V[H]$, either $V[G] \cap V[H] = V$ or there is a point $x \in V$ which is E-related to both x, y.

The class of proper-trim equivalence relations is closed under Borel reducibility. It includes in particular all equivalence relations classifiable by countable structures–Theorem 6.6.1, but also many other natural analytic equivalence relations as described in Section 6.6. Thus, the following theorem greatly extends the motivational Hjorth's ergodicity result [12, Lemma 13.3.4].

Theorem 1.0.10. (Theorem 7.1.1) *If E is an orbit equivalence relation on a Polish space X obtained from a generically turbulent action, and F is a proper-trim equivalence relation then E is generically F-ergodic.*

Chapter 7 contains a number of other ergodicity results obtained through modifications of the trim concept. They often concern the equivalence relations of the form $=_J$ on 2^ω, where J is an analytic ideal on ω and $x =_J y$ if $\{n \in \omega : x(n) \neq y(n)\} \in J$.

Theorem 1.0.11. (Corollary 7.1.17 simplified) *Let J be the ideal of subsets of ω of asymptotic density zero. Then $=_J$ is E_{K_σ}-generically ergodic.*

Investigating the possibilities for the random forcing, one can obtain measure-style ergodicity results, which are entirely out of the scope of the turbulence method. For example:

Theorem 1.0.12. (Corollary 7.2.9) *Let J be the ideal of summable subsets of ω. Let μ be the usual Borel probability measure on 2^ω. Then $=_J$ is F-μ-ergodic for every proper-trim equivalence relation F.*

Theorem 1.0.13. (Corollary 7.2.17 simplified) *Let J be the ideal of subsets of ω of asymptotic density zero. Let μ be the usual Borel probability measure on 2^ω. Then $=_J$ is E_{K_σ}-μ-ergodic.*

In Chapter 8, I introduce several other forcing-type Borel reducibility invariants that are so far not as well developed. Section 8.1 introduces an invariant which among other things leads to an exceptionally short proof of nonreducibility of E_1 to any orbit equivalence relation induced by a continuous action of a Polish group–Corollary 8.1.9. It also proves nonreducibility results complementary to the ergodicity results of Chapter 7. Section 8.2 introduces a reducibility cardinal invariant $\mathtt{non}(E)$ reminiscent of the usual invariants of the real line such as in [2], and separation-type reducibility invariant resulting from an attempt to generalize Martin-Solovay c.c.c. coding procedure to quotient spaces X/E for analytic equivalence relations E. The closely related Section 8.3 introduces ideal sequences, which connect analytic equivalence relations with ideals such as the nonstationary ideal on various regular uncountable cardinals. These concepts are all closely connected to the pinned concept:

Theorem 1.0.14. (Theorems 8.2.7 and 8.3.10) *If E is an unpinned equivalence relation then $\mathtt{non}(E) = \aleph_1$ and it has an I-sequence, where I is the nonstationary ideal on ω_1.*

Finally, Section 8.4 introduces another reducibility invariant, that of linear orderability in various choiceless models of ZF set theory. The verification of this invariant again boils down to investigation of names for elements of the underlying Polish space in various forcing notions.

Chapter 2

Preliminaries

2.1 Descriptive set theory

The notation used in the book follows the set theoretic standard of [10]. For functions p, q I write p rew q for the function r such that $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$, r is equal to q on $\mathrm{dom}(q)$, and and r is equal to p on $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$. A tree is a set of finite sequences closed under initial segment. For a tree T, $[T]$ stands for the set of all infinite branches of T. If $t \in T$ then $[t]$ is the set of all branches in $[T]$ extending the node t. For any countable set C, the set 2^C is equipped with the usual compact topology generated by all sets $O_s = \{x \in 2^C : s \subset x\}$ for every finite partial function $s : C \to 2$. The set 2^C is also equipped with the natural product Borel probability measure μ for which $\mu(O_s) = 2^{-|s|}$.

The following definition provides a catalogue of benchmark ideals on ω used in this book.

Definition 2.1.1.

1. the *summable ideal* is the ideal consisting of sets $a \subset \omega$ such that the sum $\Sigma\{1/n + 1 : n \in a\}$ is finite.
2. the *density zero ideal* is the ideal of sets $a \subset \omega$ such that $\lim_n \frac{|a \cap n|}{n} = 0$.
3. the *random graph ideal* is the ideal generated by cliques and anticliques of a random graph on ω. Since any two random graphs on ω are isomorphic, this is well-defined up to permutation of ω.
4. the *branch ideal* is the ideal on $2^{<\omega}$ generated by branches of $2^{<\omega}$.

The following definition provides a catalogue of benchmark equivalence relations used throughout the book.

Definition 2.1.2.

1. E_0 is the relation on 2^ω defined by $x\ E_0\ y$ if $\{n \in \omega : x(n) \neq y(n)\}$ is finite;
2. E_1 is the relation on $(2^\omega)^\omega$ defined by $x\ E_0\ y$ if $\{n \in \omega : x(n) \neq y(n)\}$ is finite;

3. F_2 is the relation on $(2^\omega)^\omega$ defined by $x \mathrel{F_2} y$ if $\mathrm{rng}(x) = \mathrm{rng}(y)$;

4. E_{K_σ} is the relation on ω^ω defined by $x \mathrel{E_{K_\sigma}} y$ if the numbers $|x(n) - y(n)|$ for $n \in \omega$ are bounded;

5. if J is an ideal on a countable set c then $=_J$ is the equivalence on 2^c defined by $x =_J y$ if $\{i \in c : x(i) \neq y(i)\} \in J$. The equivalence $=_J^{2^\omega}$ is defined on $(2^\omega)^c$ in the same way.

6. E_{ω_1} is the equivalence on binary relations on ω connecting x, y if either both x, y are not wellorders or they are isomorphic.

7. E_{S_∞} is the equivalence of isomorphism of binary relations on ω.

8. if E is an analytic equivalence relation on a Polish space X then its *Friedman–Stanley jump*, the equivalence relation E^+ on X^ω is defined by $y \mathrel{E^+} z$ if $[\mathrm{rng}(y)]_E = [\mathrm{rng}(z)]_E$.

The analytic equivalence relations in this book are quasi-ordered by Borel reducibility. There is also a different useful quasiorder, that of weak Borel reducibility.

Definition 2.1.3. Let E, F be analytic equivalence relations on respective Polish spaces X, Y.

1. Say that E is *Borel reducible* to F, $E \leq_{\mathrm{B}} F$, if there is a Borel *reduction* of E to F, which is a Borel function $h : X \to Y$ such that for every $x_0, x_1 \in X$, $x_0 \mathrel{E} x_1$ iff $h(x_0) \mathrel{F} h(x_1)$.

2. Say that E is *weakly Borel reducible* to F, $E \leq_{\mathrm{wB}} F$, if there is a Borel *weak reduction* of E to F, which is a Borel function $h : X \to Y$ such that for some set $A \subset X$ covered by countably many E-equivalence classes, for every $x_0, x_1 \in X \setminus A$, $x_0 \mathrel{E} x_1$ iff $h(x_0) \mathrel{F} h(x_1)$.

Clearly, weak Borel reducibility and Borel reducibility coincide if the lower equivalence relation E is Borel. For analytic equivalence relations, the two notions may differ. For example, E_{ω_1} is weakly Borel reducible to E_{S_∞} by the identity map, but there is no Borel reduction: this happens because E_{ω_1} has one non-Borel class, while all E_{S_∞}-classes are Borel.

A great deal of this book is concerned with homomorphisms between equivalence relations:

Definition 2.1.4. Let E, F be equivalence relations on respective Polish spaces X, Y. A *homomorphism* of E to F is a map $h : X \to Y$ such that for every $x_0, x_1 \in X$, if $x_0 \mathrel{E} x_1$ then $h(x_0) \mathrel{F} h(x_1)$.

In several places I will need an extension lemma for partial Borel homomorphisms. This is a straightforward application of the first reflection theorem:

Lemma 2.1.5. *Let E be an analytic equivalence relation on a Polish space X, and F a Borel orbit equivalence relation of a continuous Polish group action on a Polish space Y. Every partial Borel homomorphism of E to F can be extended to a total one.*

The difficulty is that the domain of the partial homomorphism may not be an E-invariant set.

Proof. Let I be the collection of all analytic sets $A \subset X \times Y$ such that whenever $\langle x_0, y_0 \rangle$ and $\langle x_1, y_1 \rangle$ are elements of A and $x_0 \mathrel{E} x_1$, then $y_0 \mathrel{F} y_1$. The syntax of the definition of I shows that it is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ collection of analytic sets closed under increasing countable unions. Now let $h : X \to Y$ be a partial homomorphism of E to F. By induction build sets $B_0 \subset A_0 \subset B_1 \subset A_1 \subset B_2 \subset \dots$ so that

- $A_n, B_n \in I$;
- $B_0 = h$, A_n is the $E \times F$-saturation of B_n, and B_n is Borel.

This is certainly possible–the Borel set B_{n+1} is obtained from A_n by an application of the first reflection theorem [14, Theorem 35.10] to the collection I. In the end, let $B = \bigcup_n B_n$. This is a Borel set in I whose vertical sections are either empty or else consist of a single F-equivalence class, and whose projection is E-saturated. Since F is an orbit equivalence relation, $\mathrm{id} \times F$ is idealistic in the sense of [6, Definition 5.4.9]. As $\mathrm{id} \times F \restriction B$ is smooth, by [6, Theorem 5.4.11] there is a Borel uniformization k of the set B. Now define a totalization $\bar{h}$ of the homomorphism h by setting $\bar{h}(x) = h(x)$ if $x \in \mathrm{dom}(h)$, $\bar{h}(x) = k(x)$ if $x \in \mathrm{dom}(k) \setminus \mathrm{dom}(h)$, and $\bar{h}(x) =$ any fixed element of Y if $x \notin \mathrm{dom}(k)$. This works. □

2.2 Forcing

I use the standard textbook [10] as a reference for basic forcing terminology and facts. I will start with a definition of a Cohen forcing associated with a specific Polish space.

Definition 2.2.1. Let X be a Polish space. The *Cohen* poset P_X consists of nonempty open subsets of X ordered by inclusion.

Note that for any choice of countable basis for X, the basis is dense in P_X and therefore the poset P_X has countable density. In the common case of a perfect space X, the poset P_X has no atoms and it is therefore in forcing sense equivalent to Cohen forcing. It adds a single element of the space X, typically denoted by $\dot{x}_{gen}$, which belongs to the intersection of all open sets in the generic filter. A point $x \in X$ is P_X-generic over a model of ZF if and only if it belongs to all open dense subsets of X coded in the model.

Definition 2.2.2. Let κ be a set. The *collapse* $\mathrm{Coll}(\omega, \kappa)$ is the poset consisting of all finite partial functions from ω to κ ordered by reverse inclusion.

The following fact summarizes the commonly known properties of the collapse used in this book.

Fact 2.2.3. [10, Lemma 26.7, Corollary 26.8] *Let κ be an infinite cardinal.*

1. *P is up to forcing equivalence the only poset of size κ which forces $|\kappa| = \aleph_0$;*

2. *if P is a partial order of size $\leq \kappa$ then P can be regularly embedded into $RO(\mathrm{Coll}(\omega, \kappa))$;*

3. *if P is any partial order of size $< \kappa$ regularly embedded in $RO(\omega, \kappa)$ then the remainder forcing is isomorphic to $\mathrm{Coll}(\omega, \kappa)$.*

Definition 2.2.4. Let κ be a strongly inaccessible cardinal. The *Lévy collapse* $\mathrm{Coll}(\omega, < \kappa)$ is the finite support product $\prod_{\alpha \in \kappa} \mathrm{Coll}(\omega, \alpha)$.

Fact 2.2.5. *Let κ be a strongly inaccessible cardinal.*

1. *$\mathrm{Coll}(\omega, < \kappa)$ is κ-c.c. and it forces $\check{\kappa} = \aleph_1$;*

2. *if P is a poset of size $< \kappa$, $G \subset \mathrm{Coll}(\omega, < \kappa)$ is a filter generic over V, and $H \subset P$ is a filter generic over V in $V[G]$, then $V[G]$ is a $\mathrm{Coll}(\omega, < \kappa)$-extension of $V[H]$.*

One forcing tool commonly used throughout the book is product forcing. For the terminology, if P is a poset, then the product $P \times P$ as a forcing notion adds two generic filters on P, one on the left copy of P and the other on the right copy. If P is a poset and τ is a P-name, τ_{left} is the $P \times P$ name for the evaluation of τ according to the left generic filter and τ_{right} is the $P \times P$ name for the evaluation of τ according to the right generic filter.

Fact 2.2.6. (Product forcing theorem, [10, Lemma 15.9]) *If P, Q are partial orders and $G \subset P$ and $H \subset Q$ are filters, then the following are equivalent:*

1. *$G \times H \subset P \times Q$ is a generic filter over V;*

2. *$G \subset P$ is a generic filter over V and $H \subset Q$ is a generic filter over $V[G]$.*

In either case, $V[G] \cap V[H] = V$.

Lemma 2.2.7. *Suppose that P, Q are posets and $G \times H \subset P \times Q$ is a filter generic over V. Suppose that P_0, Q_0 are posets in V and $G_0 \subset P_0$ in $V[G]$ and $H_0 \subset Q_0$ in $V[H]$ are filters generic over V. Then $G_0 \times H_0 \subset P_0 \times Q_0$ is a filter generic over V.*

Proof. Suppose that $p \in P$, $q \in Q$, τ is a P-name, σ is a Q-name, and $p \Vdash \tau \subset \check{P}_0$ is a generic filter over V, and $q \vdash \sigma \subset \check{Q}_0$ is a generic filter over V. Let $D \subset P_0 \times Q_0$ be an open dense set. I must find a condition $\langle p_0, q_0 \rangle \in D$ as well as $p' \leq p$, $q' \leq q$ such that $p' \Vdash \check{p}_0 \in \tau$ in P and $q' \Vdash \check{q}_0 \in \sigma$ in Q. Then, a straightforward density argument completes the proof of the lemma.

There are conditions $pr(p) \in P_0$ and $pr(q) \in Q_0$ such that for every condition $r \leq pr(p_0)$ in P_0 there is $p' \leq p$ such that $p' \Vdash \check{r} \in \tau$ in P, and for every condition $r \leq pr(q_0)$ in Q_0 there is $q' \leq q$ such that $q' \Vdash \check{r} \in \sigma$ in Q. Find a condition $\langle p_0, q_0 \rangle \in D$ which is below $\langle pr(p), pr(q) \rangle$ in the poset $P_0 \times Q_0$. Use the choice of $pr(p)$ and $pr(q)$ to find conditions $p' \leq p$ in P and $q' \leq q$ in Q such that $p' \Vdash \check{p}_0 \in \tau$ and $q' \Vdash \check{q}_0 \in \sigma$ as desired.

□

Lemma 2.2.8. *Whenever M is a countable model of ZF and $P \in M$ is a partial order, then there is a continuous map $f : 2^\omega \to \mathcal{P}(P)$ such that for every finite tuple $\langle z_i : i \in n \rangle$ of pairwise distinct elements of 2^ω the sets $\langle f(z_i) : i \in n \rangle$ are mutually generic over the model M.*

Proof. Let $\langle a_n, D_n : n \in \omega \rangle$ enumerate with repetitions all pairs $\langle a, D \rangle$ such that $a \subset 2^{<\omega}$ is a finite sequence of pairwise distinct binary sequences of the same length, and $D \subset P^{|a|}$ is an open dense set in M. By induction on $n \in \omega$ build conditions $p_s \in P$ for $s \in 2^n$ so that

- $s \subset t$ implies $p_t \leq p_s$;
- whenever $\langle s_i : i \in |a_n| \rangle$ is a sequence of binary strings of length n such that $a(i) \subset s_i$, then $\langle p_{s_i} : i \in |a| \rangle \in D_n$.

This is easily done. Once the induction has been performed, let $f : 2^\omega \to \mathcal{P}(P)$ assign to every binary sequence $x \in 2^\omega$ the filter on P generated by the conditions $\{p_{x \restriction n} : n \in \omega\}$. This is easily seen to work. □

Let P, Q, R be posets such that both P, Q are regular subposets of R. Say that P, Q are *independent* in R if $R \Vdash V[\dot{G} \cap \check{P}] \cap V[\dot{G} \cap \check{Q}] = V$, where $\dot{G}$ is the usual R-name for the R-generic filter over V.

Lemma 2.2.9. *Suppose that M, N are transitive models of set theory such that $M \subset N$. Suppose that $P, Q, R \in N$ are posets such that N satisfies "P, Q are regular subposets of R and they are independent in R". Then M satisfies the same sentence.*

Proof. I will first restate the independence in terms of complete Boolean algebras, removing the forcing relation. If A is a subalgebra of C and $c \in C$ is a nonzero element, write A_c for the algebra of all elements of the form $a \wedge c$ for $a \in A$; so $c = 1_{A_c}$. If A is a complete subalgebra of C then A_c is a complete subalgebra of C_c.

Claim 2.2.10. *Suppose that A, B are complete subalgebras of a complete Boolean algebra C. The following are equivalent:*

1. *A, B are independent in C;*

2. *for every $c \in C$, the algebra $A_c \cap B_c$ contains an atom.*

Observe that the intersection of two complete subalgebras is again a complete subalgebra, so $A_c \cap B_c$ is in fact a complete subalgebra of C_c.

Proof. Suppose first that (2) fails; i.e. there is $c \in C$ such that the algebra $A_c \cap B_c$ is atomless. Let $\dot{H}$ be the name for $\dot{G} \cap A_c \cap B_c$. Then $c \Vdash_C \dot{H} \notin V$, since $\dot{H}$ is a filter on an atomless Boolean algebra generic over V. Also, $c \Vdash_C \dot{H} \in V[\dot{G} \cap A]$, since it can be reconstructed there as the set $\{a \wedge c : a \in \dot{G} \cap \check{A}, a \cap c \in \check{B}_c\}$. For the same reason $c \Vdash_C \dot{H} \in V[\dot{G} \cap B]$ and so A, B are not independent in C.

Suppose on the other hand that (2) holds and $c \in C$ and τ is an A-name for a set of ordinals and σ is a B-name for a set of ordinals and $c \Vdash_C \tau = \sigma$. I must find a condition $d \leq c$ that decides the membership of all ordinals in τ; this will show that $d \Vdash \tau \in V$ and by the obvious density and $\in$-minimalization arguments it will prove the independence of A and B. Note that for every ordinal α, the Boolean values $|\check{\alpha} \in \tau|$ and $|\check{\alpha} \in \sigma|$ are the same in C_c. Since τ is an A_c-name, it must be the case that $|\check{\alpha} \in \tau| \in A_c$; since σ is an B_c-name, it must be the case that $|\check{\alpha} \in \sigma| \in B_c$ and so these Boolean values belong to $A_c \cap B_c$. Thus, if $d \leq c$ is an atom of $A_c \cap B_c$, d must decide the membership of every ordinal in τ (and σ) and so $d \Vdash \tau \in V$ as desired. $\square$

The Boolean characterization of independence in Claim 2.2.10 depends on the evaluation of the intersection of complete Boolean subalgebras. I will now show that the evaluation of intersection is absolute between models of set theory in a suitable sense.

Claim 2.2.11. *Let $M \subset N$ be ω-models of set theory. Let A, B, C be Boolean algebras in M such that $M \models C$ is complete and A, B are its complete subalgebras. In N, let $\bar{C}$ be a completion of C and $\bar{A}, \bar{B}$ be the completions of A, B inside $\bar{C}$. Then $A, B, C, A \cap B$ are respectively dense in $\bar{A}, \bar{B}, \bar{C}, \bar{A} \cap \bar{B}$.*

Proof. Every Boolean algebra is dense in its completion, so C is dense in $\bar{C}$. Since $M \models A \subset C$ is complete, it is also the case that $M \models A \subset C$ is regular. Being a regular subalgebra is a first order statement (for every $c \in C^+$ there is $a \in A^+$ such that every nonzero $a' \leq a$ in A is compatible with c), and so $N \models A \subset C$ is regular as well. A regular subalgebra is dense in its completion in the ambient algebra [10, Exercise 7.31], and this fact applied in the model N shows that $A \subset \bar{A}$ is dense. Similarly, $B \subset \bar{B}$ is dense.

The most important part of the claim is proving that $A \cap B$ is dense in $\bar{A} \cap \bar{B}$. For this, define the projection of C to A in the model M: $p^M(c, A) = \sum\{a \in A :$ every extension of a in A is compatible with $c\}$ for every $c \in C$. Define also the projection of $\bar{C}$ into $\bar{A}$ in the model N: $p^N(c, \bar{A}) = \sum\{a \in \bar{A} :$ every extension of a in $\bar{A}$ is compatible with $c\}$. As $A \subset \bar{A}$ is dense, for every $c \in C$ it is the case that $p^M(c, A) = p^N(c, \bar{A})$, and I will write $p(c, A) \in A$ for this common value. Similar usage will prevail for $p(c, B) \in B$ for $c \in C$.

To show that $A\cap B$ is dense in $\bar{A}\cap\bar{B}$, suppose that $c\in\bar{A}\cap\bar{B}$ is an arbitrary nonzero element. Find $c'\in C$ with $c'\leq c$. Construct a sequence a_n, b_n so that

- $a_n\in A, b_n\in B$, $c'\leq a_0\leq b_0\leq a_1\leq b_1\leq\dots$;
- $a_0=p(c',A)$, $b_n=p(a_n,B)$, and $a_{n+1}=p(b_n,A)$.

Observe that $b_n=p(a_n,\bar{B})$ and $a_{n+1}=p(b_n,\bar{A})$ and so by induction on n it follows that all the elements a_n, b_n must stay below c. Let $d=\sup_n a_n=\sup_n b_n$. Since the sequences a_n, b_n are both in the smaller model M, it is also the case that $d\in N$ and $d\in A\cap B$. Since $d\leq c$ and $c\in\bar{A}\cap\bar{B}$ was arbitrary, it follows that $A\cap B$ is dense in $\bar{A}\cap\bar{B}$.

$\square$

Now suppose that $M\subset N$ are transitive models and P, Q, R are posets in M such that M satisfies that P, Q are regular independent subposets of R. Passing to completions if necessary, I may assume that P, Q, R are in fact complete Boolean algebras in the model M. Work in the model N. Consider the completion $\bar{R}$ of R and the completions $\bar{P}, \bar{Q}$ of P and Q inside R. It follows from Claim 2.2.11 that forcing with R gives the same generic extensions as forcing with $\bar{R}$. Also, the property of Claim 2.2.10(2) transfers from $P\cap Q$ to $\bar{P}\cap\bar{Q}$, since $P\cap Q$ is dense in $\bar{P}\cap\bar{Q}$ by Claim 2.2.11 again. Thus, $\bar{P}$ and $\bar{Q}$ are independent subposets of $\bar{R}$ in the model N, and so P, Q are independent subposets of R in N as well. $\square$

2.3 Descriptive set theory in forcing extensions

The main proof tool used in this book is the interpretation of notions of descriptive set theory in various models of set theory and comparison of the results. The following classical absoluteness results are central:

Fact 2.3.1. (Borel absoluteness) *Let α be a countable ordinal, M a model of ZF+DC such that $\alpha+\omega+1$ is isomorphic to an initial segment of ordinals of M, and $M\models B\subset X$ is a $\mathbf{\Pi}^0_\alpha$ subset of a Polish space. If $x\in X$ is in M then $x\in B$ if and only if $M\models x\in B$.*

The proof is a straightforward induction on α and as such is left to the reader.

Fact 2.3.2. (Mostowski absoluteness) *Let ϕ be a Σ^1_1 formula with some free variables, M a transitive model of ZF+DC, and $\vec{x}$ a sequence of elements of $\omega^\omega\cap M$. Then $\phi(\vec{x})$ holds if and only if $M\models\phi(\vec{x})$.*

Fact 2.3.3. (Shoenfield absoluteness) *Let ϕ be a Π^1_2 formula with some free variables, M a transitive model of ZF+DC containing ω_1 as a subset, and $\vec{x}$ a sequence of elements of $\omega^\omega\cap M$. Then $\phi(\vec{x})$ holds if and only if $M\models\phi(\vec{x})$.*

Let X be a Polish space and $A\subset X$ its analytic subset. I must make precise what it means to reinterpret the Polish space and the analytic set in various models of set theory, typically generic extensions. This leads to a rather abstract and intelectually somewhat sterile discussion. All of these concerns trivialize if the reader ignores Polish spaces other than the recursively presented ones, whose interpretations can be simply taken to be their definitions in the respective models.

Definition 2.3.4. Let M be an ω-model of ZFC, $M \models X$ is a Polish space.

1. Let $\hat{X}$ be a Polish space. A map $\phi : X \to \hat{X}$ is an *interpretation* of X if there are an M-complete compatible metric d on X and a complete compatible metric e on $\hat{X}$ such that ϕ is a distance-preserving map from $\langle X, d\rangle$ to $\langle \hat{X}, e\rangle$ with dense range.
2. If $M \models C \subset X$ is a closed set, the interpretation of C is the closure of $\phi''C$ in $\hat{X}$.
3. If $M \models A \subset X$ is a closed set, $A = p(C)$ for some closed subset of $X \times Y$, then the interpretation of A is the set $p(\hat{C})$, where $\hat{C}$ is the interpretation of C in some product $\hat{X} \times \hat{Y}$.

The Shoenfield absoluteness provides the tools for showing that the interpretations have the uniqueness properties that one comes to expect as soon as the model M is regular enough:

Fact 2.3.5. *Suppose that the wellfounded part of M contains ω_1 as a subset. Interpretations of Polish spaces are unique up to a unique homeomorphism. Interpretations of analytic sets do not depend on the choice of the projecting closed sets.*

One way in which this fact is used is the notation for Polish spaces and their analytic subsets in generic extensions. If X is a Polish space, $A \subset X$ is an analytic set, P is a poset and $G \subset P$ is a generic filter over V, I will abuse the notation by using the letters X and A to denote the interpretations of X and A in $V[G]$, which are unique up to unique homeomorphisms.

Another way of using this fact is the following. If X is a Polish space, M is a countable elementary submodel of a large structure containing M, $\pi : M \to N$ is its transitive collapse, and $N[g]$ is a generic extension of N, then $\pi^{-1} : \pi(X) \to X$ is an interpretation of $\pi(X)$ in V. Also, if $\phi : \pi(X) \to Y$ is an interpretation of $\pi(X)$ in the model $N[g]$, there is a unique continuous map $\psi : Y \to X$ such that $\pi^{-1} = \psi \circ \phi$ and it is an interpretation of Y in V. Thus, I will abuse the notation by considering the (interpretation of) the space X in the model $N[g]$ as a subset of the space X.

The main absoluteness fact used tacitly in the book in many places is the following:

Fact 2.3.6. *If E, F are analytic equivalence relations on respective Polish spaces X, Y, $E \leq_{\mathrm{B}} F$, and $h : X \to Y$ is a Borel reduction of E to F, then h remains a Borel reduction of E to F in any generic extension. The same holds for weak Borel reductions.*

Proof. I will deal with the more difficult case of weak Borel reductions. Suppose that $a \subset X$ is a countable set and $h : X \to Y$ is a Borel function which is a reduction of E to F outside of the set $[a]_E$. An elementary complexity computation shows that this is a Π^1_2 statement in any Borel code for h, code for E, and enumeration of the set a. Therefore, by the Shoenfield absoluteness, this statement remains true in all forcing extensions as desired. □

In several places, I will use the following standard tool from the theory of definable forcing, which uses Mostowski absoluteness at every turn in its related arguments.

Definition 2.3.7. [2, Definition 3.6.1] A poset $\langle P, \leq \rangle$ is *Suslin forcing* if P is an analytic subset of some Polish space X and the relations of compatibility and incompatibility of conditions in P are both analytic in X^2, and moreover P is c.c.c.

Suslin forcings include such posets as the Cohen forcing, the random forcing, or the Hechler forcing. The main absoluteness features of Suslin forcings are captured in the following:

Fact 2.3.8. [9] *If P is a forcing with Suslin definition, M is a transitive model of a large fraction of ZFC containing the code for P, then the following are equivalent:*

1. *P is c.c.c.;*
2. *$M \models P$ is c.c.c.*

Theorem 2.3.9. *Suppose that P is a Suslin forcing, α is an ordinal, and M is a transitive model containing α as well as the code for P. Let P_α be the finite support iteration of P of length α. Then $P^M_\alpha \subset P_\alpha$, and whenever $G \subset P_\alpha$ is a filter generic over V, then $G \cap P^M_\alpha$ is a filter generic over M.*

Proof. By transfinite induction on α prove that $P^M_\alpha \subset P_\alpha$, the order and compatibility relation on P^M_α agree with same on P_α, and every maximal antichain of P^M_α in the model M is a maximal antichain in P_α. This will immediately imply the statement of the theorem.

For the successor ordinal $\alpha = \beta + 1$, observe first that as maximal antichains of P^M_β which appear in M are maximal in P_β by the induction hypothesis, the P^M_α-names in M are in fact P_α-names. Whenever $G \subset P_\beta$ is a filter generic over V, $G \cap P^M_\beta$ is generic over M and I can form the transitive model $M[G \cap P^M_\beta]$, for the sake of brevity denoted by $M[G]$. By the Mostowski absoluteness between $M[G]$ and $V[G]$, $P^{M[G]} = P^{V[G]} \cap M[G]$, and similarly for the ordering and compatibility in $P^{M[G]}$. It follows that P^M_α is a subset of P^V_α and the order and compatibility relations of the two posets agree.

To prove the maximal antichain part of the induction hypothesis, suppose that $A \subset P^M_\alpha$ is a maximal antichain in the model M and $p \in P_\alpha$ is a condition; I must find a condition $q \in A$ compatible with p. Again, let $G \subset P^V_\beta$ be a filter generic over V, containing the condition p, and form the models $M[G]$ and $V[G]$. Work in the model $M[G]$. Let $B = \{r \in P : \exists q \in A\ q \restriction \beta \in G$ and $r = q(\alpha)/G\}$. The set $B \subset P$ must be a maximal antichain of P since A was a maximal antichain of P_α. In the model $M[G]$, the poset P is c.c.c. by Fact 2.3.8 applied in $V[G]$, so B is countable. As the compatibility in P is an analytic relation, the maximality of B is a coanalytic formula in $M[G]$. By the Mostowski absoluteness between $M[G]$ and $V[G]$ again, B must be a maximal antichain of $P^{V[G]}$ in the model $V[G]$. Thus, there must be $q \in A$ such that $q \restriction \beta \in G$ and $q(\alpha)/G$ is compatible with $p(\alpha)/G$ in P. It follows that p, q must be compatible in P_α as desired.

For a limit ordinal α, since P_α is the direct limit of P_β for $\beta \in \alpha$, the same holds in M, and $P^M_\beta \subset P_\beta$ by the induction hypothesis, it is clear that $P^M_\alpha \subset P_\alpha$. Suppose that $A \in M$ is a maximal antichain of P^M_α, and $p \in P_\alpha$ is a condition. I must find an element of A compatible with p.

Let β is an ordinal $< \alpha$ such that $p \in P_\beta$. In the model M, pick a maximal antichain $B \subset P^M_\beta$ such that for every element $q \in B$, there is an element $r \in A$ such that q is smaller than the projection of r into P^M_β. By the induction hypothesis, there is an element $q \in B$ compatible with the condition p. Let $r \in A$ be a condition such that q is smaller than the projection of r into P^M_β. Then r must be compatible with r as requested. $\square$

2.4 Definability of forcing

In this section I will show that various operations which are the essence of the forcing method are Borel in a suitable sense. The resulting lemmas are perhaps more difficult to state properly than they are to prove. Nevertheless, they are quite useful in many complexity computations.

For the notation in this section, let $X = 2^{\omega\times\omega}$ be the Polish space of all binary relations on ω. Each element of X is understood as a model for a language with a single binary relational symbol. The following lemma is standard.

Lemma 2.4.1. *Suppose that $f : 2^\omega \to X$ is a Borel function, ϕ is a formula of the language with n free variables, and $g_i : 2^\omega \to \omega$ are Borel functions for every $i \in n$. The set $\{x \in X : f(x) \models \phi(g_0(x), g_1(x) \dots g_{n-1}(x))\}$ is Borel.*

Proof. By induction on complexity of the formula ϕ. Left to the reader. $\square$

Now, if M is a countable model of ZF and $P \in M$ is a poset, one may want to produce a filter $G \subset P$ generic over M and construct a forcing extension $M[G]$. This is a Borel procedure, as the next two lemmas show.

Lemma 2.4.2. *Suppose $M : 2^\omega \to X$ and $P : 2^\omega \to \omega$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a model of ZF and $M(y) \models P(y)$ is a poset. Then, there is a Borel function $G : 2^\omega \to \mathcal{P}(\omega)$ such that for every $y \in 2^\omega$, $G(y)$ is a filter on $P(y)$ which is generic over $M(y)$.*

Proof. By induction on $n \in \omega$ define Borel functions $f_n : 2^\omega \to \omega$ so that

- for every $y \in 2^\omega$, $M(y) \models f_n(y)$ is the largest element of the poset $P(y)$;
- for every $y \in 2^\omega$ and $n \in \omega$, if $M(y) \models n$ is an open dense subset of the poset $P(y)$, then $f_{n+1}(y)$ is the smallest number m such that $M(y) \models m$ is an element of n and it is smaller that $f_n(y)$ in the poset $P(y)$; otherwise, $f_{n+1}(y) = f_n(y)$.

The functions defined in this way are Borel by Lemma 2.4.1. Let $G : 2^\omega \to \mathcal{P}(\omega)$ be defined so that for every $y \in \omega$, $G(y)$ is the set of all m such that $M(y) \models m$ is an element of the poset $P(y)$ and for some $n \in \omega$, $M(y) \models f_n(y)$ is smaller than m in the poset $P(y)$. It is clear that the function G works. □

Lemma 2.4.3. *Suppose that $M : 2^\omega \to X$, $P : 2^\omega \to \omega$, $G : 2^\omega \to \mathcal{P}(\omega)$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a model of ZF, $M(y) \models P(y)$ is a poset, and $G(y) \subset P(y)$ is a filter generic over $M(y)$. Then, there are Borel functions $M[G] : 2^\omega \to X$ and $re : 2^\omega \times \omega \to \omega$ such that for every $y \in 2^\omega$, $M[G](y)$ is a generic extension of $M(y)$ by $G(y)$ and for every $n \in \omega$, $re(y)(n) = n/G(y)$ whenever $M(y) \models n$ is a $P(y)$-name.*

Proof. By induction on $n \in \omega$ build Borel functions $f_n : 2^\omega \to \omega$ such that for every $y \in 2^\omega$, $f_n(y)$ is the smallest number m such that $M(y) \models m$ is a $P(y)$-name, and for every $n' < n$ it is not the case that the filter $G(y)$ contains a condition p such that $M(y) \models p \Vdash_{P(y)} f_{n'}(y) = f_n(y)$. These functions are Borel by Lemma 2.4.1. Let $M[G] : 2^\omega \to X$ be the Borel function defined by $\langle n, m\rangle \in M[G](y)$ if the filter $G(y)$ contains a condition p such that $M[y] \models p \Vdash_{P(y)} f_n(y) \in f_m(y)$. Let $re : 2^\omega \times \omega \to \omega$ be the function defined by $re(y, n) = m$ if there is a condition p in the filter $G(y)$ such that $M(y) \models p \Vdash_{P(y)} f_m(x) = n$. These functions $M[G], re$ work by basic theorems on forcing applied in the models $M[y]$ for $y \in 2^\omega$. □

If M is a transitive countable model of set theory, $P \in M$ is a poset, $\tau \in M$ is a P-name for a transitive set, and a is a transitive set, one may ask whether there is a filter $G \subset P$ which is generic over M such that $a = \tau/G$, and attempt to produce such a filter if it exists. The following lemma shows that this is a Borel procedure. An important nontrivial case arises in applications where $P \Vdash V(\tau)$ fails the axiom of choice. The lemma apparently only works for wellfounded models as opposed to arbitrary (perhaps illfounded) models.

Lemma 2.4.4. *Suppose $M : 2^\omega \to X$, $P, \tau : 2^\omega \to \omega$ and $a : 2^\omega \to X$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a wellfounded model of ZFC, $M(y) \models P(y)$ is a poset, $\tau(y)$ is a $P(y)$-name for a transitive set. Then,*

1. *the set $B = \{y \in Y :$there is a filter $G \subset P(y)$ generic over the model $M(y)$ such that $\langle \tau(y)/G, \in\rangle$ is isomorphic to $a\}$ is Borel;*
2. *there is a Borel function $G : B \to \mathcal{P}(\omega)$ such that for every $y \in B$, $G(y) \subset P(y)$ is a filter generic over $M(y)$ such that $\langle \tau(y)/G(y), \in\rangle$ is isomorphic to $\langle a(y), \in\rangle$.*

Proof. Use Lemma 2.4.1 to find Borel functions $P_0, \sigma, \nu, P_1, P_2 : 2^\omega \to \omega$ so that for every $y \in 2^\omega$, $M(y)$ satisfies the following rest of this paragraph. $\sigma(y)$ is some $P(y) * \mathrm{Coll}(\omega, |\tau(y)|)$-name for an isomorph of $\langle \tau(y), \in \rangle$ with domain ω. Write Q be the poset of nonempty open subsets of the infinite permutation group S_∞, adding a Cohen-generic element $\dot{\pi} \in S_\infty$. Then $P_0(y)$ is the three step iteration $P(y) * \mathrm{Coll}(\omega, |\tau(y)|) * \dot{Q}$, $\nu(y)$ is the $P_0(y)$-name for the binary relation $\sigma \circ \dot{\pi}$ on ω. $P_1(y)$ is the complete Boolean algebra generated by the name $\nu(y)$, a complete subalgebra of the completion of the poset $P_0(y)$. Let $\dot{P}_2(y)$ be the P_1-name for the remainder poset $P_0(y)/P_1(y)$.

Let $D = \{\langle y, \pi, G \rangle \in 2^\omega \times S_\infty \times \mathcal{P}(\omega)$: $G \subset P_1(y)$ is a filter generic over the model $M(y)$ such that $\nu(y)/G = a(y) \circ \pi\}$. This is a Borel set by Lemma 2.4.3. The projection of D into the 2^ω coordinate is the set B by definitions. Write $C \subset 2^\omega \times S_\infty$ for the projection of D into the first two coordinates.

Claim 2.4.5. *1. The $\mathcal{P}(\omega)$-sections of D are either empty or else singletons.*

2. The S_∞-sections of C are either empty or else comeager in S_∞.

Proof. To simplify the notation, fix $y \in 2^\omega$ and omit the argument y from the expressions like $M(y), \sigma(y) \dots$.

(1) uses the wellfoundedness of the model M. Suppose that the $\mathcal{P}(\omega)$-section $D_{y,\pi}$ is nonempty. As $M \models$"P_1 is completely generated by the name ν", the filter G can be recovered by transfinite induction using infinitary Boolean expressions in M applied to $a \circ \pi$, and therefore it is unique.

(2) is more difficult, and it is the heart of the proof. Suppose that the S_∞-section C_y is nonempty. Thus, there is a filter $G_0 \subset P$ be a filter generic over M such that $\langle \tau/G_0, \in \rangle$ is isomorphic to the binary relation a on ω. Let $G_1 \subset \mathrm{Coll}(\omega, \tau/G_0)$ be a filter generic over $M[G_0]$, and let $z' = \sigma/(G_0 * G_1)$. Thus, z, z' are isomorphic binary relations on ω, an there is a permutation $\pi_0 \in S_\infty$ such that $z = z' \circ \pi_0$. Recall that Q is the poset of nonempty open subsets of S_∞. Let N be a countable elementary submodel of a large enough structure containing M, G_0, G_1, g_0. It will be enough to show that every element of S_∞ which is Q-generic over N belongs to the set C_y, since there are comeagerly many such points. Let $\pi \in S_\infty$ be a point Q-generic over N. Since the meager ideal on S_∞ is translation invariant, even the point $\pi_0^{-1}\pi$ is Q-generic over N and therefore over the smaller model $M[G_0][G_1]$ as well. Let $G_2 \subset Q$ be the filter generic over $M[G_0][G_1]$ associated with $\pi_0^{-1}\pi$. Now, $z \circ \pi = z' \circ \pi_0 \circ \pi_0^{-1} \circ \pi = z' \circ \pi_0^{-1}\pi$. Therefore, $z \circ \pi$ is equal to the point $\nu/(G_0 * G_1 * G_2)$and $\pi \in C_y$ as required. □

Now, as one-to-one projections of Borel sets are Borel [14, Theorem 15.1], the set C is Borel by Claim 2.4.5(1). As the category quantifier yields Borel sets [14, Theorem 16.1], the set B as the projection of C into the first coordinate is Borel. Borel sets with nonmeager vertical sections allow Borel uniformizations [14, Theorem 18.6], and so there is a Borel uniformization $f : B \to S_\infty$ of C. As the set D has singleton vertical sections, it is itself its uniformization $g : C \to \mathcal{P}(\omega)$. Let $G_1 : B \to \mathcal{P}(\omega)$ be the function defined by $G_1(y) = g(y, f(y))$. Thus, for every $y \in B$, $G_1(y) \subset P_1(y)$ is a filter generic over $M(y)$ such that $\nu/G_1(y)$ is isomorphic to $a(y)$.

The argument is now in its final stage. Let $P_2 : 2^\omega \to \omega$ be a Borel function such that for every $y \in 2^\omega$, $M(y) \models P_2(y)$ is a name for the quotient poset $P_0(y)/P_1(y)$. Use Lemmas 2.4.2 and 2.4.3 to find a Borel function $G_2 : B \to \mathcal{P}(\omega)$ such that $G_2(y) \subset P_2/G_1(y)$ is a filter generic over the model $M(y)[G_1(y)]$. Let $G_0 : B \to \mathcal{P}(\omega)$ be a Borel function indicating a filter on $P_0(y)$ which is the composition of G_1 and G_2. Let $G : B \to \mathcal{P}(\omega)$ be the function which indicates the first coordinate of the filter $G_0(y) \subset P_0(y)$. Recall that the poset $P_0(y)$ is a three stage iteration of which the first stage is $P(y)$, so for every $y \in B$, $G(y) \subset P(y)$ is a filter generic over the model $M(y)$. The function G has the required properties. $\square$

If M is a countable model of ZFC, $j : M \to N$ is an elementary embedding which is a class in M, and L is a linear ordering, one may form the iteration of j along the linear ordering L. This is a Borel operation as the following lemma shows.

Lemma 2.4.6. *Let $M : 2^\omega \to X$ and $U : 2^\omega \to \omega$ and $L : 2^\omega \to X$ be Borel functions. There is a Borel function $N : 2^\omega \to X$ and $j : 2^\omega \times \omega \to \omega$ such that for every $y \in 2^\omega$, if $M(y)$ is a model of ZFC, $M(y) \models U(y)$ is a normal measure on an uncountable cardinal, and $L(y)$ is a wellordering, then $N(y)$ is a model of ZFC isomorphic to the iteration of the $U(y)$ ultrapower of the model $M(y)$ along $L(y)$, and $j(y) : M(y) \to N(y)$ is the iteration embedding.*

Proof. To simplify the notation, assume that for every $y \in 2^\omega$, $M(y)$ is a model of ZFC and $M(y) \models U(y)$ is a normal measure on an uncountable cardinal. I have to show that the usual direct limit description of the iteration is Borel.

First, consider the construction of the usual ultrapower. Let $\kappa(y)$ be the cardinal on which $U(y)$ is normal measure in $M(y)$. For every $m \in \omega$ let $U^m(y)$ be the m-th Fubini product of $U(y)$; thus $M(y) \models U^m(y)$ is a $\kappa(y)$-complete ultrafilter on $\kappa^m(y)$. For $a \subset \omega$ and $y \in 2^\omega$, let $E_a(y)$ be the equivalence relation on ω defined by $n\ E_a\ m$ if $M(y) \models \{x : n(x) = m(x)\} \in U^{|a|}(y)$. Let $\in_a (y)$ be the binary relation on ω defined by $n\ E_a\ m$ if $M(y) \models \{x : n(x) \in m(x)\} \in U^{|a|}(y)$. Thus, $E_a, \in_a$ are Borel relations on $2^\omega \times \omega \times \omega$ and moreover, $\in_a (y)$ respects the equivalence E_a.

Now look at the direct limit of the ultraproducts constructed in the previous paragraph. For $a \subset b \subset \omega$ let $j_{ab}(y) : \omega \to \omega$ by the function defined by $j_{ab}(y)(n) = m$ if $M(y) \models n$ is a function with domain $\kappa^{|a|}(y)$ and m is a function with domain $\kappa^{|b|}(y)$ and $n = m \circ \phi$ where $\phi : |a| \to |b|$ is the unique map such that $\phi \circ \psi_0 = \psi_1 \circ i$, where $i : a \to b$ is the identity map, ψ_0 is the order preserving map from a with the $L(y)$-order to $|a|$ with the usual natural number order and ψ_1 is the order preserving map from b with the $L(y)$-order to $|b|$ with the usual natural number order. Thus, $j_{ab} : 2^\omega \times \omega \to \omega$ is a Borel map, $j_{ab}(y)$ respects the equivalences $E_a(y)$ and $E_b(y)$, and $j_{ac}(y) = j_{bc}(y) \circ j_{ab}(y)$ whenever $a \subset b \subset c$. Let $E(y)$ be an equivalence relation on $\omega \times [\omega]^{<\aleph_0}$ defined by $\langle n, a\rangle\ E\ (y)\langle m, b\rangle$ if $j_{a,a\cup b}(y)(n)\ E_{a\cup b}\ j_{b,a\cup b}(y)(m)$. Let $\in (y)$ be a relation on $\omega \times [\omega]^{<\aleph_0}$ defined by $\langle n, a\rangle \in (y)\langle m, b\rangle$ if $j_{a,a\cup b}(y)(n) \in_{a\cup b} (y) j_{b,a\cup b}(y)(m)$.

Now, the final considerations. It is now easy to find a Borel map $\pi : 2^\omega \times \omega \times [\omega]^{<\aleph_0}$ such that $\pi(y)$ is constant on $E(y)$-equivalence classes and distinct $E(y)$-equivalence classes are mapped to distnict numbers. Let $N : 2^\omega \to X$ be the Borel map defined by $(n, m) \in N(y)$ if $\pi^{-1}(y)(n) \in (y)\ \pi^{-1}(y)(m)$. Let $j : 2^\omega \times \omega \to \omega$ be the Borel map defined by $j(y)(n) = m$ if $\pi(y)(n, 0) = m$. This works. $\square$

Chapter 3

Pinned equivalence relations

3.1 Definitions and basic concerns

The concept of unpinned equivalence relations first appeared in the work of Greg Hjorth. The current general definition is due to Kanovei:

Definition 3.1.1. [12, Chapter 17] Let E be an analytic equivalence relation on a Polish space X. Let P be a poset and τ a P-name for an element of $\dot{X}$. The name τ is (E-)*pinned* if $P \times P \Vdash \tau_{\text{left}}\ \dot{E}\ \tau_{\text{right}}$. The name is ($E$-)*trivial* if $P \Vdash \tau\ \dot{E}\ \check{x}$ for some ground model element $x \in X$. The equivalence relation E is *pinned* if all E-pinned names on all posets are E-trivial. Otherwise, E is *unpinned*.

The reader should note the terminology discrepancy between this definition and that of [12, Chapter 17]; the reason behind it is the harmonization of the pinned concept and the trim concept introduced later in this book. The pinned concept is invariant under Borel reducibility and even weak Borel reducibility. This feature turns it into a potent tool for proving nonreducibility results.

Lemma 3.1.2. *If E, F are analytic equivalence relations on respective Polish spaces X, Y, $E \leq_{\text{wB}} F$, and F is pinned, then E is pinned as well.*

Proof. Let $a \subset X$ be a countable set and $h : X \to Y$ be a Borel map which is a Borel reduction of E to F on $X \setminus [a]_E$. Let P be a poset and τ an E-pinned name for an element of X. I must prove that τ is trivial.

If $P \Vdash \tau \in [a]_E$ then certainly τ is trivial. Suppose then that $P \Vdash \tau \notin [a]_E$. By the Shoenfield absoluteness, the function h remains a reduction of E to F outside the set $[a]_E$ even in the $P \times P$-extension of the ground model. It follows that $\dot{h}(\tau)$ is an F-pinned name. As F is pinned, this name must be F-trivial and so there is $y \in Y$ such that $P \Vdash \dot{h}(\tau)\ F\ \check{y}$. In particular, P forces that there is $x \in X \setminus [a]_E$ such that $h(x)\ F\ y$. By the Shoenfield absoluteness, such an

$x \in X$ must exist already in the ground model. It is easy to see that $P \Vdash \tau \; E \; \check{x}$ as desired. □

I will now list several results identifying some natural pinned and unpinned equivalence relations. The following two facts were proved soon after the introduction of the pinned concept:

Fact 3.1.3. [12, Theorem 17.1.3] *All Borel equivalence relations with $\mathbf{\Sigma}^0_3$ equivalence classes are pinned.*

Fact 3.1.4. [12, Theorem 17.1.3] *All orbit equivalence relations generated by continuous actions of Polish groups with complete left-invariant metric are pinned.*

Another quite different class of equivalence relations turns out to contain only pinned relations as well:

Definition 3.1.5. An equivalence relation E on a Polish space X is *treeable* if there is an analytic acyclic graph $T \subset X^2$ such that $x \; E \; y$ if and only if x, y are in the same connected component of T.

Theorem 3.1.6. (with John Clemens) *All treeable equivalence relations are pinned.*

Proof. First, I will introduce a useful graph-theoretic notation and a fact. If S is a tree on some vertex set V and $a \subset V$ is a set, the *convex closure* of a in S is the smallest set $b \supset a$ which with any two of its points also contains the shortest S-path connecting them. Reviewing all the finitely many possible configurations, it is easy to see that any set of four distinct points in V can be divided into two disjoint pairs whose convex closures have a nontrivial intersection.

Let E be a treeable equivalence relation on a Polish space X. Let $T \subset X^2$ be an analytic acyclic graph generating E. Let P be a poset and τ an E-pinned name; I must show that τ is E-trivial.

Consider the product P^4 of four copies of the poset P, a generic filter $G \subset P^4$, its associated mutually generic filters $G_i : i \in 4$ on the poset P, and the points $x_i = \tau/G_i \in X$ for $i \in 4$. As the name τ is pinned, the points x_i are pairwise E-related. As in the first paragraph, there are two disjoint pairs in the set $\{x_i : i \in 4\}$, say $\{x_0, x_1\}$ and $\{x_2, x_3\}$, whose convex closures have a nontrivial intersection, containing some point $x \in X$. Use the Shoenfield absoluteness to see that the graph T is still acyclic in $V[G_0, G_1]$ as well as in $V[G]$ and it generates E in both these models. By the Shoenfield absolutenes again, the (finite) convex closure of $\{x_0, x_1\}$ is a subset of the model $V[G_0, G_1]$, since it is the unique T-path between x_0 and x_1 which does not turn back on itself. Similarly, the convex closure of $\{x_2, x_3\}$ is a subset of $V[G_2, G_3]$. Thus, $x \in V[G_0, G_1] \cap V[G_2, G_3]$, and by the product forcing theorem it follows that $x \in V$. This implies that the name τ is E-trivial, forced to be E-related to $\check{x}$ as required. □

There are two canonical examples of analytic unpinned equivalence relations. First, let F_2 by the equivalence relation on $(2^\omega)^\omega$ defined by $x \; F_2 \; y$ if $\mathrm{rng}(x) = \mathrm{rng}(y)$. This is a Borel equivalence relation.

Example 3.1.7. F_2 is unpinned.

Proof. Let $P = \text{Coll}(\omega, 2^\omega)$ and let τ be the P-name for the generic enumeration of $(2^\omega)^V$. It is clear that this is an F_2-pinned name, and since the set $(2^\omega)^V$ is uncountable in V, the name is not trivial.

□

Let E_{ω_1} be the analytic equivalence relation on binary relations on ω connecting x, y if either both x, y are not wellfounded linear orders or else x and y are isomorphic. This is a most common example of an analytic equivalence relation with $\aleph_1$ many equivalence classes.

Example 3.1.8. E_{ω_1} is unpinned.

Proof. Let $P = \text{Coll}(\omega, \omega_1)$ and let τ be a P-name for a binary relation on ω which is a wellorder of length ω_1^V. It is not difficult to check that τ is a nontrivial E_{ω_1}-pinned name. □

As is the case with every class of equivalence relation closed under Borel reducibility, two questions immediately come to mind.

Question 3.1.9. Is there a $\leq_{\text{B}}$-smallest unpinned Borel equivalence relation?

Kechris [12, Question 17.6.1] conjectured that F_2 is this smallest unpinned Borel equivalence relation. This turns out to be false [27] and many counterexamples will be discussed below. However, in one sense the conjecture has a positive answer: it is true in choiceless contexts such as the Solovay model derived from a measurable cardinal–Theorem 3.4.1.

Question 3.1.10. Is there a $\leq_{\text{wB}}$-largest pinned analytic equivalence relation?

In general, it appears to be quite difficult to decide whether a given equivalence relation is pinned or not. The following question illustrates the extent of my ignorance in this direction:

Question 3.1.11. Characterize the collection of those compact (analytic etc.) sets $A \subset (2^\omega)^\omega$ such that $F_2 \restriction A$ is pinned.

3.2 Absoluteness

The definition of pinned equivalence relation quantifies over all partial orderings as well as names. The status of absoluteness of such a definition is a priori unclear. In this section, I will show that the evaluation of Borel equivalence relation as pinned/unpinned is suitably absolute in ZFC. For analytic equivalence relations, a similar result holds in the presence of suitable large cardinals.

Theorem 3.2.1. *Suppose that E is a Borel equivalence relation on a Polish space X. The following are equivalent:*

1. *E is pinned;*

2. For every ω-model M of ZFC containing the code for E, $M \models E$ is pinned.

Proof. For simplicity assume $X = \omega^\omega$. The implication (2)$\to$(1) is trivial. If (1) fails, then E is unpinned, and the failure of (2) is witnessed by $M = V$.

The implication (1)$\to$(2) is more difficult. Suppose that (2) fails. Fix a ω-model M of ZFC containing the code for E such that $M \models E$ is unpinned; taking an elementary submodel if necessary I may assume that M is countable. In the model M, find a nontrivial E-pinned name τ_0 on some poset Q_0. Form a transfinite sequence of models and a commuting system of elementary embeddings $\langle M_\alpha, Q_\alpha, \tau_\alpha, j_{\beta\alpha} : \beta < \alpha \leq \omega_1\rangle$ so that

1. $M_0 = M$, for every countable α M_α is a countable ω-model of ZFC, and $j_{\beta\alpha}(Q_\beta, \tau_\beta) = Q_\alpha, \tau_\alpha$;
2. for limit α the model M_α is obtained as a direct limit of the earlier models;
3. if there is $x \in X$ such that $Q_\alpha \Vdash \tau_\alpha \mathrel{E} \check{x}$ then there is such an x in the model $M_{\alpha+1}$.

After the induction is performed, I will show that τ_{ω_1} is a nontrivial E-pinned name on Q_{ω_1}, and therefore E is indeed unpinned and (1) fails.

The successor step of the induction is arranged through the following theorem of ZFC applied in the model M_α:

Claim 3.2.2. *Whenever P is a poset, then in some generic extension there is an elementary embedding $j : V \to W$ into a possibly illfounded ω-model W such that W contains $j''P$ as well as some subset of $j''P$ whose j-preimage is a P-generic filter over V.*

Proof. Consider the set $Y = [P \cup \mathcal{P}(P)]^{\aleph_0}$ and functions f, g with domain Y such that $f(a) = a \cap P$ and $g(a)$ is some filter on $a \cap P$ which meets all open dense subsets of $a \cap P$ in the set a. Let I be the σ-ideal of nonstationary subsets of the set Y and consider the poset $R = \mathcal{P}(Y)$ modulo I and the associated generic ultrapower $j : V \to W$. It is easy to see that the functions f, g represent the desired elements in the generic ultrapower: $[f] = j''P$ and $[g]$ is a filter on $j''P$ meeting $j(D)$ for every open dense subset $D \subset P$ in the ground model. $\square$

Now working in M_α, find a poset R forcing the existence of the elementary embedding as above for $P = Q_\alpha$. Let R' be the disjoint union of R and Q_α with the ordering defined by $r \leq q$ if $r \Vdash j(q) \in \dot{g}$. Then Q_α is a regular subposet of R' and R is a dense subset of R'. Now suppose that $Q_\alpha \Vdash \tau_\alpha \mathrel{E} \check{x}$; perhaps the point x is not in the model M_α. Let N be a countable elementary submodel of a large enough structure and let $h \subset R'$ be generic over N. Then $h \subset R'$ is also generic over M_α. Let $j = j_{\alpha\alpha+1} : M_\alpha \to M_{\alpha+1}$ be the generic embedding obtained by an application of the claim in M_α. Let $g = h \cap Q_\alpha$. Then $\tau/g \mathrel{E} x$ by the forcing theorem applied in the model N and the Mostowski absoluteness between $N[h]$ and V. Also, $\tau/g \in M_{\alpha+1}$ since $j''g \in M_{\alpha+1}$ and

τ/g is reconstructed as the unique point $y \in \omega^\omega$ such that for every $n \in \omega$, $y(n) = m$ if there is $p \in j''g$ which forces in the poset jQ_α that $j(\tau)(n) = m$.

Once the induction is performed, consider the poset $Q = Q_{\omega_1}$ as well as the name $\tau = \tau_{\omega_1}$ from the point of view of V as opposed to the model M_{ω_1}.

Claim 3.2.3. *τ is an E-pinned name for an element of X.*

Proof. Since M_{ω_1} is an ω-model, τ is indeed a name for an element of X. To see that it is pinned, let $G \times H \subset Q \times Q$ be a filter generic over V. Then $G \times H$ is also generic over M_{ω_1} and by the forcing theorem applied in M_{ω_1}, $M_{\omega_1}[G \times H] \models \tau/G \; E \; \tau/H$. Thus, the model $M_{\omega_1}[G \times H]$ contains a branch through the tree T witnessing $\tau/G \; E \; \tau/H$, which also witnesses that $V[G \times H] \models \tau/G \; E \; \tau/H$. □

Claim 3.2.4. *τ is a nontrivial E-pinned name.*

Proof. Suppose that there is a point $x \in X$ such that $Q \Vdash \tau \; E \; \check{x}$. Let N be a countable elementary submodel of a large enough structure containing the iteration and the point x and write $\alpha = \omega_1 \cap N$. Since the model M_{ω_1} is a direct limit of the tower of earlier models, the transitive collapse π of $N \cap M_{\omega_1}$ is an isomorphism of Q_α, τ_α with $Q \cap N, \tau \cap N$. Whenever $g \subset Q \cap N$ is a filter generic over V, it must be the case that $N[g] \models x \; E \; \tau/g$ by the forcing theorem applied in N, and $V[g] \models x \; E \; \tau_\alpha/\pi''g = \tau/g$ by the Mostowski absoluteness between $V[g]$ and $N[g]$. Thus, $Q_\alpha \Vdash \tau_\alpha \; E \; \check{x}$, and by the inductive assumption there is a point $y \in X \cap M_{\alpha+1}$ which is E-related to x. Then, $Q \Vdash \tau \; E \; \check{y}$. By the Borel absoluteness between the Q-extension of V and M_{ω_1}, it must be the case that $M_{\omega_1} \models Q \Vdash \tau \; E \; \check{y}$. This contradicts the fact that $M_0 \models \tau_0$ is a nontrivial E-pinned name together with the elementarity of the embedding $j_{0\omega_1}$. □

□

Corollary 3.2.5. *Let E be a Borel equivalence relation on a Polish space X. The collection $I = \{A \subset X\colon A$ is analytic and $E \restriction A$ is pinned$\}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal of analytic sets.*

Proof. It is not difficult to see that I is indeed a σ-ideal. To verify the definability condition, let $A \subset X$ be an analytic set. The statement "$E \restriction A$ is unpinned" is equivalent to the existence of an ω-model containing the code for A as well as E which satisfies the statement "$E \restriction A$ is unpinned" by Theorem 3.2.1, and this is an analytic statement. □

Corollary 3.2.6. *Let E be a Borel equivalence relation on a Polish space X. The statement "E is pinned" is absolute between all generic extensions.*

Proof. The validity of item (2) of Theorem 3.2.1 does not change if one only considers countable models. This follows from an immediate downward Löwenheim-Skolem argument. The countable model version of (2) is a coanalytic statement, and as such it is absolute among all forcing extensions by the Mostowski absoluteness. □

The absoluteness of the pinned status of an analytic equivalence relation is a considerably more difficult question. It does not hold in ZFC alone as the following rather primitive constructible example shows.

Example 3.2.7. In the constructible universe, there is an analytic equivalence relation E which is pinned, while in some generic extension it becomes unpinned.

Proof. The domain of E consists of structures with universe ω and language including one binary relation $\in$ and one ternary relation symbol R. The equivalence relation is defined by the following formula: $x \mathrel{E} y$ if either x, y both fail to be wellfounded models of the $L_{\omega_1\omega}$ sentence $\phi \wedge \psi$ where ϕ says "$V = L_\alpha$ and $\mathcal{P}(\omega)$ exists" and ψ says "for every infinite ordinal β, $R(\beta, \cdot, \cdot)$ orders the ordinals smaller than β in ordertype ω", or x is isomorphic to y. It is not difficult to see that E is indeed an analytic equivalence relation.

Now in L, there are no uncountable wellfounded models of $\phi \wedge \psi$, since ψ implies that such model would have to be L_{ω_1}, and $L_{\omega_1} \models \mathcal{P}(\omega)$ does not exist. This means that in L, E is pinned. On the other hand, in the $\mathrm{Coll}(\omega, \omega_1)$-extension of L, the relation E becomes unpinned, as $\phi \wedge \psi$ has an uncountable wellfounded model ($L_{\omega_2^L}$ with an appropriate relation R) and the $\mathrm{Coll}(\omega, \omega_2^L)$-name for an isomorph of this model with universe ω is a nontrivial E-pinned name. $\square$

Still, in the presence of sufficiently large cardinals the pinned status of every analytic equivalence relation is absolute:

Theorem 3.2.8. *Assume that there is a measurable cardinal larger than a Woodin cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is pinned;*
2. *for every wellfounded, linearly iterable model M of ZFC+there is a measurable cardinal larger than a Woodin cardinal containing the code for E, $M \models E$ is pinned.*

Proof. The argument is entirely parallel to the proof of Theorem 3.2.1 and I will only indicate the minor changes. In the proof of (1)$\to$(2), From a failure of (2) extract a countable wellfounded linearly iterable model $M_0 \models E$ is not pinned, and build the iteration so that the encountered models M_α are wellfounded. This is possible by using the stationary tower forcing associated with the Woodin cardinal at successor stages of the construction [16, Theorem 2.7.7]. In the end, the model M_{ω_1} is wellfounded as well, and by the Mostowski absoluteness it and its generic extensions are correct about the analytic equivalence relation E. This is all that is necessary to push the essentially identical argument through. $\square$

Corollary 3.2.9. *Suppose that there are class many Woodin cardinals. Let E be an analytic equivalence relation on a Polish space X. Then E is pinned if and only if every poset forces E to be pinned.*

Proof. The truth value of the statement (2) in Theorem 3.2.8 does not change if one considers countable models only. This follows from a downward Löwenheim-Skolem argument. The countable version of (2) is $\mathbf{\Sigma}^1_3$ and so the corollary follows from $\mathbf{\Sigma}^1_3$-absoluteness between the ground model any any forcing extension under the given large cardinal assumption. $\square$

3.3 Restrictions on forcings

It is now natural to ask which forcings can carry nontrivial pinned names. This is in fact a complicated issue. The following theorem records most of the general facts known to me at this point in this direction. I will need the following standard definition:

Definition 3.3.1. (Foreman, Magidor [5]) A poset P is *reasonable* if for every ordinal λ and for every function $f : \lambda^{<\omega} \to \lambda$ in the P-extension there is a set $a \subset \lambda$ which is closed under f, belongs to the ground model, and it is countable in the ground model.

In particular, all c.c.c. and all proper forcings are reasonable. Good examples of unreasonable forcings are posets which collapse $\aleph_1$, Namba forcing and Prikry forcing.

Theorem 3.3.2. *Let E be an analytic equivalence relation on a Polish space X.*

1. *If E is unpinned then there is a nontrivial E-pinned name on every poset collapsing $\aleph_1$ to $\aleph_0$;*
2. *if $V = L$ then there are no nontrivial E-pinned names on $\aleph_1$-preserving posets;*
3. *there are no nontrivial E-pinned names on reasonable posets;*
4. *if E is an orbit equivalence relation then there are no nontrivial E-pinned names on $\aleph_1$-preserving posets.*

Proof. For (1), let τ be a nontrivial E-pinned name on some poset P. Let $\langle M_\alpha : \alpha \in \omega_1\rangle$ be a continuous $\in$-tower of countable elementary submodels of a large structure containing X and E. Let $M_{\omega_1} = \bigcup_\alpha M_\alpha$, let $Q = P \cap M_{\omega_1}$ and let $\sigma = \tau \cap M_{\omega_1}$.

Claim 3.3.3. *σ is a nontrivial E-pinned name on the poset Q.*

Proof. I will just show that σ is nontrivial. Suppose for contradiction that there is a point $x \in X$ such that $Q \Vdash \sigma\ E\ \check{x}$. I will show that there then must be $y \in M_{\omega_1} \cap X$ which is E-related to x. Then $Q \Vdash \sigma\ E\ \check{y}$, by the Mostowski absoluteness between the Q-extensions of M_{ω_1} and V $M_{\omega_1} \models P \Vdash \tau\ E\ \check{y}$, and this contradicts the elementarity of the model M_{ω_1} and the nontriviality of the name τ.

To find the point $y \in M_{\omega_1} \cap X$, let N be a countable elementary submodel of a large structure containing $\langle M_\alpha : \alpha \in \omega_1 \rangle, Q, x$. Since the tower of models $\langle M_\alpha : \alpha \in \omega_1 \rangle$ is continuous, there is a limit ordinal $\alpha \in \omega_1$ such that $M_\alpha = N \cap M_{\omega_1}$. Let $Q_\alpha = Q \cap M_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \sigma \cap M_\alpha = \tau \cap M_\alpha$. By elementarity of the model N and analytic absoluteness between the Q_α-extension of N and V, $Q_\alpha \Vdash \sigma_\alpha \ E \ \check{x}$. Since $Q_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \tau \cap M_\alpha$, both Q_α, τ_α belong to the model $M_{\alpha+1}$. By the elementarity of the model $M_{\alpha+1}$, there must be a point $y \in X \cap M_{\alpha+1}$ such that $Q_\alpha \Vdash \sigma_\alpha \ E \ \check{y}$ (since the point x is such). By the transitivity of E, it follows that $x \ E \ y$. The point $y \in M_{\alpha+1} \subset M_{\omega_1}$ works. $\square$

Now, suppose that R is a poset collapsing $\aleph_1$. Since $|M_{\omega_1}| = \aleph_1$, in the R-extension there is a filter Q-generic over M_{ω_1}. Let $\dot{H}$ be an R-name for such a filter and let ν be the R-name for $\sigma/\dot{H}$. I claim that ν is a nontrivial E-pinned R-name. For this, suppose that $K_{\text{left}} \times K_{\text{right}} \subset R \times R$ is a generic filter, consider the associated filters $H_{\text{left}}, H_{\text{right}} \subset Q$ generic over M_{ω_1}, and write $x_{\text{left}} = \sigma/H_{\text{left}}$, $x_{\text{right}} = \sigma/H_{\text{right}}$. I must show that x_{left} and $x_{\text{right}} \in X$ are E-related, and at the same time not E-related to any element of the ground model.

To this end, let $F \subset Q$ be a filter generic over $V[K_{\text{left}} \times K_{\text{right}}]$ and let $y = \sigma/F$. By the forcing theorem in the ground model and the claim, y is not E-related to any element of the ground model. By the forcing theorem in the model M_{ω_1}, $M_{\omega_1}[H_{\text{left}}, F] \models x_{\text{left}} \ E \ y$ and $M_{\omega_1}[H_{\text{right}}, F] \models x_{\text{right}} \ E \ y$. By the transitivity of E and the Mostowski absoluteness between these wellfounded models and $V[K_{\text{left}} \times K_{\text{right}}]$, it follows that $V[K_{\text{left}} \times K_{\text{right}}] \models x_{\text{left}} \ E \ x_{\text{right}}$ and these two points are not related to any element of the ground model. This completes the proof of (1).

For (2), assume that $V = L$ and let P be an $\aleph_1$-preserving poset and τ an E-pinned name on P. Let $G \subset P$ be a filter generic over V and work in $V[G]$. I must find a point $x_1 \in X \cap V$ which is E-related to $x_0 = \tau/G$.

Let M be a countable elementary submodel of a large structure containing P, τ. Let N be the transitive isomorph of $M \cap V$. Let $\pi : M \cap V \to N$ be the transitive collapse map. Note that $N = L_\alpha$ for some countable ordinal α, in particular $N \in V$ and it is countable there. So, there is a filter $H_1 \subset \pi(P)$ generic over N which belongs to V, and the point $x_1 = \pi(\tau)/H \in X$ also belongs to V. I will show that $x_1 \ E \ x_0$.

By the elementarity of the model M, the filter $H_0 = \pi''G \subset \pi(P)$ is generic over N and $x_0 = \pi(\tau)/H_0$. Let $H_2 \subset \pi(P)$ be a filter generic over both countable models $N[H_0]$ and $N[H_1]$ and let $x_2 = \pi(\tau)/H_2$. Now, by the elementarity of the model M and the transitive collapse π, $N \models \pi(\tau)$ is an E-pinned name on $\pi(P)$ and so $N[H_0, H_2] \models x_0 \ E \ x_2$ and $N[H_1, H_2] \models x_1 \ E \ x_2$. Mostowski absoluteness together with the transitivity of the relation E shows that $x_0 \ E \ x_1$ holds as required.

For (3), suppose that P is a reasonable poset and τ is an E-pinned name on P. I will produce a condition $p \in P$ and a point $x \in X$ such that $p \Vdash \tau \ E \ \check{x}$. Towards this end, choose a large structure and use the reasonability of P to find a countable elementary submodel M of it containing P, E and τ and a condition

$p \in P$ such that $p \Vdash \dot{G} \cap \check{M}$ is generic over $\check{M}$, where $\dot{G}$ is the canonical P-name for its generic ultrafilter. As M is countable, there is a filter $H \subset P \cap M$ generic over M in the ground model V. Let $x = \tau/H \in X$; I claim that $p \Vdash \check{x}\ E\ \tau$.

Let $G \subset P$ is a filter generic over V containing the condition p and write $y = \tau/G$. Note that $G \cap M \subset P$ is a filter generic over M. Let $K \subset P \cap M$ be a filter generic over $V[G]$; so it is also generic over the smaller models $M[H]$ and $M[G \cap M]$. Write $z = \tau/K$. By the product forcing theorem applied in M, the filters $H \times K$ and $(G \cap M) \times K \subset P \times P$ are both generic over M. By the forcing theorem in M, $M[G, K] \models y\ E\ z$ and $M[H, K] \models x\ E\ z$. By the transitivity of the relation E and Mostowski absoluteness, $V[G] \models x\ E\ y$. This completes the proof of (3).

The proof of (4) is apparently much harder and it is conducted in Corollary 5.2.8. □

Theorem 5.3.3 provides a consistent example of a (simple) Borel equivalence relation E such that there is a nontrivial E-pinned name on Namba forcing. The following remains open though.

Question 3.3.4. Can there be an analytic equivalence relation E and a poset P such that every countable set of ordinals in the P-extension is covered by a countable set of ordinals in the ground model, and there is a nontrivial E-pinned name on P?

3.4 Unpinned relations without choice

The features of the class of unpinned equivalence relations heavily depend on the underlying set theory; this is one of the points of the present book. In this section, I will show that in the absence of the axiom of choice the class of unpinned equivalence relations may greatly simplify and admit a natural two-element basis. The proof uses the tools developed in Chapter 4 in an indispensable way.

Recall the construction of the choiceless Solovay model. If κ is an inaccessible cardinal and $G \subset \mathrm{Coll}(\omega, <\kappa)$ is a generic filter over V, then the Solovay model derived from κ is the model $V(2^\omega \cap V[G])$.

Theorem 3.4.1. *The following holds in the Solovay model derived from a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is unpinned;*
2. *$F_2 \leq_{\mathrm{B}} E$ or $E_{\omega_1} \leq_{\mathrm{wB}} E$.*

Corollary 3.4.2. *The following holds in the Solovay model derived from a measurable cardinal. Let E be a Borel equivalence relation on a Polish space X. E is unpinned if and only if $F_2 \leq_{\mathrm{B}} E$.*

Proof. This follows from Theorem 3.4.1 once I show that the option $E_{\omega_1} \leq_{\mathrm{wB}} E$ is not available for any Borel equivalence relation E. This in turn follows easily from results on the pinned cardinal $\kappa(E)$ obtained in Chapter 4: $\kappa(E) < \infty$ by Theorem 4.1.4(1), $\kappa(E_{\omega_1}) = \infty$ by Example 4.1.8, and the pinned cardinal is monotone with respect to the reducibility ordering $\leq_{\mathrm{wB}}$–Theorem 4.1.3. $\square$

Proof of Theorem 3.4.1. Let κ be a measurable cardinal and let W be the Solovay model derived from κ. In the model W, (2) certainly implies (1) as the proofs that F_2, E_{ω_1} are unpinned work in ZF, and the proof that pinned equivalence relations persist downwards in the orderings $\leq_{\mathrm{B}}$ and $\leq_{\mathrm{wB}}$ works in ZF+DC.

For the implication (1)$\to$ (2), assume that $W \models E$ is unpinned. There must be a poset P and an E-pinned E-nontrivial name τ on the poset P, both in W. Both P and τ must be definable from a ground model parameter and a real in W by the respective formulas ϕ_P and ϕ_τ. For simplicity assume that both these reals as well as the real defining the relation E belong to the ground model. Let Q be the two-step iteration $\mathrm{Coll}(\omega, <\kappa) * \dot{P}$, and write σ for the Q-name obtained from the $\dot{P}$-name τ. There are two cases.

Case 1. There is a condition $q \in Q$ such that the $Q \restriction q$-name σ is E-pinned. In this case, I will conclude that $E_{\omega_1} \leq_{\mathrm{wB}} E$ and use the Shoenfield absoluteness to transfer the weak reducibility the Solovay model. The most conceptual proof of the reducibility uses the tools of Chapter 4, namely the equivalence $\bar{E}$ and the cardinal $\kappa(E)$. To simplify the notation assume that q is the largest element of the poset Q.

First, observe that the name σ cannot be $\bar{E}$-equivalent to any name on a poset of size $< \kappa$. Suppose for contradiction that R is a poset of size $< \kappa$ and χ an E-pinned R-name such that $\langle R, \chi\rangle \ \bar{E}\ \langle Q, \sigma\rangle$. Let $G \times H \subset Q \times R$ be mutually generic filters over the ground model, and decompose G into the filters $G_0 * G_1$ corresponding to the iteration $Q = \mathrm{Coll}(\omega, <\kappa) * \dot{P}$. Since $|R| < \kappa$, there must be a filter $K \subset R$ in $V[G_0]$ generic over the ground model. By Lemma 2.2.7, the filters $H, K \subset R$ are mutually generic over the ground model. Thus, $\chi/H\ E\ \chi/K$ as χ is a pinned name. Also, $\chi/H\ E\ \sigma/G = \tau/G_1$ as $\langle R, \chi\rangle\ E\ \langle Q, \sigma\rangle$. The transitivity of E then implies that $\tau/G_1\ E\ \chi/K$, which contradicts the assumption that $\mathrm{Coll}(\omega, <\kappa) \Vdash \dot{P} \Vdash \tau$ is not E-related to any point in the $\mathrm{Coll}(\omega, <\kappa)$-extension.

Now, it follows that $\kappa(E) \geq \kappa$. By Theorem 4.1.4(2), since κ is a measurable cardinal, $\kappa(E) = \infty$. By Theorem 4.2.1, $E_{\omega_1} \leq_{\mathrm{wB}} E$ as desired.

Case 2. For every condition $q \in Q$, the $Q \restriction q$-name σ is not E-pinned. In this case, I will conclude that $F_2 \leq_{\mathrm{B}} E$. Then, a Shoenfield absoluteness argument shows that the Borel reduction of F_2 to E remains a Borel reduction also in the Solovay model.

Fix a countable elementary submodel M of a large structure containing the code for E, the posets P, Q and the name τ. I will start with an auxiliary lemma. A collection $\langle g_i : i \in I\rangle$ of filters on $\mathrm{Coll}(\omega, <\kappa)$ is called *mutually generic* over M if for every finite set $a \subset I$ the filter $\prod_{i\in a} g_i \subset \mathrm{Coll}(\omega, <\kappa)^{|a|}$ is generic over M. For every set $a \subset I$ write $2^\omega_a = \bigcup\{2^\omega \cap M[\prod_{i\in b} g_i] : b \subset a$ finite$\}$, $M_a = V(2^\omega_a)$, P_a and τ_a for the poset and name in M_a defined in the model $M(2^\omega_a)$ by the formulas ϕ_P and ϕ_τ. Similar usage will prevail for functions $y : \omega \to I$, writing $P_y = P_{\mathrm{rng}(y)}$ etc.

Lemma 3.4.3. *Suppose that $\{g_i : i \in I\}$ is a mutually generic collection of filters on* $\mathrm{Coll}(\omega, <\kappa)$*.*

1. *whenever $a \subset I$ is a nonempty set then in there is a filter h $\mathrm{Coll}(\omega, <\kappa)$-generic over M such that $2^\omega \cap V[h] = 2^\omega_a$;*
2. *whenever $a, b, c \subset I$ are pairwise disjoint countable nonempty sets then there is a filter $h_a \times h_b \times h_c \subset \mathrm{Coll}(\omega, <\kappa)^3$ generic over M such that $2^\omega_a = 2^\omega \cap V[h_a]$ and similarly for 2^ω_b and 2^ω_c;*
3. *whenever a, b are distinct countable subsets of I then $P_a \times P_b \Vdash \neg\tau_a \mathrel{E} \tau_b$.*

Proof. For (1), let $R = \{k : \exists\alpha \in \kappa\ \exists b \subset a$ finite $k \subset \mathrm{Coll}(\omega, <\alpha)$ is a filter generic over M and $k \in V[g_i : i \in b]\}$ and order R by inclusion. Let $K \subset R$ be a sufficiently generic filter; I claim that $h = \bigcup K$ works as desired. Indeed, a simple density argument shows that $h \subset \mathrm{Coll}(\omega, <\kappa)$ is an ultrafilter all of whose proper initial segments are generic over M. By the κ-c.c. of $\mathrm{Coll}(\omega, <\kappa)$, the filter h is in fact generic over V itself. A straightforward genericity argument then shows that $2^\omega_a = 2^\omega \cap V[h]$ as desired.

(2) follows easily from (1). Let $h_a, h_b, h_c \subset \mathrm{Coll}(\omega, <\kappa)$ be any filters obtained from (1); I will show that these filters are in fact mutually generic over the model V. Since $\mathrm{Coll}(\omega, <\kappa)^3$ has κ-c.c.c., it is enough to show that for every ordinal $\alpha \in \kappa$, the filters $h^\alpha_a = h_a \cap \mathrm{Coll}(\omega, <\alpha)$, $h^\alpha_b = h_b \cap \mathrm{Coll}(\omega, <\alpha)$, and $h^\alpha_c = h_c \cap \mathrm{Coll}(\omega, <\alpha)$ are mutually generic over V. Since the filters h^α_a, h^α_b and h^α_c are coded by reals in the models $M[h_a]$, $M[h_b]$, and $M[h_c]$,there are finite sets a', b', c' of a, b, c respectively such that $h^\alpha_a \in M[\prod_{i \in a'} g_i]$ etc. The mutual genericity now follows from the general Lemma 2.2.7 about product forcing.

(3) is proved in several parallel cases depending on the mutual position of the sets a, b vis-a-vis inclusion. I will treat the case in which all three sets $a \cap b$, $a \setminus b$, $b \setminus a$ are nonempty. Suppose for contradiction that $P_a \times P_b \Vdash \tau_a \mathrel{E} \tau_b$. From (2), it follows that in V, the triple product $\mathrm{Coll}(\omega, <\kappa)^3$ forces $\dot{P}_{\{0,1\}} \times \dot{P}_{\{1,2\}} \Vdash \dot{\tau}_{\{0,1\}} \mathrel{E} \dot{\tau}_{\{1,2\}}$. Then, the quadruple product $\mathrm{Coll}(\omega, <\kappa)^4$ forces in V that $\dot{P}_{\{0,1\}} \times \dot{P}_{\{1,2\}} \times \dot{P}_{\{2,3\}} \Vdash \tau_{\{0,1\}} \mathrel{E} \tau_{\{1,2\}} \mathrel{E} \tau_{\{2,3\}}$, in particular $\dot{P}_{\{0,1\}} \times \dot{P}_{\{2,3\}} \Vdash \tau_{\{0,1\}} \times \tau_{\{2,3\}}$. In view of (2) again, this means that the product $\mathrm{Coll}(\omega, <\kappa) \times \mathrm{Coll}(\omega, <\kappa)$ forces $\dot{P}_{\mathrm{left}} \times \dot{P}_{\mathrm{right}} \Vdash \tau_{\mathrm{left}} \mathrel{E} \tau_{\mathrm{right}}$. In other words, $(\mathrm{Coll}(\omega, <\kappa) * \dot{P}) \times (\mathrm{Coll}(\omega, <\kappa) * \dot{P})$ forces $\sigma_{\mathrm{left}} \mathrel{E} \sigma_{\mathrm{right}}$, contradicting the case assumption. □

Use Lemma 2.2.8 to find a continuous map $f : 2^\omega \to \mathcal{P}(\mathrm{Coll}(\omega, <\kappa) \cap M)$ such that its range consists of mutually generic filters over M. Write $Y = (2^\omega)^\omega = \mathrm{dom}(F_2)$. It is easy to find a Borel map $g : Y \to (2^\omega)^\omega$ such that for every $y \in Y$, $g(y)$ enumerates the set 2^ω_y. Use Lemmas 3.4.3(1) and 2.4.4 to find a Borel map $h : Y \to \mathcal{P}(Q \cap M)$ such that for every $y \in Y$, $h(y) \subset Q$ is a filter generic over M and $\mathrm{rng}(g(y)) = 2^\omega \cap V[h_0(y)]$, where $h_0(y) \subset \mathrm{Coll}(\omega, <\kappa)$

is the filter generic over M obtained from $h(y)$. Let $k : Y \to X$ be given by $k(y) = \tau/h(y)$; this is a Borel map by Lemma 2.4.3. I will show that k is a reduction of F_2 to E.

First, assume that $y_0, y_1 \in Y$ are F_2-related. Then $\mathrm{rng}(y_0) = \mathrm{rng}(y_1)$, $\mathrm{rng}(g(y_0)) = \mathrm{rng}(g(y_1))$, and so $M_{y_0} = M_{y_1}$, $P_{y_0} = P_{y_1}$ and $\tau_{y_0} = \tau_{y_1}$. Let $H \subset P_{y_0}$ be a filter generic over both countable models $M_{y_0}[k(y_0)]$ and $M_{y_1}[k(y_1)]$ and let $x = \tau_{y_0}/H$. By the forcing theorem applied in the model $M_{y_0} = M_{y_1}$ and the fact that τ_{y_0} is an E-pinned name, conclude that $x\ E\ k(y_0)$ and $x\ E\ k(y_1)$ and so $k(x_0)\ E\ k(x_1)$ as desired.

Second, assume that $y_0, y_1 \in Y$ are not F_2-related. Choose a sufficiently generic filter $H_0 \times H_1 \subset P_{y_0} \times P_{y_1}$ so that H_0 is generic over $M_{y_0}[k(y_0)]$ and H_1 is generic over $M_{y_1}[k(y_1)]$. As the names τ_{y_0} and τ_{y_1} are E-pinned, the forcing theorem in the models M_{y_0} and M_{y_1} implies that $k(y_0)\ E\ \tau_{y_0}/H_0$ and $k(y_1)\ E\ \tau_{y_1}/H_1$. Now, $\tau_{y_0}/H_0\ E\ \tau_{y_1}/H_1$ fails by Lemma 3.4.3(3), and so $k(y_0)\ E\ k(y_1)$ must fail as well. This completes the proof.

$\square$

Chapter 4

The pinned cardinal

4.1 Definitions and basic concerns

There is a cardinal number intrinsically connected with every (unpinned) equivalence relation on a Polish space. Its definition begins with an extension of the equivalence relation to the space of all pinned names.

Definition 4.1.1. Let E be an analytic equivalence relation on a Polish space X, and let τ, σ be E-pinned names on respective posets P, Q. Say that $\langle P, \tau\rangle \ \bar{E} \ \langle Q, \sigma\rangle$ holds if $P \times Q \Vdash \tau \ E \ \sigma$. In case when the orderings are clear from context, write also $\tau \ \bar{E} \ \sigma$.

It is not difficult to verify that $\bar{E}$ is an equivalence relation. If $\langle P, \tau\rangle \ \bar{E} \ \langle Q, \sigma\rangle$ and $\langle Q, \sigma\rangle \ \bar{E} \ \langle R, \chi\rangle$ then the product $P \times Q \times R$ forces $\tau \ E \ \sigma \ E \ \chi$ and so by the transitivity of E it forces $\tau \ E \ \chi$. By the Mostowski absoluteness between the $P \times Q \times R$ extension and $P \times R$ extension, $P \times R \Vdash \tau \ E \ \chi$ and so $\tau \ \bar{E} \ \chi$ as desired.

It turns out that it is interesting to count the number of $\bar{E}$-equivalence classes, or investigate other invariants of $\bar{E}$-equivalence classes. Already the simplest approach carries a lot of information:

Definition 4.1.2. Let E be an analytic equivalence relation on a Polish space X. The *pinned cardinal* of E, $\kappa(E)$ is the smallest cardinal κ such that every E-pinned name is $\bar{E}$-equivalent to a name on a poset of size $< \kappa$ if such κ exists; otherwise $\kappa(E) = \infty$. If E is pinned then write $\kappa(E) = \aleph_1$.

The last sentence of the definition is justified by the fact that there are no nontrivial pinned names on countable posets. A countable poset is certainly c.c.c., therefore reasonable, and Theorem 3.3.2(3) then excludes all nontrivial pinned names on it. Thus, an equivalence relation is unpinned if and only if $\kappa(E) \geq \aleph_2$.

The pinned cardinal of an equivalence relation can attain interesting values, from cardinals of the form $\aleph_\alpha$ for α a countable ordinal, to inaccessible and

Mahlo cardinals. It is also a potent tool for proving irreducibility results, as outlined in the following basic result.

Theorem 4.1.3. *Suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y. If $E \leq_{\rm wB} F$ then $\kappa(E) \leq \kappa(F)$.*

Proof. Let $a \subset X$ be a countable set and $h : X \to Y$ be a Borel function which is a reduction of E to F on the set $X \setminus [a]_E$. By the Shoenfield absoluteness, h maintains this property in every forcing extension. Let P be a poset and τ an E-pinned P-name; I must produce a poset Q of size $< \kappa(F)$ and a Q-name τ' $\bar{E}$-related to τ. Either $P \Vdash \tau \in [a]_E$. In this case, the name τ is E-trivial and there is nothing to prove. Or, $P \Vdash \tau \notin [a]_E$. Then, consider the P-name $\sigma = \dot{h}(\tau)$ for an element of the space Y. As h is a weak Borel reduction in the $P \times P$-extension, σ is an F-pinned name. Find a poset Q of size $< \kappa(F)$ and a Q-name σ' which is $\bar{F}$-related to σ. Since $P \times Q \Vdash \exists x \in X\ \dot{h}(x)\ F\ \sigma'$ (consider $x = \tau$), the Shoenfield absoluteness between the $P \times Q$-extension and Q-extension implies that $Q \Vdash \exists x \in X \setminus [a]_E\ \dot{h}(x)\ F\ \sigma'$. Let τ' be a Q-name for such x and observe that $\langle Q, \tau' \rangle\ \bar{E}\ \langle P, \tau \rangle$ as desired. □

An important feature of the pinned cardinal is that there are a priori bounds on its size:

Theorem 4.1.4. *Let E be an analytic equivalence relation on a Polish space X.*

1. *if E is Borel of rank $\mathbf{\Pi}^0_\alpha$, then $\kappa(E) \leq (\beth_\alpha)^+$;*
2. *if E is arbitrary analytic and $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first measurable cardinal.*

Proof. For (1), suppose that E is Borel of rank $\alpha \in \omega_1$ and let τ be a pinned P-name; I must produce a $\mathrm{Coll}(\omega, \beth_\alpha)$-name σ which is $\bar{E}$-related to τ. Note that $[\tau]_E$ is a P-name for a Borel set of rank $\leq \alpha$. As is the case for every name for a Borel set, [24, Corollary 2.9] shows that in the $\mathrm{Coll}(\omega, \beth_\alpha)$ extension $V[G]$ there is a Borel code for a Borel set $B \subset X$ such that in every further forcing extension $V[G][H]$ and every $x \in X \cap V[G][H]$ in that extension, $x \in B$ if and only if $V[x] \models P \Vdash \check{x} \in [\tau]_E$. Note that if $H \subset P$ is generic over $V[G]$, then the set B is nonempty in $V[G][H]$, containing the point τ/H; this follows from the fact that τ is E-pinned. Thus, the set B is nonempty already in $V[G]$ by the Mostowski absoluteness between $V[G]$ and $V[G][H]$. Back in V, let σ be any $\mathrm{Coll}(\omega, \beth_\alpha)$-name for an element of the set B. This clearly works.

For (2), let κ be a measurable cardinal and suppose that there is a poset P and an E-pinned name τ on P which is not $\bar{E}$-related to any name on a poset of size $< \kappa$. It will be enough to show that $\kappa(E) = \infty$ in this case.

First note that the poset P and the name τ can be selected so that $|P| = \kappa$. Simply take an elementary submodel M of size κ of large structure with $V_\kappa \subset M$ and consider $Q = P \cap M$ and $\sigma = \tau \cap M$. Since $|Q| = \kappa$, it will be enough to show that σ is an E-pinned name which is not $\bar{E}$-related to any name on a poset of size $< \kappa$. To see that σ is pinned, let $G \times H \subset Q \times Q$ be a generic filter over V. Then, it is also a generic filter over M, by the elementarity of M and the forcing theorem in M $M[G, H] \models \sigma/G\ E\ \sigma/H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, also $V[G, H] \models \sigma/G\ E\ \sigma/H$. This proves that σ is pinned. Now suppose that R is a poset of size $< \kappa$ and ν is an E-pinned name on R; without loss $R, \nu \in V_\kappa$. Let $G \times H \subset Q \times R$ be a generic filter over V. Then, as R is both an element and a subset of the model M, the filter is generic over M as well, and by the elementarity of M and the assumption that τ is not $\bar{E}$-related to ν, $M[G, H] \models \neg\sigma/G\ E\ \nu/H$. By the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$, $V[G, H] \models \neg\sigma/G\ E\ \nu/H$. This proves that σ is not $\bar{E}$-related to any name on a poset of size $< \kappa$ as desired.

Thus, assume that the poset P has size κ. Let $j : V \to N$ be any elementary embedding into a transitive model with critical point equal to κ. Note that $H(\kappa) \subset N$ and so both P, τ are (isomorphic to) elements of N. Let $\langle N_\alpha, j_{\beta\alpha} : \beta \in \alpha\rangle$ be the usual system of iteration of the elementary embedding j along the ordinal axis. Let $P_\alpha = j_{0\alpha}(P)$ and $\tau_\alpha = j_{0,\alpha}(\tau)$. It will be enough to show that no set meets all $\bar{E}$-classes, and for that it is enough to show that the pairs $\langle P_\alpha, \tau_\alpha\rangle$ for $\alpha \in \mathrm{Ord}$ are pairwise $\bar{E}$-unrelated.

Suppose that $\beta \in \alpha$ are ordinals and $G \times H \subset P_\beta \times P_\alpha$ is a generic filter over V. As initially $|P| = \kappa$, the poset P_β and the name τ_β are (isomorphic to) elements of the model N_α, the poset P_α is an element of M_α by the definitions, and so the filter $G \times H \subset P_\beta \times P_\alpha$ is also generic over N_α. By the elementarity of the embedding $j_{0\alpha}$, $N_\alpha \models \langle P_\alpha, \tau_\alpha\rangle$ is not $\bar{E}$-related to any name on a poset of size $< |j_{0\alpha}(\kappa)|$, in particular it is not $\bar{E}$-related to $\langle P_\beta, \tau_\beta\rangle$. It follows by the forcing theorem applied in N_α that $N_\alpha[G, H] \models \neg\tau_\beta/G\ \tau_\alpha\ /H$. By the Mostowski absoluteness between $N_\alpha[G, H]$ and $V[G, H]$, $V[G, H] \models \neg\tau_\beta/G\ \tau_\alpha\ /H$, and so $\langle P_\alpha, \tau_\alpha\rangle$ is not $\bar{E}$-related to $\langle P_\beta, \tau_\beta\rangle$ as desired. $\square$

The bound presented in the first item is nearly optimal, as shown by Corollary 4.4.7. One interesting corollary is that $\kappa(E) < \infty$ for every Borel equivalence relation E. Note that the definition of $\kappa(E)$ does not depend on the Polish topology chosen for the space X, and so we conclude that if the value of the cardinal $\kappa(E)$ is too large, then there is no topology on X inducing the same Borel structure that makes E into a $\mathbf{\Pi}^0_\alpha$ subset of $X \times X$. This connects the pinned cardinal with the subject of potential Borel classes studied by Lecomte [18].

Finally, the stage is ready for the first elementary computations of the cardinal $\kappa(E)$. Much more substantial examples will be presented in Section 4.4.

Example 4.1.5. $\kappa(F_2) = \mathfrak{c}^+$.

Proof. For the inequality $\kappa(F_2) \geq \mathfrak{c}^+$, consider the poset $P = \mathrm{Coll}(\omega, 2^\omega)$ and the name τ for the generic enumeration of $(2^\omega)^V$. It is clear that τ is an F_2-pinned name. Moreover, if $\langle Q, \sigma\rangle$ are a poset and a name $\bar{E}$-related to $\langle P, \sigma\rangle$, it follows that $Q \Vdash (2^\omega)^V$ is countable and therefore $|Q| \geq \mathfrak{c}$.

For the other inequality and later reference, I will prove a useful claim.

Claim 4.1.6. *Let P be a poset and τ a P-name for an element of* $\operatorname{dom}(F_2)$. *τ is F_2-pinned if and only if there is a set $A \subset 2^\omega$ such that $P \Vdash \operatorname{rng}(\tau) = \check{A}$.*

Proof. The right-to-left direction is immediate as $P \times P \Vdash \operatorname{rng}(\tau_{\text{left}}) = \operatorname{rng}(\tau_{rr}) = \check{A}$. For the left-to-right direction, to find the set A, first note that $P \Vdash \operatorname{rng}(\tau) \subset V$. If this failed then there would be some condition $p \in P$ and a P-name ν $p \Vdash \nu \in \operatorname{rng}(\tau) \setminus V$. If $G \times H \subset P \times P$ is a generic filter over V such that $p \in G$, then $\nu/G \in V[G] \setminus V[H]$ by the product forcing theorem, and so certainly $\operatorname{rng}(\tau/G) \neq \operatorname{rng}(\tau/H)$, contradicting the assumption that τ was a pinned name. Note also that for every point $x \in 2^\omega$, either $P \Vdash \check{x} \in \operatorname{rng}(\tau)$ or $P \Vdash \check{x} \notin \operatorname{rng}(\tau)$. If this failed, then there would be a point $x \in 2^\omega$ as well as conditions $p, q \in P$ such that $p \Vdash \check{x} \in \operatorname{rng}(\tau)$ and $q \Vdash \check{x} \notin \operatorname{rng}(\tau)$. In such a case, $\langle p, q\rangle \Vdash \operatorname{rng}(\tau_{\text{left}}) \neq \operatorname{rng}(\tau_{\text{right}})$ since the two sets differ in the membership of x. This again contradicts the assumption that τ is an F_2-pinned name. All summed up, the set $A = \{x \in 2^\omega : P \Vdash \check{x} \in \operatorname{rng}(\tau)\}$ works as desired. $\square$

Now suppose that P is a poset and τ is an F_2-pinned P-name. Let $A \subset 2^\omega$ be the set as in Claim 4.1.6. Certainly $\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$ where $Q = \operatorname{Coll}(\omega, A)$ and σ is the Q-name for the generic enumeration of the set A. Since $|A| \leq \mathfrak{c}$, I conclude $|Q| \leq \mathfrak{c}$ and also $\kappa(F_2) \leq \mathfrak{c}^+$.

$\square$

Example 4.1.7. Let E be the equivalence relation on the space X of all binary relations on ω connecting x, y if either both x, y are not wellfounded models of ZFC+there is no inaccessible cardinal or x is isomorphic to y. Then $\kappa(E)$ is the successor of the first weakly inaccessible cardinal.

Proof. Let κ be the first weakly inaccessible cardinal. To show that $\kappa(E) \geq \kappa^+$, consider the structure $M = \langle L_\kappa, \in\rangle$, the poset $P = \operatorname{Coll}(\omega, \kappa)$ and the P-name τ for any binary relation on ω isomorphic to M. Clearly, τ is an E-pinned name. Moreover, if $\langle Q, \sigma\rangle$ are a poset and a name $\bar{E}$-related to $\langle P, \tau\rangle$ then $Q \Vdash |M| = \kappa = \aleph_0$ and therefore $|Q| \geq \kappa$.

For the inequality $\kappa(E) \leq \kappa^+$, suppose that P is a poset and τ is an E-pinned name on P. If τ is not forced to be a wellfounded model of extensionality, then τ is forced to belong to the single nonwellfounded E-class which is already represented in the ground model, and therefore τ is trivial. Suppose that $P \Vdash \tau$ is wellfounded and extensional. I will prove that there is a (possibly uncountable) transitive set M in the ground model such that $P \Vdash \tau$ is isomorphic to $\langle M, \in\rangle$. Since M must be a model of ZFC+no inaccessible, it cannot contain the ordinal κ (which would be inaccessible in M) and therefore $|M| \leq \kappa$. Thus, $\langle P, \tau\rangle \bar{E} \langle Q, \sigma\rangle$ where $Q = \operatorname{Coll}(\omega, M)$ and σ is a Q-name for some binary relation on ω isomorphic to M. This will conclude the proof of $\kappa(E) \leq \kappa^+$.

To find the set M, use the Mostowski collapse theorem to find a P-name $\dot{N}$ for the unique transitive isomorph of τ. First argue that $P \Vdash \dot{N} \subset V$. Otherwise, by a $\in$-minimalization argument, there would be a condition $p \in P$ and a P-name ν such that $p \Vdash \nu \in \dot{N}$ and $\nu \subset V$. If $G \times H \subset P \times P$ is a generic filter over V such that $p \in G$, then $\nu/G \in V[G] \setminus V[H]$ by the product forcing theorem, and therefore τ/G is not isomorphic to τ/H since the transitive isomorphs of the two relations differ in the membership of ν/G. This contradicts the asumption that τ is an E-pinned name.

Thus, $P \Vdash \dot{N} \subset V$. I will now show that for every $x \in V$, $P \Vdash \check{x} \in \dot{N}$ or $P \Vdash \check{x} \notin \dot{N}$. Otherwise, there would be a set x and conditions $p, q \in P$ such that $p \Vdash \check{x} \in \dot{N}$ and $q \Vdash \check{x} \notin \dot{N}$. In such a case, $\langle p, q\rangle \Vdash \tau_{\text{left}}$ is not isomorphic to τ_{right} since their respective transitive isomorphs differ in the membership of x. This again contradicts the assumption that τ is an E-pinned name. Thus, $M = \{x \in V : P \Vdash \check{x} \in \dot{N}\}$ works as desired. □

Example 4.1.8. $\kappa(E_{\omega_1}) = \infty$.

Proof. For any cardinal κ consider the poset $P = \mathrm{Coll}(\omega, \kappa)$ and a P-name for some binary relation on ω isomorphic to the ordinal ordering on κ. Clearly, τ is an E_{ω_1}-pinned name. If $\langle Q, \sigma\rangle$ are a poset and a name $\bar{E}$-related to $\langle P, \tau\rangle$, then $Q \Vdash |\kappa| = \aleph_0$ and so $|Q| \geq \kappa$. □

4.2 A characterization

It seems to be quite difficult to characterize those analytic equivalence relations E such that $\kappa(E)$ attains a prescribed set of cardinal values (say $\kappa(E) \geq \mathfrak{c}^+$ or $\kappa(E) \leq \mathfrak{c}^+$) in descriptive set theoretic terms. It is a priori even not clear if the collections of equivalence relations defined in this way are say in $L(\mathbb{R})$. In this section, I will provide such a characterization for the class of analytic equivalence relations with the largest possible value of the pinned cardinal.

Theorem 4.2.1. *Assume that there is a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. $\kappa(E) = \infty$*;*
2. $E_{\omega_1} \leq_{\mathrm{wB}} E$*.*

Proof. (2) implies (1) by Example 4.1.8 and Theorem 4.1.3. The large cardinal assumption is not needed for this direction. For the (1)→(2) implication, suppose that $\kappa(E) = \infty$. Let κ be a measurable cardinal.

Claim 4.2.2. *There is a poset P of size κ and an E-pinned P-name τ such that τ is not $\bar{E}$-related to any name on a poset of size $< \kappa$.*

Proof. Since $\kappa(E) = \infty$, there is a poset Q and an E-pinned P-name σ which is not $\bar{E}$-equivalent to any name on a poset of size $< \kappa$. Choose an elementary submodel M of a large enough structure such that $|M| = \kappa$, $V_\kappa \subset M$ and $Q, \sigma \in M$, and let $P = Q \cap M$ and let $\tau = \sigma \cap M$. I claim that P, τ works as in the claim.

Indeed, if $G \times H \subset P \times P$ is a filter generic over V, then it is also generic over M, by the elementarity of M and the forcing theorem in M $M[G, H] \models \tau/G\ E\ \tau/H$, and by the Mostowski absoluteness between $M[G, H]$ and $V[G, H]$,

$V[G,H] \models \tau/G\ E\ \tau/H$. This proves that the name τ is E-pinned. If R is a poset in V of size $< \kappa$ and ν is an E-pinned R-name and $G \times H \subset P \times R$ is a generic filter over V, then $R, \nu \in M$, the filter $G \times H$ also generic over M, by the elementarity of M and the forcing theorem in M $M[G,H] \models \neg\tau/G\ E\ \nu/H$, and by the Mostowski absoluteness between $M[G,H]$ and $V[G,H]$, $V[G,H] \models \neg\tau/G\ E\ \sigma/H$. This proves that $\nu\ \bar{E}\ \tau$ fails and completes the proof of the claim. $\square$

Choose a poset P of size κ and an E-pinned name τ as in the claim. Let M be a countable elementary submodel of a large enough structure. Let Y be the space of binary relations on ω, so $Y = \mathrm{dom}(E_{\omega_1})$. By Lemma 2.4.6 and 2.4.2, there are Borel functions $f : Y \to Y$, $g : Y \times M \to \omega$, $h : Y \to \mathcal{P}(\omega)$ and $k : Y \to X$ such that whenever $y \in Y$ is a wellorder then $f(y)$ is an isomorph of the iteration of the model M of length y, $g(y)$ is the iteration elementary embedding of M into $f(y)$, $h(y)$ is a filter on $g(y)(P)$ generic over $f(y)$, and $k(y) = g(y)(\tau)/h(y)$. It will be enough to show that k is a Borel reduction of E_{ω_1} to E on the set of $y \in Y$ which code well-orders.

Suppose first that $y, z \in Y$ are well-orders of the same length. Then $f(y), f(z)$ are wellfounded and isomorphic. Write N for their common transitive isomorph, $j : M \to N$ for the iteration map, and let $Q = j(P)$ and $\sigma = j(\tau)$. By the elementarity of the embedding j, $N \models \sigma$ is an E-pinned Q-name. Identify $h(y), h(z)$ with filters on Q separately generic over N, so $k(y) = \sigma/h(y)$ and $k(z) = \sigma/h(z)$. Let $h' \subset Q$ be a filter generic over both countable models $N[h(y)]$ and $N[h(z)]$. By the product forcing theorem, the filters $h' \times h(y)$ and $h' \times h(z)$ are both $Q \times Q$-generic over N. As $N \models \tau$ is an E-pinned name, it follows that $N[h', h(y)] \models k(y) = \sigma/h(y)\ E\ \sigma/h'$ and $N[h', h(z)] \models k(z) = \sigma/h(z)\ E\ \sigma/h'$. By the Mostowski absoluteness between these two models and V, and the transitivity of the relation E, $k(y)\ E\ k(z)$ follows.

Suppose now that $y, z \in Y$ are well-orders of different lengths; say that y is shorter than z. Let N_y be the transitive isomorph of $f(y)$, $j_y : M \to N_y$ the iteration map, $Q_y = j_y(P)$, $\sigma_y = j_y(\tau)$; similarly for N_z, j_z, Q_z, σ_y. Identify $h(y), h(z)$ with filters on Q_y, Q_z separately generic over N_y, N_z, so $k(y) = \sigma/h(y)$ and $k(z) = \sigma/h(z)$. Now, as $P, \tau \in H_{\kappa^+}$, it is the case that $Q_y, \sigma_y \in N_z$ and $N_z \models |Q_y| < j_z(\kappa)$. Find a filter $h' \subset Q_y$ generic over both the countable models $N_y[h(y)]$ and N_z, and let $h'' \subset Q_z$ be a filter generic over both the countable models $N_z[h']$ and $N_z[h(z)]$. Let $x' = \sigma_y/h'$ and $x'' = \sigma_z/h''$, both elements of the space X. Since $N_y \models \sigma_y$ is E-pinned, the Mostowski absoluteness between V and $N_y[h(y), h']$ implies that $k(y)\ E\ x'$ holds. Since $N_z \models \sigma_z$ is E-pinned, the Mostowski absoluteness between V and $N_y[h(z), h'']$ implies that $k(z)\ E\ x''$ holds. Finally, since $N_z \models \langle Q_z, \tau_z\rangle\ \bar{E}\ \langle Q_y, \tau_y\rangle$ fails by the choice of P and the elementarity of the embedding j_z, the Mostowski absoluteness between V and $N_z[h', h'']$ implies that $x'\ E\ x''$ fails. In conclusion $k(y)\ E\ k(z)$ fails as required. $\square$

4.3 Operations

In this section, I will prove that the pinned cardinal behaves naturally under the usual operations on analytic equivalence relations such as the various products and the Friedman–Stanley jump.

Theorem 4.3.1. *Suppose that J is a Borel ideal on ω such that the equivalence relation $=_J$ is pinned. Let $\langle E_i : i \in \omega\rangle$ be a sequence of analytic equivalence relations on respective Polish spaces X_i, and let E be the product of E_i modulo J. Then $\kappa(E) \leq (\sup_i \kappa(E_i))^+$.*

Theorem 4.3.2. *Let E be an analytic equivalence relation on a Polish space X. Then $\kappa(E^+) \leq (2^{<\kappa(E)})^+$.*

To package the proofs in a most efficient way, the following technical lemma will be useful.

Lemma 4.3.3. *Let E be an analytic equivalence relation on a Polish space X. Let κ be an infinite cardinal. The following are equivalent:*

1. $\kappa^+ \geq \kappa(E)$;
2. $\mathrm{Coll}(\omega, \kappa) \Vdash$ *if τ is an E-pinned name on a poset P, both in the ground model, then τ is E-trivial.*

Proof. For (1)→(2), suppose that $\kappa^+ \geq \kappa(E)$. Every $\bar{E}$-class then contains a name on the poset $\mathrm{Coll}(\omega, \kappa)$: each E-pinned name τ on some poset P is $\bar{E}$-equivalent to a name on a poset of size some $\kappa(E)$. That poset regularly embeds into $\mathrm{Coll}(\omega, \kappa)$ by Fact 2.2.3, and so τ has a $\bar{E}$-equivalent σ on $\mathrm{Coll}(\omega, \kappa)$. Now, $P \times \mathrm{Coll}(\omega, \kappa) \Vdash \tau\ E\ \sigma$, which by the forcing theorem can be rewritten as $\mathrm{Coll}(\omega, \kappa) \Vdash P \Vdash \tau\ E\ \sigma$. This confirms (2).

Suppose on the other hand that (2) holds. This means that for every E-pinned name τ on a poset P there is a $\mathrm{Coll}(\omega, \kappa)$-name σ such that $\mathrm{Coll}(\omega, \kappa) \Vdash P \Vdash \sigma\ E\ \tau$. By the product forcing theorem, this means that $P \times \mathrm{Coll}(\omega, \kappa) \Vdash \tau\ E\ \sigma$, in other words $\langle \mathrm{Coll}(\omega, \kappa), \sigma\rangle\ \bar{E}\ \langle P, \tau\rangle$. This confirms (1). □

Proof of Theorem 4.3.1. Let $X = \prod_i X_i$ be the domain of E. Let P be a poset and τ an E-pinned name on P, both in V. Consider the P-name $\dot{a} = \{i \in \omega : \exists p$ in the P-generic filter such that $\tau(i)$ is an E_i-pinned name on $P \restriction p\}$.

Claim 4.3.4. *$P \Vdash \omega \setminus \dot{a} \in J$.*

Proof. Suppose the opposite is forced by some condition $p \in P$. As τ is E-pinned, there must be conditions $p_0, p_1 \leq p$ and a number $i \in \omega$ such that $\langle p_0, p_1\rangle \Vdash_{P\times P} \tau_{\mathrm{left}}(i)\ E_i\ \tau_{\mathrm{right}}(i)$ and $i \notin \dot{a}_{\mathrm{left}}$. Then, however, $\tau(i)$ is E-pinned on $P \restriction p_0$ and so $p_0 \Vdash_P i \in \dot{a}$. This is a contradiction. □

Now, let $\kappa = \sup_i \kappa(E_i)$, let $G \subset \mathrm{Coll}(\omega, \kappa)$ be a filter generic over V. By Lemma 4.3.3 it will be enough to produce an element $x \in X$ in $V[G]$ such that $P \Vdash \check{x}\ E\ \tau$. Work in $V[G]$. Let $f_i : 2^\omega \to X_i/E_i$ be a bijection for each $i \in \omega$, and consider the P-name ν for an element of $(2^\omega)^\omega$ defined by the following formula: if there is $p \in P$ in the generic filter on P such that $\tau(i)$ is an E_i-pinned name on $P \restriction p$, then by the application of Lemma 4.3.3 to E_i there must be a point $x_i \in X_i$ such that $P \restriction p \Vdash \check{x}_i\ E_i\ \tau(i)$ and then let $\nu(i)$ be the point y such that $x_i \in f_i(y)$; otherwise let $\nu(i) = \text{trash}$. Claim 4.3.4 shows that ν is an $=_J^{2^\omega}$-pinned name on P and the set $\{i \in \omega : \nu(i) = \text{trash}\} \in J$.

Since the relation $=_J^{2^\omega}$ is pinned in V, it is pinned also in $V[G]$ by Corollary 3.2.6 and so there must be a point $y \in (2^\omega)^\omega$ in $V[G]$ such that $P \Vdash \nu =_J \check{y}$. Then, let $x \in X$ be any point such that for every $i \in \omega$, $x(i) \in f_i(y(i))$ and observe that $P \Vdash \check{x}\ E\ \tau$. This completes the proof. ☐

Proof of Theorem 4.3.2. Let τ be an E^+-pinned name on a poset P. Let $\kappa = 2^{<\kappa(E)}$ and let $G \subset \mathrm{Coll}(\omega, \kappa)$ be a generic filter. By Lemma 4.3.3, it will be enough to produce a point $y \in X^\omega$ in the model $V[G]$ such that $P \Vdash \check{y}\ E^+\ \tau$.

Back in the ground model, consider the set $A = \{\langle \sigma, \mu \rangle : \mu < \kappa(E), \sigma$ is a nice E-pinned $\mathrm{Coll}(\omega, \mu)$-name and $\mathrm{Coll}(\omega, \mu) \times P \Vdash \exists i\ \sigma\ E\ \tau(i)\}$. A simple counting argument shows that the set A has size $2^{<\kappa(E)}$. In $V[G]$, for every $\mu < \kappa(E)$ there is a filter $H_\mu \subset \mathrm{Coll}(\omega, \mu)$ generic over V. Let $y \in X^\omega$ be any point which enumerates the set $\{\sigma/H_\mu : \langle \sigma, \mu \rangle \in A\}$. Note that this set is countable in $V[G]$. I claim that the point $y \in X^\omega$ works as required.

The definition of the set A together with Fact 2.2.6 implies that $P \Vdash \forall j \exists i\ y(j)\ E\ \tau(i)$. All that remains to be shown is that $P \Vdash \forall j \exists i\ y(i)\ E\ \tau(j)$. Suppose for contradiction that some condition $p \in P$ forces this to fail as witnessed by some specific number $j \in \omega$. As τ is E^+-pinned, there are conditions $p_0, p_1 \le p$ and a number $k \in \omega$ such that $\langle p_0, p_1 \rangle \Vdash_{P \times P} \tau_{\text{left}}(j)\ E\ \tau_{\text{right}}(k)$. Then, $\tau(j)$ is an E-pinned name on $P \restriction p_0$. By the definition of $\kappa(E)$, there is a cardinal $\mu < \kappa(E)$ and an E-pinned name σ such that $\langle \mathrm{Coll}(\omega, \mu), \sigma \rangle\ E\ \langle P \restriction p_0, \tau \rangle$. This means that $\langle p, 1 \rangle \Vdash_{P \times \mathrm{Coll}(\omega, \mu)} \exists i\ \sigma\ E\ \tau(i)$, and since τ is an E^+-pinned name, this implies that $P \times \mathrm{Coll}(\omega, \mu) \Vdash \exists i\ \sigma\ E\ \tau(i)$. In other words, $\langle \sigma, \mu \rangle \in A$ and $p_0 \Vdash \tau(j)\ \bar{E}\ \sigma/H_\mu$, and therefore $p_0 \Vdash \exists i\ \tau(j)\ E\ \check{y}(i)$. This contradicts the choice of the condition p and the number j. ☐

4.4 Model-theoretic examples

In this section, I will provide a number of orbit equivalences E for actions of closed subgroups of S_∞ with interesting values of $\kappa(E)$.

Definition 4.4.1. Let ψ be an $L_{\omega_1\omega}$-sentence. E_ψ is the the equivalence relation of isomorphism of models of ψ with universe ω.

I do not know how to evaluate the cardinal $\kappa(E_\psi)$ for a general $L_{\omega_1\omega}$ sentence ψ. However, in a rather broad class of sentences, such an evaluation is readily available.

Definition 4.4.2. Let ψ be an $L_{\omega_1\omega}$ sentence. Say that ψ is *set-like* if there is a symbol R for a binary relation in the language of ψ and a countable ordinal $\alpha \in \omega_1$ such that ψ implies the statement "the relation R satisfies the axiom of extensionality, and it is well-founded of rank $< \alpha$".

Note that wellfoundedness of at most a fixed countable rank is expressible in the language $L_{\omega_1\omega}$.

Theorem 4.4.3. *Suppose that ψ is a set-like $L_{\omega_1\omega}$ sentence. Then E_ψ is Borel, and $\kappa(E_\psi)$ is the least uncountable cardinal κ such that ψ has no model of size κ.*

Proof. The argument depends on the transitive collapse theorem of Mostowski [10, Theorem 6.15]: a wellfounded extensional relation is isomorphic to the $\in$-relation on some transitive set, and both the transitive set and the isomorphism are unique. In particular, if two well-founded extensional relations are isomorphic, the isomorphism must be unique. Thus, the equivalence relation E_ψ is the one-to-one projection of the Borel set $\{\langle x, y, \pi\rangle : x, y \in X, \pi \in \omega^\omega$ and π is an isomorphism of $x, y\}$, and as such is Borel by a classical result of Lusin and Suslin [14, Theorem 15.1].

For the computation of $\kappa(E_\psi)$, find a binary relational symbol R witnessing that ψ is set-like.

Lemma 4.4.4. *If P is a poset and τ an E_ψ-pinned name, then there is a model $M \models \psi$ such that $P \Vdash \tau$ is isomorphic to $\check{M}$.*

Proof. Let $\langle \dot{N}, \in\rangle$ be the P-name for the transitive isomorph of $\langle \omega, R^\tau\rangle$.

Claim 4.4.5. *There is a set $M \in V$ such that $P \Vdash \dot{N} = \check{M}$.*

Proof. First argue that $P \Vdash \dot{N} \subset V$. Otherwise, by a $\in$-minimalization argument, there would be a condition $p \in P$ and a P-name ν such that $p \Vdash \nu \in \dot{N}$ and $\nu \subset V$. If $G \times H \subset P \times P$ is a generic filter over V such that $p \in G$, then $\nu/G \in V[G] \setminus V[H]$ by the product forcing theorem, and therefore τ/G is not isomorphic to τ/H since the transitive isomorphs of the two differ in the membership of ν/G. This contradicts the asumption that τ is an E-pinned name.

Thus, $P \Vdash \dot{N} \subset V$. I will now show that for every $x \in V$, $P \Vdash \check{x} \in \dot{N}$ or $P \Vdash \check{x} \notin \dot{N}$. Otherwise, there would be a set x and conditions $p, q \in P$ such that $p \Vdash \check{x} \in \dot{N}$ and $q \Vdash \check{x} \notin \dot{N}$. In such a case, $\langle p, q\rangle \Vdash \tau_{\text{left}}$ is not isomorphic to τ_{right} since their respective transitive isomorphs differ in the membership of x. This again contradicts the assumption that τ is an E-pinned name. Thus, $M = \{x \in V : P \Vdash \check{x} \in \dot{N}\}$ works as desired. $\square$

I will now equip the transitive set M with relations that will turn it into a model of ψ. Clearly, $R^M = \in$. For the other relations, first let $\dot{\pi} : \check{M} \to \omega$ be a P-name for inverse of the unique Mostowski transitive collapse map between $\langle \omega, R^\tau\rangle$ and $\langle M, \in\rangle$.

Claim 4.4.6. *Let S be an n-ary relational symbol in the language for the sentence ψ and let $v \in M^n$ be an n-tuple. Then $P \Vdash \pi \circ v \in S^\tau$ or $P \Vdash \pi \circ v \notin S^\tau$.*

Proof. If this failed, there would be conditions $p, q \in P$ such that $p \Vdash \pi \circ v \in S^\tau$ and $q \Vdash \pi \circ v \notin S^\tau$. Then the condition $\langle p, q \rangle$ in the product $P \times P$ forces that τ_{left} and τ_{right} are not isomorphic: the map $\pi_{\text{left}} \circ \pi_{\text{right}}^{-1}$ is the only candidate for an isomorphism between τ_{right} and τ_{left}, and it carries the tuple $\pi_{\text{right}} \circ v$ (which is not in $S^{\tau_{\text{right}}}$) to $\pi_{\text{left}} \circ v$ (which is in $S^{\tau_{\text{left}}}$). This is a contradiction. □

Similar claim clearly holds for all functional symbols of the language of ψ. Now for every relational symbol S let $S^M = \{v : P \Vdash \pi \circ v \in S^\tau\}$, and for every functional symbol F let $F^M(v) = m$ if $P \Vdash F^\tau(\pi \circ v) = \pi(m)$. The claim immediately implies that $P \Vdash \pi$ is an isomorphism of the model M with τ as desired.

□

The theorem now easily follows. Suppose that κ is an uncountable cardinal. If ψ has a model M of size κ then consider the poset $P = \text{Coll}(\omega, \kappa)$ and a P-name τ for some isomorph of M with universe ω. Clearly, this is an E_ψ-pinned name. If Q is a poset and σ is a Q-name such that $\langle P, \tau \rangle \; \bar{E}_\psi \; \langle Q, \sigma \rangle$, then $Q \Vdash$ the transitive isomorph of R_σ must be equal to the transitive isomorph of R^M. Since the universe of M has size κ, this means that $Q \Vdash |\kappa| = \aleph_0$ and so $|Q| \geq \kappa$. Ergo, $\kappa < \kappa(E_\psi)$.

On the other hand, suppose that ψ has no model of size κ. By a downward Löwenheim–Skolem argument, every model of ψ has size $< \kappa$. Let P be a poset and τ be an E_ψ-pinned name. By the claim, there is a model M of ψ such that $P \Vdash \tau$ is isomorphic to $\check{M}$. Certainly, the pair $\langle P, \tau \rangle$ is $\bar{E}_\psi$ related to the pair $\langle Q, \sigma \rangle$, where $Q = \text{Coll}(\omega, M)$ and σ is some Q-name for an isomorph of M with universe ω. Since $|M| < \kappa$, $|Q|$ must be smaller than κ and $\kappa(E) \leq \kappa$ follows.

□

Corollary 4.4.7. *For every countable ordinal α there is a Borel equivalence relation E_α such that $\kappa(E_\alpha) = (\beth_\alpha)^+$.*

Proof. Let ψ_α be an $L_{\omega_1\omega}$ sentence in the language with one binary relational symbol R saying "R satisfies the axiom of extensionality and has rank $\leq \omega + \alpha$". ψ_α is clearly set-like. For every model M of ψ_α, its unique transitive isomorph is a subset of $V_{\omega+\alpha}$ and therefore has size $\leq \beth_\alpha$. At the same time, $\langle V_{\omega+\alpha}, \in \rangle$ is a model of ψ_α of size $\beth_\alpha$. Theorem 4.4.3 then shows that $\kappa(E_{\psi_\alpha}) = (\beth_\alpha)^+$ as desired. □

Corollary 4.4.8. *For every countable ordinal $\alpha > 0$ there is a Borel equivalence relation E_α such that (provably) $\kappa(E_\alpha) = \aleph_\alpha$. For $\alpha > 2$, such an equivalence relation E_α cannot be Borel reducible to F_2 and F_2 cannot be Borel reducible to E_α.*

Proof. First argue that for every countable ordinal $\alpha > 0$ there is an $L_{\omega_1\omega}$ sentence ϕ_α which has models of all infinite cardinalities $< \aleph_\alpha$ but no model of size $\aleph_\alpha$. The proof goes by induction on α.

For $\alpha = 1$ just let ϕ_α be any sentence which describes the natural ordering on ω. For a limit ordinal α let ϕ_α be the disjunction of ϕ_β for $\beta \in \alpha$. For a successor ordinal $\alpha = \beta + 1$ distinguish the case of β limit or β successor. If β is limit, then let the language of ϕ_α contain new unary predicates A_γ for $\gamma \in \beta$ and let ϕ_α say "the predicates A_γ for $\gamma \in \beta$ partition the universe and $A_\gamma \models \phi_\beta$". If $\beta = \gamma + 1$ is a successor ordinal, then let the language of ϕ_α contain new unary predicates A, B, a binary predicate $<$ and a binary functional symbol F and let ϕ_α say "the predicates A, B partition the universe, $A \models \phi_\beta$, $B \models <$ is a linear order, and $F : A \times B \to B$ is a function such that for every $x \in B$, the initial segment of $<$ up to x is a subset of the range of $F(\cdot, x)$". This clearly works. For example, in the latter case, in any model M of ϕ_α, the predicate A^M has size at most $\aleph_\gamma$, and the predicate B^M has a linear order on it whose proper initial segments have size $\leq |A^M|$, so $|B^M| \leq \aleph_{\gamma+1}$ as desired.

Now, for every ordinal $\alpha > 0$ let ψ_α be the sentence in the language of ϕ_α together with a new binary relational symbol R which says "ϕ_α holds and R is a relation satisfying the axiom of extensionality, which is wellfounded of rank $\leq \omega + \alpha$". Clearly, ψ_α is set-like. Moreover, every model of ϕ_α can be equipped with an additional relation R with which it becomes a model of ψ_α, since $|M| < \aleph_\alpha \leq \beth_\alpha = |V_{\omega+\alpha}|$ and the $\in$-relation on V_α is extensional and wellfounded of rank $\omega + \alpha$. Thus, the sentence ψ_α has models of all infinite cardinalities $< \aleph_\alpha$. Theorem 4.4.3 now says that $E_{\psi_\alpha} = \aleph_\alpha$.

To see that F_2 cannot be Borel reducible to any E_α, suppose for contradiction that $h : \mathrm{dom}(E) \to \mathrm{dom}(F)$ is a Borel reduction. Pass to a generic extension in which $\mathfrak{c} > \aleph_{\omega_1}$. There, h is still a reduction of E to F, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 4.1.3. To see that E_α cannot be reducible to F_2 for any $\alpha > 2$, pass to a generic extension in which the Continuum Hypothesis holds instead. □

To motivate the next corollary, recall a rare guest indeed in a book devoted to descriptive set theory.

Definition 4.4.9. [10, page 58] The *Singular Cardinal Hypothesis* is the statement that for every singular cardinal κ, if $2^{\mathtt{cof}(\kappa)} < \kappa$ then $\kappa^{\mathtt{cof}(\kappa)} = \kappa^+$.

The powerset of singular cardinals is much harder to manipulate than the powerset of regular cardinals. As a result, the status of the singular cardinal hypothesis remained open long after similar questions for regular cardinals were resolved. In a major breakthrough, Magidor [19] proved that the singular cardinal hypothesis can fail at $\aleph_\omega$. The following corollary shows that Magidor's result can be injected into the Borel reducibility ordering of Borel equivalence relations classifiable by countable structures.

Corollary 4.4.10. *There are Borel equivalence relations E, F such that (provably) $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$ and $\kappa(F) = \max\{\mathfrak{c}, \aleph_{\omega+1}\}^+$. Under suitable large cardinal hypothesis, E cannot be Borel reducible to F.*

Proof. For E, the first step is the construction of an $L_{\omega_1\omega}$ sentence which has models of size $\aleph_\omega^{\aleph_0}$ but no larger. First, let χ be an $L_{\omega_1\omega}$ sentence that has models of size $\aleph_\omega$ but not any larger, as obtained in the previous example. Let ϕ be a sentence in the language of χ with additional unary predicates A, B, C and a binary functionaly symbol f. ϕ will say: "the predicates A, B, C partition the universe, $A \models \chi$, B is ordered in type ω, and $f : B \times C \to A$ is a function such that for distinct $i \neq j \in C$ the sets $f''(\cdot, i)$ and $f''(\cdot, j)$ are distinct". The sentence ϕ clearly works as desired.

Let ψ be a sentence in the language of ϕ with an additional binary relational symbol R which says "ϕ holds and the relation R satisfies the axiom of extensionality and it is well-founded with rank $< \omega + \omega + 2$". The sentence ψ is clearly set-like, and it has models of size $\aleph_\omega^{\aleph_0}$ but no larger. Theorem 4.4.3 now shows that $\kappa(E_\psi) = (\aleph_\omega^{\aleph_0})^+$ as desired.

For F, just use the previous two examples to find set-like sentences ϕ_0, ϕ_1 such that ϕ_0 has models of size $\mathfrak{c}$ but no larger, and ϕ_1 has models of size $\aleph_{\omega+1}$ but no larger. It is not difficult to see that $F = E_{\phi_0 \vee \phi_1}$ works as required.

To see that E cannot be Borel reducible to F, suppose for contradiction that $h : \mathrm{dom}(E) \to \mathrm{dom}(F)$ is a Borel reduction. Use a classical result of Magidor [19] and pass to a generic extension in which the Singular Cardinals Hypothesis fails at $\aleph_\omega$: $\aleph_\omega^{\aleph_0} > \max(\mathfrak{c}, \aleph_{\omega+1})$. There, h is still a reduction of E to F, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 4.1.3. $\square$

Thus, one can (oh horror!) encode the status of the Singular Cardinal Hypothesis at $\aleph_\omega$ into the value of cardinal invariants of Borel equivalence relations E, F which are even classifiable by countable structures, and turn the proof of independence of SCH into a proof of Borel nonreducibility of E to F.

4.5 Combinatorial examples

Not all examples of equivalence relations with exotic values of $\kappa(E)$ are related to model theory. There are Borel equivalence relations reducible to F_2 whose pinned cardinal depends on the fine partition properties of small uncountable cardinals. Some of them may be naturally defined from common objects of interest in mathematical analysis.

Definition 4.5.1. Let $X = (2^\omega)^\omega$. For every natural number $n > 1$, let $B_n \subset X$ be the Borel set of all $x \in X$ such that for every n-tuple $\langle x_i : i \in n \rangle$ of distinct elements of $\mathrm{rng}(x)$ there is $i \in n$ such that x_i is recursive in $\langle x_j : j \in n, j \neq i \rangle$.

Theorem 4.5.2. *Let $n > 1$. If Martin's Axiom holds and $\mathfrak{c} > \aleph_n$, then $\kappa(F_2 \restriction B_n) = \aleph_n$.*

Proof. The argument is based on a classical partition theorem of Sierpiński. Recall that if A is a set, $n \in \omega$ is a number and $f : [A]^n \to \mathcal{P}(A)$ is a function, then a set $a \subset A$ is *free* for f if for every $b \in [a]^n$ and $i \in a \setminus b$, $i \notin f(b)$.

Fact 4.5.3. (Sierpiński [8]) *Let $n > 1$ be a number.*

1. *Every function* $f : [\omega_n]^{n-1} \to [\omega_n]^{\aleph_0}$ *has a free* n*-tuple.*

2. *There is a function* $f : [\omega_{n-1}]^{n-1} \to [\omega_{n-1}]^{\aleph_0}$ *without a free* n*-tuple.*

To see that $\kappa(F_2 \restriction B_n) \leq \aleph_n$ holds in ZFC, suppose that P is a poset and τ is an F_2-pinned P-name for an element of the set B_n. Claim 4.1.6 yields a set $A \subset 2^\omega$ such that $P \Vdash \mathrm{rng}(\tau) = \check{A}$. Consider the map $f : [A]^{n-1} \to [A]^{\aleph_0}$ given by $f(b) = \{y \in A : y \text{ is recursive in } b)\}$. The map f cannot have a free set of size n, as this would contradict the definition of the set B_n and the assumption that $P \Vdash \tau \in \dot{B}_n$. By Fact 4.5.3(1), it must be the case that $|A| < \aleph_n$. Now, since $\langle P, \tau \rangle$ is $\bar{F}_2$-equivalent to the pair $\langle \mathrm{Coll}(\omega, A), \sigma \rangle$, where σ is the usual $\mathrm{Coll}(\omega, A)$-name for the generic enumeration of the set A, and $|\mathrm{Coll}(\omega, A)| < \aleph_n$, it follows that $\kappa(F_2 \restriction B_n) \leq \aleph_n$.

To see that $\kappa(F_2 \restriction B_n) = \aleph_n$ under MA+$\mathfrak{c} \geq \aleph_n$, I will need a simple ZFC coding lemma.

Lemma 4.5.4. *Let* $n \in \omega$ *be a nonzero natural number, let* κ *be a cardinal and* $f : [\kappa]^n \to [\kappa]^{\aleph_0}$ *any function. Then there is a c.c.c. forcing adding an injection* $\pi : \kappa \to \mathcal{P}(\omega)$ *such that for every set* $a \in [\kappa]^n$ *and every* $\alpha \in f(a)$*,* $\pi(\alpha)$ *is recursive in the set* $\pi''a$*.*

Proof. Let $\langle k_m : m \in \omega \rangle$ be a recursive sequence of increasing functions in ω^ω with disjoint ranges. For a finite set $b \subset \mathcal{P}(\omega)$ let e_b be the increasing enumeration of the set $\bigcap b$, for every $m \in \omega$ let $h_m(b) \subset \mathcal{P}(\omega)$ be the set of all l such that $e_b \circ k_m(l)$ is an odd number. I will produce the map π such that for every set $a \in [\kappa]^n$ and every $\alpha \in f(a)$, there is a number $m \in \omega$ such that $\pi(\alpha)$ is modulo finite equal to $h_m(\pi''a)$.

Let R be the poset of all tuples $r = \langle n_r, \pi_r, \nu_r \rangle$ so that

- $n_r \in \omega$, π_r is a partial function from κ to $\mathcal{P}(n_r)$ with finite domain $\mathrm{dom}(r)$;
- ν_r is a finite partial function from $[\mathrm{rng}(r)]^n \times \omega$ to $\mathrm{rng}(r)$ such that $\nu_r(a, m) \in f(a)$ whenever $\langle a, m \rangle \in \mathrm{dom}(\nu_r)$.

The ordering on R is defined by $s \leq r$ if $n_r \leq n_s$, $\mathrm{dom}(r) \subset \mathrm{dom}(s)$, $\forall \alpha \in \mathrm{dom}(r)\ \pi_r(\alpha) = \pi_s(\alpha) \cap n_r$, $\nu_r \subset \nu_s$, and for every $\langle a, m \rangle \in \mathrm{dom}(\nu_r)$, whenever l is a number in the domain of $(e_{\pi_s''a} \setminus e_{\pi_r''a}) \circ k_m$ then $e_{\pi_s''a} \circ k_m(l)$ is odd if and only if $l \in \pi_s(\nu_r(a, m))$. It is not difficult to see that R is indeed an ordering. If $G \subset R$ is a generic filter, in the model $V[G]$ define the map $\pi : \kappa \to \mathcal{P}(\omega)$ by setting $\pi(\alpha) = \bigcup_{r \in G} \pi_r(\alpha)$. I claim that this function works. This immediately follows from the following claims.

Claim 4.5.5. *Whenever* $a \in [\kappa]^n$ *and* $\beta \in f(a)$*, the set* $D_{a,\beta} = \{r \in R : a \cup \{\beta\} \subset \mathrm{dom}(r), \exists m\ \nu_r(a, m) = \beta\}$ *is dense in* R*.*

Proof. Let $r \in R$; I must find a condition $s \leq r$ in the set $D_{a,\beta}$. For definiteness assume that $\beta \notin \mathrm{dom}(r)$. Choose $m \in \omega$ such that $\langle a, m \rangle \notin \mathrm{dom}(\nu_r)$. Consider the condition $s \leq r$ defined by $n_s = n_r$, $\pi_s = \pi_r \cup \{\langle \alpha, 0 \rangle : \alpha \in a \setminus \mathrm{dom}(r), \langle \beta, 0 \rangle\}$, $\nu_s = \nu_r \cup \{\langle a, m, \beta \rangle\}$. The condition $s \leq r$ is in the set $D_{a,\beta}$ as required. $\square$

Claim 4.5.6. *For every $a \in [\kappa]^n$ and every $k \in \omega$, the set $D_{a,k} = \{r \in R : a \subset \mathrm{dom}(r)$ and the set $\bigcap \pi_r'' a$ has at least k elements$\}$ is dense in R.*

Proof. Fix a, k and let $r \in R$ be an arbitrary condition. I must find a condition $s \leq r$ in the set $D_{a,k}$. First of all, the previous claim shows that one can strengthen r to include all ordinals in a. Increasing n_r if necessary, I may also assume that $k < n_r$.

Consider the set $b = \pi_r'' a$ and the function e_b; write $k' = \mathrm{dom}(e_b)$. If $k \leq k'$ then $s = r$ will work. Otherwise, it is easy to find an increasing sequence $d = \langle m_i : k' \leq i < k \rangle\rangle$ of numbers larger than n_r such that, writing $e = e_b \cup d$, for every natural number m such that $\langle a, m\rangle \in \mathrm{dom}(\nu_r)$ and every l such that $k' \leq k_m(l) < k$, $m_{k_m(l)}$ is odd if and only if $l \in r(\nu_r(a, m))$. The condition $s \leq r$ defined by $n_s = m_{k-1} + 1$, $\mathrm{dom}(\pi_s) = \mathrm{dom}(\pi_r)$, $\forall \beta \in a\ pi_s(\beta) = \pi_r(a) \cup \{m_i : k' \leq i < k\}$, $\forall \beta \in \mathrm{dom}(\pi_r) \setminus a\ \pi_s(\beta) = \pi_r(\beta)$, and $\nu_s = \nu_r$, is in the set $D_{a,k}$ as desired. □

Claim 4.5.7. *The poset R is c.c.c.*

Proof. Let $\langle r_\alpha : \alpha \in \omega_1\rangle$ be conditions in R. The usual Δ-system and counting arguments can be used to thin down the collection if necessary so that the sets $\mathrm{dom}(r_\alpha)$ for $\alpha \in \omega_1$ form a Δ-system with root b and for all $a \in [b]^n$ and all $\alpha \in \omega_1$, $f(a) \cap \mathrm{dom}(r_\alpha) \subset b$. Moreover, I can require that the increasing bijection between $\mathrm{dom}(r_\alpha)$ and $\mathrm{dom}(r_\beta)$ extends to an isomorphism of r_α and r_β for every $\alpha, \beta \in \omega_1$.

I claim that any two conditions in such a collection are compatible. Indeed, whenever $\alpha, \beta \in \omega_1$, then the condition s defined by $n_s = n_{r_\alpha}$, $\pi_s = \pi_{r_\alpha} \cup \pi_{r_\beta}$ and $\nu_s = \nu_{r_\alpha} \cup \nu_{r_\beta}$ is easily checked to be a common lower bound of the conditions r_α, r_β. □

□

Now, fix a function $f : [\omega_{n-1}]^{n-1} \to [\omega_{n-1}]^{\aleph_0}$ without a free n-element set as in Fact 4.5.3(2). Use Martin's Axiom and the lemma to produce an injection $\pi : \omega_{n-1} \to \mathcal{P}(\omega)$ such that for every set $a \in [\omega_{n-1}]^{n-1}$ and every $\beta \in f(a)$, $\pi(\beta)$ is recursive in $\pi'' a$. Write $A = \mathrm{rng}(\pi)$, consider the poset $P = \mathrm{Coll}(\omega, A)$ and the P-name τ for the generic enumeration of the set A in ordertype ω. A brief review of definitions shows that P is an F_2-pinned name for an element of the set B_n. It is not $\bar{F}_2$-related to any name on a poset of size $< \aleph_{n-1}$ since it necessitates the collapsing of the cardinal $|A| = \aleph_{n-1}$ to $\aleph_0$. This completes the proof of the theorem. □

One can ask whether the recursivity can be replaced by some other conditions more intimately tied to some preexisting mathematical structures. For example,

Question 4.5.8. Let G be a Borel group and $n \in \omega$ be a number; write $X = G^\omega$. Consider the set $D_n \subset X$ consisting of those $x \in X$ such that for every n-tuple of distinct elements of $\mathrm{rng}(x)$, one element of the tuple is in the algebraic closure of the remainder of the tuple. What is $\kappa(F_2 \restriction D_n)$? Can it be used to distinguish between various Borel groups?

The next example relies on a partition theorem discovered by Komjáth and Shelah.

Definition 4.5.9. The function $g : [\mathcal{P}(\omega)]^{<\aleph_0} \to [\omega]^{<\aleph_0}$ is defined by $g(a) = \{\min(x \setminus m+1) : x \in a\}$ if a is a set of size at least two and consists of pairwise almost disjoint sets and m is the largest number which appears in at least two of them; $g(a) = \min(x)$ if $a = \{x\}$ is a singleton; and otherwise $g(a) = 0$. Let n be a nonzero natural number, write $X = (\mathcal{P}(\omega))^\omega$ and let $B_n = \{x \in X :$ no finite set $a \subset \operatorname{rng}(x)$ can be written in more than $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $g(b) = g(c)\}$.

Theorem 4.5.10. *Assume MA+$\mathfrak{c} > \aleph_\omega$. For every nonzero number $n \in \omega$, $\kappa(F_2 \restriction B_n) = \aleph_{n+1}$.*

Proof. I will use the following partition theorem:

Fact 4.5.11. (Komjáth, Shelah[15]) *Let n be a nonzero natural number.*

1. *For every function $f : [\omega_n]^{<\aleph_0} \to \omega$ there is a finite set $a \subset \omega_m$ which can be written in at least $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $f(b) = f(c)\}$.*
2. *If $MA_{\aleph_n}$ holds then there is a function $f : [\omega_n]^{<\aleph_0} \to \omega$ such that every finite set $a \subset \omega_m$ can be written in at most $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $f(b) = f(c)\}$.*

It is now easy to argue in ZFC that for every $n > 0$, $\kappa(F_2 \restriction B_n) \leq \aleph_{n+1}$. Suppose that P is a poset and τ is a F_2-pinned element of the Borel set B_n. By Claim 4.1.6, there is a set $A \subset \mathcal{P}(\omega)$ such that $P \Vdash \operatorname{rng}(\tau) = \check{A}$. Since $P \Vdash \tau \in \dot{B}$, it must be the case that no finite set $a \subset A$ can be written in more than $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $g(b) = g(c)\}$. By Fact 4.5.11(1), it must be the case that $|A| < \aleph_{n+1}$. Consider the poset $Q = \operatorname{Coll}(\omega, A)$ and the Q-name σ for the generic enumeration of the set A in ordertype ω. It is clear that $|Q| < \aleph_{n+1}$ and $\langle P, \tau \rangle \bar{F}_2 \langle Q, \sigma \rangle$ as desired.

For the other inequality, a simple coding lemma is necessary.

Lemma 4.5.12. *Suppose that κ is a cardinal and F is an equivalence relation on $[\kappa]^{<\aleph_0}$ with countably many classes. Then there is a c.c.c. poset R adding an injection $\pi : \kappa \to \mathcal{P}(\omega)$ such that the equivalence relation on $[\kappa]^{<\aleph_0}$ induced by $g \circ \pi$ is finer than F.*

Proof. Let $f : [\kappa]^{<\aleph_0} \to \omega$ be a map inducing the equivalence relation F. Let $\nu : [\omega]^{<\aleph_0} \to \omega$ be sufficiently generic map such that $\nu(g(0)) = f(0)$. Let R be the poset of all maps r such that

- $\operatorname{dom}(r) \subset \kappa$ is a finite set;
- for every $\alpha \in \operatorname{dom}(r)$ the value $r(\alpha)$ is a nonempty subset of ω;
- for every $\alpha \in \operatorname{dom}(r)$, $\nu(\min(r(\alpha))) = f(\alpha)$;

- for every set $a \subset \mathrm{dom}(r)$ of size at least 2 there is a number which belongs to at least two sets $r(\alpha), r(\beta)$ for $\alpha \neq \beta \in a$, and writing m for the largest such number, $r(\alpha) \setminus m+1 \neq 0$ holds for every $\alpha \in a$, and $\nu(\{\min(r(\alpha) \setminus m+1) : \alpha \in a\}) = f(a)$.

The ordering is defined by $s \leq r$ if for every $\alpha \in \mathrm{dom}(r)$, $s(\alpha)$ end-extends $r(\alpha)$, and the sets $s(\alpha) \setminus r(\alpha)$ are pairwise disjoint for $\alpha \in \mathrm{dom}(r)$. If $G \subset R$ is a generic filter, in the model $V[G]$ let $\pi : \kappa \to \mathcal{P}(\omega)$ be defined by $\pi(\alpha) = \bigcup_{r \in G} r(\alpha)$; I claim that this is the function with the requested properties. This follows immediately from the following claims.

Claim 4.5.13. *The set $D_\alpha = \{r \in R : \alpha \in \mathrm{dom}(r)\}$ is dense in R for every $\alpha \in \kappa$.*

Proof. Let $\alpha \in \kappa$ and $r \in R$; I must produce $s \leq r$ such that $\alpha \in \mathrm{dom}(s)$. Enumerate $\mathrm{dom}(r)$ as β_i for $i \in k$ and write $\alpha = \beta_k$. Use the genericity of the function ν to find numbers $m < m_0 < m_1 < \dots m_{k-1}$ and pairwise distinct n_i^j for $i \in k, j \in k+1$ so that

- $\nu(m) = f(\alpha)$ and $m_0 > \max \bigcup \mathrm{rng}(r)$;
- $m_i < n_i^j$ for every $j \in k+1$;
- for every nonempty set $a \subset \mathrm{dom}(r) \cup \{\alpha\}$ and every $i \in k$, $\nu(\{n_i^j : \beta_j \in a\}) = f(a)$.

Once this is done, just consider the function s defined by $\mathrm{dom}(s) = \mathrm{dom}(r) \cup \{\alpha\}$, $s(\alpha) = \{m, m_i : i \in k, n_i^k : i \in k\}$, and for every $i \in k$, $s(\beta_i) = r(\beta_i) \cup \{m_i, n_j^i : j \in k\}$. It is not difficult to observe that $s \in R$ and $s \leq r$ as desired. □

Claim 4.5.14. *R has c.c.c.*

Proof. In fact, R is semi-Cohen in the sense of [1], but we will not need that fact here. By the usual Δ-system arguments, it is enough to show that any two conditions $r, s \in R$ such that $r \restriction \mathrm{dom}(r) \cap \mathrm{dom}(s) = s \restriction \mathrm{dom}(r) \cap \mathrm{dom}(s)$, are compatible. To find the lower bound, enumerate $\mathrm{dom}(r) \cup \mathrm{dom}(s)$ as β_i for $i \in k$, enumerate $(\mathrm{dom}(r) \setminus \mathrm{dom}(s)) \times (\mathrm{dom}(s) \setminus \mathrm{dom}(r))$ as u_j for $j \in l$. Use the genericity of the function ν to build numbers $m_0 < m_1 < \dots < m_{l-1}$ and pairwise distinct numbers n_i^j for $i \in k$ and $j \in l$ so that

- $m_0 > \max(\bigcup \mathrm{rng}(r) \cup \bigcup \mathrm{rng}(s))$;
- $m_j < n_i^j < m_{j-1}$ for every $j \in l$;
- for every set $a \subset \mathrm{dom}(r) \cup \mathrm{dom}(s)$, $\nu(\{n_i^j : \beta_i \in a\}) = f(a)$.

The lower bound is then a function t defined by $\mathrm{dom}(t) = \mathrm{dom}(r) \cup \mathrm{dom}(s)$, for $\alpha \in \mathrm{dom}(r)$, $\alpha = \beta_i$ set $t(\alpha) = r(\alpha) \cup \{n_i^j : j \in l\} \cup \{m_j : \alpha$ appears in the pair $u_j\}$. Similarly, for $\alpha \in \mathrm{dom}(s)$, $\alpha = \beta_i$ set $t(\alpha) = s(\alpha) \cup \{n_i^j : j \in l\} \cup \{m_j : \alpha$ appears in the pair $u_j\}$. It is not difficult to check that $t \leq r, s$ as required. □

Claim 4.5.15. *R forces $f = \nu \circ g \circ \pi$.*

Proof. This is clear from the definition of the poset R. $\square$

It follows that the equivalence relation on $[\kappa]^{<\aleph_0}$ induced by $g \circ \pi$ must be finer than the one induced by f, i.e. F. This completes the proof of the lemma. $\square$

Now, suppose that $\mathrm{MA}_{\aleph_n}$ holds, and use Fact 4.5.11(2) to find a function $f : [\omega_n]^{<\aleph_0} \to \omega$ such that every finite set $a \subset \omega_m$ can be written in at most $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $f(b) = f(c)\}$. Use Lemma 4.5.12 to find an injection $\pi : \omega_n \to \mathcal{P}(\omega)$ such that on $[\omega_n]^{<\aleph_0}$, the equivalence relation induced by $g \circ \pi$ is finer than that induced by f. It follows that the $\mathrm{Coll}(\omega, \omega_n)$-name τ for a generic enumeration of $\mathrm{rng}(\pi)$ is an F_2-pinned name for an element of $\dot{B}_n$ which is not equivalent to any name on a smaller poset since $|\mathrm{rng}(\pi)| = \aleph_n$. $\square$

As a last example I will present a simple analytic equivalence relation weakly reducible to F_2 whose pinned cardinal depends on the status of Chang's conjecture. We will not need anything about Chang's conjecture except for one equivalent restatement quoted below. Consistency of Chang's conjecture requires some modest large cardinals.

Definition 4.5.16. The *Chang's conjecture* is the statement that every first order model of type $(\aleph_2, \aleph_1)$ has an elementary submodel of type $(\aleph_1, \aleph_0)$.

Definition 4.5.17. Let $\omega = \bigcup_{n,m\in\omega} a_{n,m}$ be a partition of ω into infinite sets. For almost disjoint sets $b, c \subset \omega$ such that b is lexicographically less than c let $f(b, c) = n$ and $f(c, b) = m$ if $\max(b \cap c) \in a_{n,m}$. Let $X = (\mathcal{P}(\omega))^\omega$, and let $B \subset X$ be the coanalytic set of all $x \in X$ such that $\mathrm{rng}(x)$ consists of pairwise almost disjoint subsets of ω and there are no infinite subsets $d, e \subset \mathrm{rng}(x)$ such that $f \restriction d \times e$ is constant. Let E be the equivalence relation on X connecting x, y if either both of them fail to belong to B, or else $\mathrm{rng}(x) = \mathrm{rng}(y)$.

Theorem 4.5.18. *Assume MA+$\mathfrak{c} > \aleph_1$. The following are equivalent:*

1. *Chang's conjecture;*
2. *$\kappa(E) \leq \aleph_2$.*

Proof. The argument is based on the following partition theorem:

Fact 4.5.19. (Todorcevic, [25])

1. *If Chang's conjecture holds, then for every partition of ω_2^2 into countably many pieces, one piece of the partition contains a product of infinite sets.*
2. *If MA holds and Chang's conjecture fails, then there is a partition of ω_2^2 into countably many pieces such that no piece of the partition contains a product of infinite sets.*

To show in ZFC that Chang's conjecture implies $\kappa(E) \leq \aleph_2$, let P be a poset and let τ be an E-pinned P-name. I must produce a poset Q of size $\aleph_1$ and a name on it $\bar{E}$-related to τ. If $P \Vdash \tau \notin B$, then the name τ is trivial and so a trivial poset Q will work. Suppose on the other hand that $P \Vdash \tau \in \dot{B}$. Then τ must be an F_2-pinned name and Claim 4.1.6 provides a set $A \subset \mathcal{P}(\omega)$ such that $P \Vdash \mathrm{rng}(\tau) = \check{A}$. Let $f : A^2 \to \omega$ be the map defined by $f(b, c) = n$ and $f(c, b) = m$ if $\max(b \cap c) \in a_{n,m}$ and b is lexicographically smaller than c. This map is not constant on any product of two infinite subsets of A since otherwise $P \Vdash \tau \in B$ would fail. Fact 4.5.19(1) together with the Chang's conjecture assumption show that $|A| < \aleph_2$. Consider the poset $Q = \mathrm{Coll}(\omega, A)$ with its canonical name σ for a generic enumeration of the set A. Clearly, $\tau \; \bar{E} \; \sigma$, and $|Q| < \aleph_2$ as required.

Now, assume that MA holds and Chang's conjecture fails; I must conclude that $\kappa(E) > \aleph_2$. I will use a simple ZFC coding lemma.

Lemma 4.5.20. *Let κ be a cardinal and $g : \kappa^2 \to \omega$ be a function. Then there is a c.c.c. poset R adding an injection $\pi : \kappa \to \mathcal{P}(\omega)$ such that $g = f \circ \pi$.*

Proof. Let R be the poset of all functions r such that

- $\mathrm{dom}(r) \subset \kappa$ is a finite set;
- $\mathrm{rng}(r)$ consists of finite subsets of ω such that neither of them is an initial segment of another;
- for every $\alpha \neq \beta$ such that $r(\alpha)$ is lexicographically smaller than $r(\beta)$, the set $r(\alpha) \cap r(\beta)$ is nonempty, and its maximum belongs to the set $a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$.

The ordering on R is defined by $s \leq r$ if $\mathrm{dom}(r) \subset \mathrm{dom}(s)$, for every $\alpha \in \mathrm{dom}(r)$ the set $r(\alpha)$ is an initial segment of $s(\alpha)$, and the sets $\{s(\alpha) \setminus r(\alpha) : \alpha \in \mathrm{dom}(r)\}$ are pairwise disjoint. If $G \subset R$ is a generic filter, in the model $V[G]$ let $\pi : \kappa \to \mathcal{P}(\omega)$ be defined by $\pi(\alpha) = \bigcup_{r \in G} r(\alpha)$. The function π is as required. This follows immediately from the following claims.

Claim 4.5.21. *For every $\alpha \in \kappa$ the set $D_\alpha = \{r \in R : \alpha \in \mathrm{dom}(r)\}$ is dense in R.*

Proof. Let $\alpha \in \kappa$ be an ordinal and $r \in R$ be a condition; I must produce a condition $s \leq r$ such that $\alpha \in \mathrm{dom}(s)$. It will be the case that $s(\alpha) \cap \max(\bigcup \mathrm{rng}(r)) + 1 = 0$; this way, $s(\alpha)$ will be lexicographically smaller than all $s(\beta)$ for $\beta \in \mathrm{dom}(r)$. List $\mathrm{dom}(r)$ as β_i for $i \in j$, and find pairwise distinct numbers m_i for $i \in j$ so that

- $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\alpha, \beta) = n$;
- each m_i is greater than $\max(\bigcup \mathrm{rng}(r))$.

Then, let s be the function defined by $\mathrm{dom}(s) = \mathrm{dom}(r) \cup \{\alpha\}$ and $s(\beta_i) = r(\beta_i) \cup \{m_i\}$ and $s(\alpha) = \{m_i : i \in j\}$. It is immediate that the condition s works. □

Claim 4.5.22. *The poset R is c.c.c.*

Proof. By the usual Δ-system arguments it is only necessary to show that any two conditions $r, s \in R$ such that $r \restriction \mathrm{dom}(r) \cap \mathrm{dom}(s) = s \restriction \mathrm{dom}(r) \cap \mathrm{dom}(s)$ are compatible in the poset R. Strengthening the conditions r, s on $\mathrm{dom}(r) \setminus \mathrm{dom}(s)$ and $\mathrm{dom}(s) \setminus \mathrm{dom}(r)$ respectively if necessary, I may assume that no set in $\mathrm{rng}(r) \cup \mathrm{rng}(s)$ is an initial segment of another. Enumerate $(\mathrm{dom}(r) \setminus \mathrm{dom}(s)) \times (\mathrm{dom}(s) \setminus \mathrm{dom}(r))$ as u_i for $i \in j$ and find pairwise distinct numbers m_i for $i \in j$ such that

- if $u_i = \langle \alpha, \beta \rangle$ and $r(\alpha)$ is lexicographically smaller than $s(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$;
- if $u_i = \langle \alpha, \beta \rangle$ and $r(\alpha)$ is lexicographically greater than $s(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = n$ and $g(\beta, \alpha) = m$;
- all numbers m_i are greater than $\max(\bigcup \mathrm{rng}(r) \cup \bigcup \mathrm{rng}(s))$.

In the end, let t be the function defined by $\mathrm{dom}(t) = \mathrm{dom}(r) \cup \mathrm{dom}(s)$, for all $\alpha \in \mathrm{dom}(r)$ let $t(\alpha) = r(\alpha) \cup \{m_i : \alpha \text{ appears in } u_i\}$, and for all $\beta \in \mathrm{dom}(s)$ let $t(\beta) = s(\beta) \cup \{m_i : \beta \text{ appears in } u_i\}$. It is not difficult to check that t is a common lower bound of the conditions r, s as desired. □

□

Let $g : \omega_2^2 \to \omega$ be a function which is not constant on the product of any two infinite sets, as obtained by Fact 4.5.19. Use Martin's Axiom and the previous lemma to produce an injection $\pi : \omega_2 \to \mathcal{P}(\omega)$ such that $g = f \circ \pi$. Let $A = \mathrm{rng}(\pi)$, let $P = \mathrm{Coll}(\omega, A)$, and let τ be the P-name for the generic enumeration of the set A in ordertype ω. Clearly, τ is an F_2-pinned name, and the choice of the function g shows that $P \Vdash \tau \in \dot{B}$. The name τ is not equivalent to any name on a smaller poset since it necessitates the collapse of the size of the set A to ω. This completes the proof of the theorem. □

As was the case with previous examples, one can adjust the definition of E to shed light on various mathematical structures.

Question 4.5.23. Let X be any Borel linearly ordered metric space, let $\mathbb{R}^+ = \bigcup_{m,n \in \omega} a_{m,n}$ be a partition into dense Borel sets, let $f : X^2 \to \omega$ be the function defined by $f(x, y) = m$ if $x < y$ and $d(x, y) \in a_{m,n}$ and $f(y, x) = n$ if $x < y$ and $d(x, y) \in a_m, n$, and let $B \subset X^\omega$ be the collection all elements $z \in X^\omega$ such that f is not constant on any product of infinite subsets of $\mathrm{rng}(z)$. What is $\kappa(E_2 \restriction B)$? Can it be used to distinguish various metric spaces?

As a final remark in this section, with such wealth of analytic equivalence relations with interesting values of their pinned cardinal, one can naturally ask about limitations of the search for new examples. The most obvious question of this type is the following.

Question 4.5.24. Characterize the class $\mathfrak{P}$ of all cardinals κ such that there is an analytic (Borel, classifiable by countable structures etc.) equivalence relations such that $\kappa(E) = \kappa$.

There are many natural, weaker yes-no questions that would shed much light on the whole subject if answered in any direction.

Question 4.5.25. Is the class $\mathfrak{P}$ closed under the cardinal successor function?

Question 4.5.26. Let E be an analytic equivalence relation and κ the first weakly compact cardinal. Does $\kappa(E) > \kappa$ imply $\kappa(E) = \infty$?

Question 4.5.27. Is it consistent that there is a Borel equivalence relation E such that $\kappa(E)$ is a limit cardinal of uncountable cofinality?

Chapter 5

Cardinalistic equivalence relations

5.1 Definition and basic concerns

One can ask how much forcing sophistication is really necessary to produce pinned names. After all, the pinned names in the previous sections all were essentailly cardinal collapse names. It turns out that there is a class of equivalence relations for which this is indeed always the case, and it includes all orbit equivalence relations and many more. At the same time, I will produce a simple Borel equivalence relation for which more sophisticated pinned names can occur.

Definition 5.1.1. An analytic equivalence relation E on a Polish space X is *cardinalistic* if for every E-pinned name τ on a poset P there is a cardinal $\kappa(\tau)$ such that for every poset Q, Q carries a name $\bar{E}$-related to τ if and only if it collapses $\kappa(\tau)$ to $\aleph_0$. For trivial names write $\kappa(\tau) = \aleph_0$.

The definition reflects a common feature of arguments presented earlier–a pinned name for equivalence relations associated with isomorphism of structures is often associated with an isomorphism type of an uncountable structure in the ground model. Then, introducing an element of a Polish space coding it is equivalent to collapsing the size of that structure to $\aleph_0$.

The notion of cardinalistic equivalence relation, as compared with say pinned equivalence relation, has the disadvantage that it is not absolute between various forcing extensions. For example, the mutual domination equivalence relation of Definition 5.3.1 is cardinalistic in L, with the possible values $\kappa(\tau)$ just $\aleph_0$ (for trivial pinned names) and $\aleph_1$ (for nontrivial pinned names); however, under Martin's Axiom it is not cardinalistic as there is a nontrivial pinned name on Namba forcing under those conditions. Still, the class of cardinalistic equivalence relations is interesting and leads to ergodicity results.

Theorem 5.1.2. *If E, F are analytic equivalence relations, $E \leq_{\mathrm{wB}} F$, and F is cardinalistic, then E is cardinalistic.*

Proof. Let E, F be analytic equivalence relations on respective Polish spaces X, Y, let $a \subset X$ be a countable set and let $h : X \to Y$ be a Borel function which is a reduction of E to F on $X \setminus [a]_E$. Suppose that F is cardinalistic. Let P be a poset and τ an E-pinned P-name. Either, $P \Vdash \exists x \in \check{a}\ \tau\ E\ x$, and then $\kappa(\tau) = \aleph_0$. Or, $P \Vdash \neg\exists x \in \check{a}\ \tau\ E\ x$. Then, $\dot{h}(\tau)$ is an F-pinned P-name, and so $\kappa = \kappa(\dot{h}(\tau))$ exists. It will be enough to show that $\kappa(\tau) = \kappa$.

Let $V[G]$ be a generic extension of V, and work in $V[G]$. I must verify the equivalence in Lemma 5.2.10(2). If κ is countable, then there is $y \in Y$ such that $P \Vdash \dot{h}(\tau)\ F\ \check{y}$ by an application of Lemma 5.2.10 to F and $\dot{h}(\tau)$. By the Shoenfield absoluteness between $V[G]$ and its P-extension there is an element $x \in X \setminus [a]_E$ such that $h(x)\ F\ y$. As the function h remains a reduction, it must be the case that $P \Vdash \tau\ E\ \check{x}$ as desired. Suppose on the other hand that there is $x \in X$ in $V[G]$ such that $P \Vdash \tau\ E\ \check{x}$. As the function h remains a reduction in the P-extension of $V[G]$, it is also the case that $P \Vdash h(\tau)\ F\ h(x)$. By Lemma 5.2.10(2) applied to F and $\dot{h}(\tau)$, κ must be countable. This completes the proof of (1). $\square$

5.2 Operations

The class of cardinalistic equivalence relations contains all orbit equivalence relations, and it is closed under a good number of natural operations. This is what I prove in this section.

Theorem 5.2.1. *Every orbit equivalence relation generated by a continuous Polish group action is cardinalistic.*

Proof. Let $G \curvearrowright X$ be a Polish group continuously acting on a Polish space and let E denote the resulting orbit equivalence relation on X. Let τ be an E-pinned name on some poset P. The main point of the proof is that the $\bar{E}$-equivalence class of τ has a canonical representative up to the forcing equivalence. Let P_G be the poset of nonempty open subsets of G ordered by inclusion, adding a single point $\dot{g}_{\mathrm{gen}} \in G$. Let σ be the $P \times P_G$-name for the element $\dot{g}_{\mathrm{gen}} \cdot \tau$–so clearly $\tau\ \bar{E}\ \sigma$. Let Q be the regular subalgebra of the completion of $P \times P_G$ completely generated by the name σ. It turns out that the pair $\langle Q, \sigma\rangle$ up to forcing isomorphism does not depend on the initial choice of $\langle P, \tau\rangle$ in its $\bar{E}$-equivalence class, and Q is in fact the poset $\mathrm{Coll}(\omega, \kappa)$ for some cardinal κ. Then $\kappa = \kappa(\tau)$ will work as required in the definition of cardinalistic equivalence relation.

I will start with a seemingly unrelated pure forcing lemma. Whenever μ is an ordinal and $f, g : \mu^{<\omega} \to \mathcal{P}(\mu)$ are functions, write $f \wedge g$ for the function $x \mapsto f(x) \cap g(x)$. A set $a \subset \mu$ is said to be *closed* under f if for every tuple $x \in a^{<\omega}$ the value $f(x)$ is a subset of a. Let λ be a cardinal. Two forcing extensions $V[G_0]$ and $V[G_1]$ are said to be μ, λ-*perpendicular* over V if for every

$f_0 : \mu^{<\omega} \to [\mu]^{\aleph_0}$ in $V[G_0]$ and $f_1 : \mu^{<\omega} \to [\mu]^{\aleph_0}$ in $V[G_1]$ there is a set $a \in V$, of size $< \lambda$ in V, closed under the function $f_0 \wedge f_1$.

Lemma 5.2.2. *Suppose λ is a regular cardinal, R_0, R_1 are posets preserving regularity of λ, and μ is an ordinal. In some forcing extension, there are filters $G_0 \subset R_0$, $G_1 \subset R_1$ separately generic over V, such that the models $V[G_0]$ and $V[G_1]$ are μ, λ-perpendicular over V.*

Proof. Suppose first that R is a poset preserving regularity of λ, $a \subset \mu$ is a set, u is a countable set of R-names for functions from $\mu^{<\omega}$ to $[\mu]^{\aleph_0}$, and $r \in R$ is a condition. Say that a is *good* for r, u if for every set $b \subset \mu$ of size $< \lambda$ disjoint from a there is a condition $r' \leq r$ forcing that for no $x \in a^{<\omega}$, no $\beta \in b$ and no $\sigma \in u$ it is the case that $\beta \in \sigma(x)$.

Claim 5.2.3. *For every r, u and every set $a \subset\in [\mu]^{<\lambda}$ there is a set $a' \supset a$ in $[\mu]^{<\lambda}$ which is good for r, u.*

Proof. Suppose that this fails and by induction on $\alpha \in \lambda$ build pairwise disjoint sets $a_\alpha \in [\mu]^{<\lambda}$ such that $a_0 = a$ and a_α witnesses that $\bigcup_{\beta\in\alpha} a_\beta$ is not good for r, u. Since P preserves the regularity of λ,a pressing down argument shows there must be a condition $s \leq r$ and an ordinal $\alpha \in \lambda$ such that no element of $\bigcup\{a_\beta : \beta \geq \alpha\}$ belongs to any set $\sigma(x)$ where $x \in (\bigcup_{\beta\in\alpha} a_\beta)^{<\omega}$ and $\sigma \in u$. This contradicts the choice of the set a_α. □

Towards the proof of the lemma, let S be the poset consisting of quintuples $s = \langle r^0_s, r^1_s, u^0_s, u^1_s, a_s\rangle$ such that r^0_s, r^1_s are conditions in R_0, R_1 respectively, u^0_s, u^1_s are countable collections of R_0 and R_1-names for functions from $\mu^{<\omega}$ to $\mu^{\aleph_0}$, $a_s \in [\mu]^{<\lambda}$, and either a_s is good for r^0_s and u^0_s in R_0, or a_s is good for r^1_s and u^1_s in R_1. The ordering is defined by $t \leq s$ if $r^0_t \leq r^0_s$, $r^1_t \leq r^1_s$, $u^0_s \subset u^0_t$, $u^1_s \subset u^1_t$, $a_s \subset a_t$, and (*) for every ordinal $\alpha \in a_t \setminus a_s$ and every finite tuple $u \in a_s^{<\omega}$, either $r^0_t \Vdash_{R_0} \forall\sigma \in u^0_s\ \check{\alpha} \notin \sigma(u)$ or $r^1_t \Vdash_{R_1} \forall\sigma \in u^1_s\ \check{\alpha} \notin \sigma(u)$. It is not difficult to see that S is a partial order with largest condition $\langle 1_{R_0}, 1_{R_1}, 0, 0, 0\rangle$. If $H \subset S$ is a generic filter over V, let $G_0 \subset R_0$ be the filter generated by $\{r^0_s : s \in H\}$, and let $G_1 \subset R_1$ be the filter generated by $\{r^1_s : s \in H\}$. I will show that the two filters have the desired properties. The following is the required density claim in the ground model.

Claim 5.2.4. *The following sets are dense in S:*

1. *the set $C_1 = \{s \in S : a_s$ is good for r^0_s and u^0_s in $R_0\}$, and similarly on the R_1 side;*
2. *if $D \subset R_0$ is open dense, the set $C_2 = \{s \in S : r^0_s \in D\}$, and similarly on the R_1 side;*
3. *if σ is an R_0 name for a function from $\mu^{<\omega}$ to $[\mu]^{\aleph_0}$, the set $C_3 = \{s \in S : \sigma \in u^0_s\}$, and similarly on the R_1 side;*
4. *if $a \in [\mu]^{<\lambda}$, the set $C_4 = \{s \in S : a \subset a_s\}$.*

Proof. For (1), fix a condition $s \in S$ and work to strengthen s to get a condition in the set C_1. If a_s is good for r^0_s and u^0_s in R_0, then we are done. Thus, assume instead that a_s is good for r^1_s and u^1_s in R_1. Use Claim 5.2.3 to find a set $b \supset a_s$ in $[\mu]^{<\lambda}$ which is good for r^0_s and u^0_s in R_0. Use the goodness of a_s to find a condition $r \leq r^1_s$ forcing in R_1 that for no $x \in a_s^{<\omega}$, no $\beta \in b \setminus a_s$ and no $\sigma \in u^0_s$ it is the case that $\beta \in \sigma(x)$. The condition $\langle r^0_s, r, u^0_s, u^1_s, b\rangle$ is as required.

For (2), suppose that $D \subset R_0$ is open dense and $s \in S$ is a condition, and work to strengthen s to a condition in the set C_2. First, use (1) to strengthen s if necessary so that a_s is good for r^1_s and u^1_s. Then, find a condition $r \leq r^0_s$ in the poset R_0 in D. The condition $\langle r, r^1_s, u^0_s, u^1_s, a_s\rangle$ is as required.

For (3), suppose that σ is an R_0 name, and $s \in S$ is a condition. Strengthen s if necessary so that a_s is good for r^1_s and a^1_s in R_1. The condition $\langle r^0_s, r^1_s, a^0_s \cup \{\sigma\}, a^1_s, a_s\rangle \leq s$ is in the set C_3.

For (4), suppose that $a \in [\mu]^{<\lambda}$ and $s \in S$. Use (1) to strengthen s if necessary so that a_s is good for r^1_s and a^1_s in R_1. Use Claim 5.2.3 to find a set $b \supset a_s \cup a$ is good for r^0_s, and use the goodness of a_s to find a condition $r \leq r^1_s$ forcing in R_1 that for no $x \in a_s^{<\omega}$, no $\beta \in b \setminus a_s$ and no $\sigma \in u^0_s$ it is the case that $\beta \in \sigma(x)$. The condition $\langle r^0_s, r, u^0_s, u^1_s, b\rangle \leq s$ is in the set C_4. $\square$

Now, Claim 5.2.4(2) shows that the filters $G_0 \subset R_0$ and $G_1 \subset R_1$ are separately generic over V. Now, suppose that $f_0 \in V[G_0]$ and $f_1 \in V[G_1]$ are functions from $\mu^{<\omega}$ to $[\mu]^{\aleph_0}$, and find names σ_0, σ_1 such that $f_0 = \sigma_0/G_0$ and $f_1 = \sigma_1/G_1$. Use Claim 5.2.4(4) to find a condition $s \in S$ such that $\sigma_0 \in u^0_s$ and $\sigma_1 \in u^1_s$. I claim that the set a_s is closed under the function $f_0 \wedge f_1$. Indeed, Claim 5.2.4(3) shows that the union $\bigcup\{a_t : t \in H\}$ is equal to μ, and so for every ordinal $\alpha \in \mu \setminus a_s$ and every finite sequence $x \in a_s^{<\omega}$ itit is either the case that $\alpha \notin f_0(x)$ or $\alpha \notin f_1(x)$ from the condition (*) in the definition of the poset S.

$\square$

Let E be a orbit equivalence relation on a Polish space X. Let P be a poset and τ and E-pinned name on P. Find the smallest cardinal λ such that there is a poset Q and an E-pinned name σ on Q such that $\langle P, \tau\rangle \bar{E} \langle Q, \sigma\rangle$ and $Q \Vdash \lambda = \omega_1$. Switching from P to Q if necessary, we may assume that in fact $P \Vdash \lambda = \omega_1$. I will show that λ is a successor cardinal, and its predecessor is the cardinal $\kappa(\tau)$ with the required properties.

Let G be a Polish group, and let $G \curvearrowright X$ be a continuous action such that E is its orbit equivalence. Consider the poset $P \times P_G$, where P_G is the poset of all nonempty basic open subsets of G ordered by inclusion, adding a generic element $\dot{g}_{\text{gen}}$. Consider the poset Q generated by the $P \times P_G$-name $\sigma = \dot{g}_{\text{gen}} \cdot \tau$.

Lemma 5.2.5. *For every cardinal μ, $Q \Vdash ([\mu]^{<\lambda})^V$ is a stationary set.*

Proof. In other words $P \times P_G$ forces $V[\sigma] \models ([\mu]^{<\lambda})^V$ is a stationary set. Suppose for contradiction that some condition $\langle p, q\rangle \in P \times P_G$ forces the opposite, and identifies some ordinal μ such that the set $([\mu]^{\lambda})^V$ is not stationary in $V[\sigma]$. Apply Lemma 5.2.2 to pass to a generic extension $V[H]$ in which there are two filters $G_0, G_1 \subset P$ containing p and separately generic over V, such that the extensions $V[G_0]$ and $V[G_1]$ are μ, λ-perpendicular over V. Write $x_0 = \tau/G_0$ and $x_1 = \tau/G_1$.

Claim 5.2.6. *$x_0 \; E \; x_1$.*

Proof. In some further forcing extension there is a filter $G_2 \subset P$ which is generic over $V[G_0, G_1]$. As τ is an E-pinned name, the forcing theorem in V implies $V[G_0, G_2] \models x_0 = \tau/G_0 \; E \; \tau/G_2$ and $V[G_1, G_2] \models x_1 = \tau/G_1 \; E \; \tau/G_2$; by transitivity of E, $x_0 \; E \; x_1$ as desired. $\square$

Let $g_{gen} \in G \cap q$ be a point P_G-generic over $V[H]$, and consider the model $W = V[g_{gen} \cdot x_0]$. I will reach a contradiction by showing that every function $e : \mu^{<\omega} \to \mu$ in W has a closure point in V which is of size $< \lambda$ in V.

As $x_0 \; E \; x_1$, there is a group element $g \in G$ in the model $V[H]$ such that $g \cdot x_1 = x_0$. As the multiplication by g on the right induces an automorphism of the poset P_G, the point $g_{gen}g$ is P_G-generic over $V[H]$. Moreover, the point $g_{gen} \cdot x_0 \in X$ is equal to $g_{gen}g \cdot x_1$. Thus, the model W is a subset of P_G-extensions $V[G_0][g_{gen}]$ and $V[G_1][g_{gen}g]$ of $V[G_0]$ and $V[G_1]$ respectively. Thus, a standard c.c.c. argument shows that there are functions $f_0 \in V[G_0]$ and $f_1 \in V[G_1]$ from $\mu^{<\omega}$ to $[\mu]^{\aleph_0}$ such that for every $x \in \mu^{<\omega}$, $e(x) \in f_0(x)$ and $e(x) \in f_1(x)$. Use the perpendicularity of the extensions $V[G_0]$ and $V[G_1]$ to find a set $a \in [\mu]^{<\lambda}$ in V closed under $f_0 \wedge f_1$. This set must then be closed under e as well, completing the proof. $\square$

Now, use Lemma 5.2.5 to find a condition $q \in Q$ to find an elementary submodel M of a large structure of size $< \lambda$ and a condition $q \in Q$ which is master for M. It will be enough to show that for every poset R collapsing the size of $|M|$ to $\aleph_0$ there is an R-name which is $\bar{E}$-related to τ'. Then, by the minimal choice of λ and the possibility that $R = \text{Coll}(\omega, |M|)$, it follows that $|M|^+ = \lambda$, and $\kappa(\tau) = |M|$ works as required. Let R be a poset collapsing $|M|$ to $\aleph_0$ and find an R-name for a filter $\dot{K} \subset M \cap Q$ generic over M, and let ν be the R-name for the evaluation $\sigma/\dot{K}$; this is a name for an element of the space Y.

Claim 5.2.7. *$\nu \; \bar{E} \; \sigma$.*

Proof. If $G_0 \times G_1 \subset (Q \restriction q) \times R$ is a generic filter over V, in some further generic extension there is a filter $G_2 \subset Q \cap M$ generic over $V[G_0, G_1]$. As σ is an E-pinned name, the forcing theorem applied in the model M shows that $M[G_0, G_2] \models \sigma/G_0 \; F \; \sigma/G_2$ and $M[\dot{K}/G_1, G_2] \models \sigma/\dot{K}/G_1 \; F \; \sigma/G_2$. The transitivity of E and the Mostowski absoluteness for the models just mentioned then shows that σ/G_0 is F-related to $\nu/G_1 = \sigma/K/G_1$. $\square$

Now by the definition of σ, $\sigma \; \bar{E} \; \tau$ and so $\nu \; \bar{E} \; \tau$ as desired. $\square$

Corollary 5.2.8. *If E is an orbit equivalence relation and P is an $\aleph_1$-preserving poset, then every E-pinned name on P is trivial.*

Proof. Let τ be an E-pinned name. Since P preserves $\aleph_1$, the cardinal $\kappa(\tau)$ must be equal to $\aleph_0$. Thus, $\mathrm{Coll}(\omega, \omega)$ must contain an E-pinned name $\bar{E}$-related to τ. However, $\mathrm{Coll}(\omega, \omega)$ is just the Cohen forcing, therefore reasonable, and so all E-pinned names in it are trivial by Theorem 3.3.2. $\square$

Theorem 5.2.9. *The class of cardinalistic equivalence relations contains all pinned equivalence relations. It is closed under*

1. *the Friedman–Stanley jump;*
2. *product modulo every Borel ideal J on ω such that $=_J$ is pinned.*

In order to efficiently package the proof of Theorem 5.2.9, I will use a technical Lemma similar to Lemma 4.3.3.

Lemma 5.2.10. *Suppose that τ is an E-pinned name on a poset P and κ is an infinite cardinal. The following are equivalent:*

1. *$\kappa = \kappa(\tau)$;*
2. *in every forcing extension $V[G]$ of V, $|\kappa| = \aleph_0$ iff there is $x \in X$ such that $P \Vdash \tau\ E\ \check{x}$.*

Proof. First assume (1). To verify (2), suppose that Q is any poset. If some condition $q \in Q$ forces that κ is countable, then there is a name σ on $Q \restriction q$ such that $\langle P, \tau \rangle\ \langle\ Q \restriction q, \sigma \rangle$ by (1). Then, $q \Vdash P \Vdash \tau\ E\ \sigma$ by the forcing theorem. On the other hand, if some condition $q \in Q$ forces that there is $x \in X$ such that $P \Vdash \tau\ E\ \check{x}$ and σ is a Q-name for such an x, then $\langle P, \tau \rangle\ \bar{E}\ \langle Q \restriction q, \sigma \rangle$ by the product forcing theorem. By (1), q must force $|\kappa| = \aleph_0$. This verifies (2).

Now, assume (2). To verify (1), suppose that Q is a poset. If $Q \Vdash |\kappa| = \aleph_0$ then by (2) $Q \Vdash \exists x\ P \Vdash \tau\ E\ \check{x}$. If σ is a Q-name for any such a point $x \in X$, then the product forcing theorem shows that $P \times Q \Vdash \tau\ E\ \sigma$ and so $\tau\ \bar{E}\ \sigma$. On the other hand, if some condition $q \in Q$ forces κ to remain uncountable, then by (2) it also forces that there is no $x \in X$ such that $P \Vdash \tau\ E\ \check{x}$, which excludes the existence of a Q-name σ such that $\langle P, \tau \rangle\ \bar{E}\ \langle Q, \sigma \rangle$. $\square$

Proof of Theorem 5.2.9. For (1), suppose that E is a cardinalistic equivalence relation on a Polish space X and τ is an E^+-pinned name for an element of X^ω on some poset P. Let $A = \{\langle P \restriction p, \tau(i) \rangle : p \in P, i \in \omega$ and $\tau(i)$ is an E-pinned name on the poset $P \restriction p\}$. Let λ be the number of $\bar{E}$-classes represented in the set A and let $\kappa = \sup\{\lambda, \kappa(z) : z \in A\}$. Since E is assumed to be a cardinalistic equivalence relation, κ is well-defined. I claim that $\kappa = \kappa(\tau)$ works. Suppose that $V[G]$ is a generic extension of V and work in $V[G]$; I must verify the equivalence in (1) of Lemma 5.2.10.

Suppose first that there is $y \in X^\omega$ such that $P \Vdash \tau\ E^+\ \check{y}$. Then for each $\bar{E}$-class c represented by $z = \langle P \restriction p, \tau(i) \rangle \in A$, the countable set $\mathrm{rng}(y)$ must contain an element $x_c \in X$ such that $p \Vdash \tau(i)\ E\ \check{x}_c$. As for distinct $\bar{E}$-classes c the elements x_c must be non-E-related and therefore distinct, it follows that λ must be countable in $V[G]$. Also, Lemma 5.2.10 applied to E and x_c shows that the cardinal $\kappa(z)$ must be countable in $V[G]$, this for every $z \in A$. It follows that κ is countable in $V[G]$ as required.

Suppose now that κ is countable in $V[G]$. Applying Lemma 5.2.10 to E, for every pair $z = \langle P \restriction p, \tau(i)\rangle \in A$ there is $x_z \in X$ in $V[G]$ such that $p \Vdash \tau(i)\ E\ \check{x}_z$. Since the number of $\bar{E}$-classes represented in the set A is countable in $V[G]$, the set $\{x_z : z \in A\}$ contains only countably many E-classes. Let $y \in X^\omega$ be some point in $V[G]$ visiting exactly these classes. I claim that $P \Vdash \tau\ E^+\ \check{y}$ as desired.,

First of all, it is clear that $P \Vdash \forall i \exists j\ \check{y}(i)\ E\ \tau(j)$, since for every $i \in \omega$ there is $j \in \omega$ and $p \in P$ such that $p \Vdash \check{y}(i)\ E\ \tau(j)$ by the choice of y, and the name τ is E^+-pinned. To prove that $P \Vdash \forall i \exists j\ \check{y}(j)\ E\ \tau(i)$, suppose for contradiction that it fails as forced by some condition $p \in P$ and a specific number $i \in \omega$. As the name τ is E^+-pinned, there must exist conditions $p_0, p_1 \leq p$ and a number $k \in \omega$ such that $\langle p_0, p_1\rangle \Vdash_{P\times P} \tau_{\mathrm{left}}(i)\ E\ \tau_{\mathrm{right}}(k)$. Then, $\tau(i)$ is an E-pinned name on $P \restriction p_0$, and so $\langle P \restriction p, \tau(i)\rangle \in A$. By the choice of the point $y \in X^\omega$, there is $j \in \omega$ such that $P \restriction p \Vdash \tau(i)\ E\ \check{y}(j)$, contradicting the choice of the condition p and the number i.

For (2), suppose that J is a Borel ideal on ω such that $=_J$ is pinned, and for every $i \in \omega$, E_i is a cardinalistic equivalence relation on a Polish space X_i, $X = \prod_i X_i$ and $E = \prod_J E_i$. Suppose that P is a poset and τ is an E-pinned P-name for an element of X. Consider the following P-names. For each $i \in \omega \setminus a$, if there is a condition p in the generic filter such that τ_i is an E_i-pinned name on $P \restriction p$, then $\kappa_i = \kappa(\tau_i)$. Also, let κ be the least cardinal such that the set $\{i \in \omega : \kappa_i > \kappa\}$ is in J.

Claim 5.2.11. *$P \Vdash \{i \in \omega : \kappa_i$ is not defined$\} \in J$.*

Proof. Suppose there is a condition $p \in P$ forcing the opposite. Since τ is a E-pinned name, there must be conditions $p_0, p_1 \leq p$ and $i \in \omega$ such that $\langle p_0, p_1\rangle \Vdash \tau_{i,\mathrm{left}}\ E_i\ \tau_{i,\mathrm{right}}$ and $p_0 \Vdash \kappa_i$ is not defined. However, this is impossible since τ_i is an E_i-pinned name on $P \restriction p_0$ and E_i is cardinalistic. □

Claim 5.2.12. *The value of κ is decided by the largest condition in P.*

Proof. Suppose for contradiction that p_0, p_1 are conditions in P that decide the value of $\dot{\kappa}$ to be the respective distinct values λ_0, λ_1, say $\lambda_0 < \lambda_1$. Since E is an E-pinned name, strengthening the conditions $p_, p_1$ if necessary I may find an index $i \in \omega$ such that $\langle p_0, p_1\rangle \Vdash \tau_{i,\mathrm{left}}\ E_i\ \tau_{i,\mathrm{right}}$ and $p_0 \Vdash \kappa_i \leq \lambda_1$ and $p_1 \Vdash \kappa_i > \lambda_1$. Strengthening further if necessary, I can make sure that p_0, p_1 decide the value of κ_i. This contradicts the fact that κ_i is an invariant of the $\bar{E}$-equivalence. □

Now, I will show that $\kappa = \kappa(\tau)$ works as required. Suppose that $V[G]$ is a generic extension and work in $V[G]$; I must verify the equivalence in Lemma 5.2.10(2). If $|\kappa| = \aleph_0$ then by the definition of κ and Lemma 5.2.10 applied to E_i, $P \Vdash \tau(i)$ is E_i-equivalent to some element of $V[G]$ for all but J-many numbers $i \in \omega$. For each $i \in \omega$ let $f_i : 2^\omega \to X_i/E_i$ be any enumeration of all E_i-classes and let ν be the P-name for an element of $(2^\omega)^\omega$ given by the following formula: if $\tau(i)$ is E_i-equivalent to some element of $V[G]$ then $\nu(i)$ is the unique $y \in 2^\omega$ such that $\tau(i)$ is E_i-equivalent to the elements of $f_i(y)$, and $\nu(i)$ =trash otherwise. The definitions show that this is a $=_J$-pinned name on P. Since the equivalence relation $=_J$ is pinned in V by the assumptions, it is pinned in $V[G]$ as well by Corollary 3.2.6, and so the $=_J$-pinned name ν is trivial, forced to be $=_J$-equivalent to some element $z \in (2^\omega)^\omega$. Let $x \in X$ be any point such that for every $i \in \omega$, $x(i) \in f_i(z(i))$. It follows directly from definitions that $P \Vdash \tau\ E\ \check{x}$.

If, on the other hand, there is an element $y \in X$ such that $P \Vdash \tau\ E\ \check{y}$, then for al but J-many $i \in \omega$, the equivalence class $[\tau(i)]_{E_i}$ is V-E_i-pinned. By the minimal choice of κ, it also must be the case that the cardinals $\kappa([y(i)]_{E_i})$ are cofinal in κ or perhaps include κ. These cardinals are all countable by Lemma 5.2.10 applied to E_i, so κ must be countable as well. $\square$

5.3 An ergodicity result

In this section, I will isolate a simple Borel equivalence relation which is not cardinalistic in some forcing extension. This equivalence relation then possesses a strong ergodicity property with respect to all provably cardinalistic relations, in particular all orbit equivalence relations.

Definition 5.3.1. The *mutual domination* Borel equivalence relation E on $X = (\omega^\omega)^\omega$ connects points $x, y \in X$ if for every $n \in \omega$ there is $m \in \omega$ such that $y(m)$ modulo finite dominates $x(n)$ and vice versa, for every $n \in \omega$ there is $m \in \omega$ such that $x(m)$ modulo finite dominates $y(n)$.

To state the ergodicity result, I must first identify a suitable σ-ideal on the space X.

Definition 5.3.2. Let $A \subset X$ be a set. The game $G(A)$ is played by Players I and II alternately choosing points $y_n \in \omega^\omega$ for $n \in \omega$ so that y_{n+1} modulo finite dominates y_n for every $n \in \omega$. Player I wins if there is $x \in A$ such that $\langle y_n : n \in \omega\rangle\ E\ x$. The *mutual domination ideal* is the σ-ideal generated by sets $X \setminus [A]_E$ where $A \subset X$ is an analytic set such that Player I has a winning strategy in $G(A)$.

The purpose of this section is to prove the following theorem:

Theorem 5.3.3. *Let E be the mutual domination equivalence relation on a Polish space X.*

1. *$\kappa(E) = \mathfrak{c}^+$;*
2. *in some generic extension, E is not cardinalistic;*
3. *If F is an analytic equivalence relation on a Polish space Y which is cardinalistic in every forcing extension and $h : X \to Y$ is a Borel homomorphism from E to F, then there is an $y \in Y$ such that $X \setminus h^{-1}[y]$ belongs to the mutual domination ideal.*

Corollary 5.3.4. *If F is an orbit equivalence relation of a Polish group action on a Polish space Y, if $h : X \to Y$ is a Borel homomorphism from E to F, then there is an $y \in Y$ such that $X \setminus h^{-1}[y]$ belongs to the mutual domination ideal.*

Before the proof of Theorem 5.3.3, I will spend some time outlining the main features of the mutual domination ideal. It turns out that it is closely related to the σ-ideal naturally associated with ω-iteration of Hechler forcing.

Definition 5.3.5. [2, Definition 3.1.9] The *Hechler forcing* is the poset P consisting of pairs $p = \langle t_p, x_p \rangle$ where $t_p \in \omega^{<\omega}$ and $x_p \in \omega^\omega$. The ordering is defined by $q \leq p$ if $t_p \subset t_q$, $x_p \leq x_q$ coordinatewise, and $\forall n \in \mathrm{dom}(t_q) \setminus \mathrm{dom}(t_p)\ t_q(n) \geq x_q(n)$. The poset P adds a *Hechler real*, which is the point $\dot{x}_{gen} = \bigcup_{p \in G} t_p \in \omega^\omega$ whenever $G \subset P$ is a generic filter.

The letter P in this section always denotes the Hechler forcing. P is one of the most frequently used Suslin forcings. The Hechler real $\dot{x}_{gen}$ modulo finite dominates all functions in the ground model. It can also modulo finite dominate any function which is not in the ground model. This is the contents of the following technical lemma, which I will use repeatedly:

Lemma 5.3.6. *Suppose that M is a countable transitive model, $p \in P^M$, and $y \in \omega^\omega$ is an arbitrary function. Then there is a filter $g \subset P^M$ generic over M such that $p \in g$ and $\dot{x}_{gen}/g$ modulo finite dominates y.*

Proof. Let N be a countable elementary submodel of a large structure containing all the named objects. Let $p = \langle t_p, x_p \rangle$, let $q \leq p$ be the condition $q = \langle t_p, \max(x_p, y)$, let h be a Hechler generic filter over N containing h, and let $g = h \cap P^M$. By Theorem 2.3.9, g is P^M-generic over M, it contains the condition p, and by the choice of the condition q, $\dot{x}_{gen}/g(n) > y(n)$ for all $n > \mathrm{dom}(t_p)$. This completes the proof. □

Definition 5.3.7. The *ω-Hechler ideal* on the space X is the collection of those Borel sets B such that $P_\omega \Vdash \vec{x}_{gen} \notin \dot{B}$. Here, P_ω is the usual c.c.c. finite support iteration of length ω of Hechler forcing and $\vec{x}_{gen} \in (\omega^\omega)^\omega$ its generic sequence.

On the collection of E-saturated subsets of the space X, the ω-Hechler ideal has a forcing free description coming from a natural infinite game.

Theorem 5.3.8. *For every analytic set $A \subset X$, the game $G(A)$ is determined. If $A \subset X$ is an analytic set then $\omega^\omega \setminus [A]_E$ is in the ω-Hechler ideal if and only if it is in the mutual domination ideal.*

Proof. For the determinacy, if $A \subset X$ is an analytic set, fix a continuous function $f : \omega^\omega \to X$ such that $\mathrm{rng}(f) = [A]_E$. Let H be the unraveled version of the game $G(A)$: with each move y_{2n} of the game $G(A)$ Player I indicates a number $m_n \in \omega$, and he wins if $f(m_n : n \in \omega) = (y_{2n} : n \in \omega)$. The game H is closed for Player I, therefore determined by the Gales-Stewart theorem. If Player I has a winning strategy in H, the same strategy without revealing the additional numbers will win for Player I in $G(A)$. It will be enough to show that if Player

II has a winning strategy in H then he has a winning strategy in the game $G(A)$ as well.

Indeed, if σ is a winning strategy for Player II in the game $H(A)$, he can transform it to a winning strategy τ in the game $G(A)$. Just let τ answer a given finite sequence of moves of player I in $G(A)$ with a function in ω^ω that modulo finite dominates all answers the strategy σ can give to the same sequence enriched with some choices of natural numbers in $H(A)$. Since there are only countably many natural numbers, this is possible. Such a strategy τ must be winning for Player II. Indeed, if $\langle y_n : n \in \omega\rangle$ is any counterplay against τ that Player I won, then there would have to be an element $z \in \omega^\omega$ such that $f(z) = \langle y_{2n} : n \in \omega\rangle$, and then Player I would also win against the strategy σ with the moves $y_{2n} : n \in \omega$ and $z(n) : n \in \omega$.

For the comparison between the ω-Hechler ideal and the mutual domination ideal, assume first that $A \subset X$ is an analytic set and $\omega^\omega \setminus [A]_E$ is in the mutual domination ideal. I must show that $P_\omega \Vdash \dot{x}_{gen} \in [A]_E$. Suppose for contradiction that $p \in P_\omega$ is a condition forcing the opposite. Let M be a countable elementary submodel of a large enough structure containing p. I will construct a filter $g \subset P_\omega$ generic over M such that $p \in g$ and $\dot{x}_{gen}/g \in [A]_E$. By the Mostowski absoluteness for the model $M[g]$, this means that $M[g] \models \dot{x}_{gen}/g \in [A]_E$, contradicting the forcing theorem.

For the construction of the filter g, let σ be a winning strategy for Player I in the game $G(A)$, let $\langle D_n : n \in \omega\rangle$ enumerate all open dense subsets of P_ω in the model M, and by induction on $n \in \omega$ build filters $g_n \subset P$ and conditions $q_n \in P_\omega \cap M$ such that:

- writing $M_n = M[g_i : i \in n]$, the filter $g_n \subset P^{M_n}$ is P-generic over M_n;
- $p = q_0 \geq q_1 \geq \dots$ are conditions in $P_\omega \cap M$ such that $q_{n+1} \in D_n$ and $p_n \restriction n \in g_0 * g_1 * \dots * g_{n-1}$;
- writing x_n for the n-th Hechler real, $g(x_n)$ modulo finite dominates the function $\sigma(x_i : i \in n)$.

This is not difficult to do. Given p_n and $g_i : i \in n$, use Lemma 5.3.6 to find a filter $g_n \subset P^{M_n}$ generic over M_n such that $p_n(n) \in g_n$ and its generic real dominates the function $\sigma(x_i : i \in n)$. By the genericity of the filter $g_0 * \dots * g_n$ on the iteration of Hechler forcing of length $n+1$, there must be a condition $q_{n+1} \leq q_n$ in the model M such that $q \restriction n+1 \in g_0 * \dots * g_n$ and $q_{n+1} \in D_{n+1}$. This completes the induction step.

In the end, the filter generated by the conditions q_n for $n \in \omega$ is P_ω-generic over M, and its generic sequence $\vec{x}_{gen}$ equals to $\langle x_n : n \in \omega\rangle$. $\vec{x}_{gen}$ is a legal counterplay of Player II against the strategy σ by the last item of the induction hypothesis. As the strategy σ was winning for Player I, $\vec{x}_{gen} \in [A]_E$. At the same time, the forcing theorem applied in the model M shows that $M[\vec{x}_{gen}] \models \vec{x}_{gen} \notin [A]_E$, and by the Mostowski absoluteness then $\vec{x}_{gen} \notin [A]_E$ holds even in V, which is a contradiction.

□

Proof of Theorem 5.3.3. For (1), to show that $\kappa(E) \geq \mathfrak{c}^+$, let $C \subset \mathcal{P}(\omega)$ be an almost disjoint family of size continuum and let $a \subset \omega^\omega$ be the set of characteristic functions of finite unions of sets in C. Consider the poset $\mathrm{Coll}(\omega, a)$ and its name σ for an enumeration of the set a. I will show that σ is not $\bar{E}$-related to any name on a poset of size $< \mathfrak{c}$. Towards a contradiction, let Q be such a poset and τ be such a name. Each entry of τ is dominated by a characteristic function of a finite union of some elements of c, and by the small size of Q there is $c \in C$ which is forced by C not to appear in any of these finite unions. Then the characteristic function of c is forced to be not dominated by any element of τ, contradicting the assumption $\tau \; \bar{E} \; \sigma$.

To show that $\kappa(E) \leq \mathfrak{c}^+$, suppose that σ is a pinned name on some poset P. Let $a = \{z \in \omega^\omega : \exists p \; p \Vdash \exists n \; \check{z}$ is modulo finite dominated by $\sigma(n)\}$. Since the name σ is pinned, we have that in fact $a = \{z \in \omega^\omega : P \Vdash \exists n \; \check{z}$ is modulo finite dominated by $\sigma(n)\}$. We will show that the name σ is $\bar{E}$-equivalent to any $\mathrm{Coll}(\omega, a)$-name τ for a generic enumeration of the set a.

It is enough to show that $P \Vdash \forall n \; \exists z \in a \; \sigma(n)$ is modulo finite dominated by z. Suppose for contradiction that $p \in P$ is a condition and $n \in \omega$ is a number such that $p \Vdash \sigma(n)$ is not modulo finite dominated by any function in a. Use the pinned condition to find $m, l \in \omega$ and $q \in P$ and strengthen p if necessary so that $\langle p, q\rangle \Vdash \forall k > l \; \sigma_{\mathrm{left}}(n)(k) \leq \sigma_{\mathrm{right}}(m)(k)$. Let $z \in \omega^\omega$ be a function that assigns to each $k \in \omega$ the maximal number $h \in \omega$ such that there is $p' \leq p$ forcing $\sigma(n)(k) = h$ if such number exists. Note that for all $k > l$, such number must exist as otherwise it would be possible to find $p', q' \leq \langle p, q\rangle$ such that $\langle p', q'\rangle \Vdash \sigma_{\mathrm{left}}(n)(k) > \sigma_{\mathrm{right}}(m)(k)$. Now, $z \notin a$ since p forces $\sigma(n)$ to be modulo finite dominated by z. This means that $q \Vdash \sigma(m)$ does not dominate $\check{z}$ modulo finite and so there is $q' \leq q$ and $k > l$ such that $q' \Vdash \sigma(m)(k) < z(k)$. Find $p' \leq p$ forcing $\sigma(n)(k) = \check{z}(k)$; then $\langle p', q'\rangle \Vdash \sigma_{\mathrm{left}}(n)(k) > \sigma_{\mathrm{right}}(m)(k)$, contradicting the assumed properties of $\langle p, q\rangle$.

For (2), let $V[G]$ be some forcing extension in which there is a modulo finite increasing sequence $z = \langle z_\alpha : \alpha \in \omega_2\rangle$ of elements of ω^ω. Work in $V[G]$. Let R be the Namba forcing, adding a cofinal sequence $f : \omega \to \omega_2^V$ and let τ be the R-name for the composition $z \circ f$. The following three items show complete the proof of (2):

- The name τ is E-pinned. For any two functions $f, g : \omega \to \omega_2$ with cofinal rangle, the compositions $z \circ f$ and $z \circ g$ are E-related.
- The name τ is E-nontrivial. If $R \Vdash \tau \; E \; \check{x}$ for some $x \in X$, then the function $g : \omega \to \omega_2$ given by $g(n) =$the least $\alpha \in \omega_2$ such that z_α modulo finite dominates $x(n)$ would have to have cofinal range in ω_2.
- $\kappa(\tau)$ does not exist. The poset R preserves $\aleph_1$, so the only option for $\kappa(\tau)$ is $\aleph_0$. However, $\mathrm{Coll}(\omega, \omega)$ is just Cohen forcing, all of its pinned names are trivial by Theorem 3.3.2, and so none of them can be equivalent to the E-nontrivial name τ.

For (3), consider the poset $P = P_{\omega_2} * \dot{R}$, where P_{ω_2} is the finite support iteration of Hechler forcing of length ω_2, adding a sequence $z \in (\omega^\omega)^{\omega_2}$, and

$\dot{R}$ is the Namba forcing, adding an increasing cofinal function $f : \omega \to \omega_2$. Consider the P-name τ for the element of X defined as $\tau = z \circ f$. Consider also the name $h(\tau)$ for an element of Y.

First, move to the P_{ω_2} extension. The remainder of the name τ is an E-pinned $\dot{R}$-name by (2); the E-equivalence class of τ does not depend on the particular cofinal function from ω to ω_2 as the points on the sequence z are modulo finite increasing. As h remains a homomorphism, the remainder of the name $h(\tau)$ must be an F-pinned name. Now, $\dot{R}$ preserves $\aleph_1$ and F is still a cardinalistic equivalence, it follows that $\kappa(\text{remainder of } h(\tau)) = \aleph_0$; in other words, the remainder of $h(\tau)$ is a trivial F-pinned name, whose equivalence class is represented by some element of Y in the P_{ω_2}-extension. Let $\dot{y}$ be some P_{ω_2} name for this element.

Back to the ground model. I will argue that $\dot{y}$ is an F-pinned name in the poset P_{ω_2}. First, use the c.c.c. of the poset to find an ordinal $\alpha \in \omega_2$ such that $\dot{y}$ is in fact a P_α-name. Suppose for contradiction that $P_\alpha \times P_\alpha \Vdash \dot{y}_{\text{left}}\ F\ \dot{y}_{\text{right}}$ fails. Then, obtain filters $G_0 \times G_1 \subset P_\alpha \times P_\alpha$ generic over V such that $\dot{y}/G_0\ F\ \dot{y}/G_1$. Write $z_0, z_1 \in (\omega^\omega)^\alpha$ for the sequences of Hechler reals added by G_0 and G_1 respectively. Find a filter $H \subset P_{\omega_2 \setminus \alpha}$ generic over $V[G_0, G_1]$ and write $z_2 \in (\omega^\omega)^{\omega_2 \setminus \alpha}$ for its sequence of Hechler reals. Since the iteration $P_{\omega_2 \setminus \alpha}$ is a finite support iteration of Suslin forcings, z_2 is in fact $V[G_0]$-generic for that iteration as evaluated in the model $V[G_0]$, and similarly for $V[G_1]$ by Theorem 2.3.9. Thus, the sequences $z_0^\frown z_2$ and $z_1^\frown z_2$ are both V-generic for P_{ω_2}. If $f_0, f_1 : \omega \to \omega_2$ are Namba-generic sequences over the models $V[z_0, z_2]$ and $V[z_1, z_2]$ respectively with minimal value greater than α, then the points $z_2 \circ f_0, z_2 \circ f_1 \in X$ are E-related. As h is a homomorphism, the points $h(z_2 \circ f_0), h(z_2 \circ f_1) \in Y$ are F-related. By the forcing theorem applied in the respective models $V[z_0, z_2]$ and $V[z_1, z_2]$ though, the former is F-related to $\dot{y}/z_0$ and the latter is F-related to $\dot{y}/z_1$, which are non-F-related points, a contradiction.

Since the poset P_{ω_2} is c.c.c., all F-pinned names in it must be trivial by Theorem 3.3.2. In particular, there is an element $y_0 \in Y$ such that $P_{\omega_2} \Vdash \dot{y}\ F\ \check{y}_0$. Together with the choice of the name $\dot{y}$, this gives $P_{\omega_2} * \dot{R} \Vdash h(\tau)\ F\ \check{y}_0$. Write $A = h^{-1}[y_0]_F$; this is an analytic E-invariant set. I will prove that $X \setminus A \in I$.

Suppose for contradiction that it is not, and so Player II has a winning strategy σ in the game $G(A)$. Let M be a countable elementary submodel of a large enough structure, let $\langle \alpha_n : n \in \omega \rangle$ be an increasing sequence cofinal in $\omega_2 \cap M$ with $\alpha_0 = 0$, and by induction on $n \in \omega$ build sequences $\langle z_\alpha : \alpha_n \leq \alpha < \alpha_{n+1} \rangle$ so that

- $\langle z_\alpha : \alpha_n \leq \alpha < \alpha_{n+1} \rangle$ is a sequence generic over the model $M_n = M[z_\alpha : \alpha \in ga_n]$ for the poset S_n which is the finite support iteration of Hechler forcing along the interval $[\alpha_n, \alpha_{n+1})$.

- the point z_{α_n} modulo finite dominates the response of the strategy σ to the moves $z_{\alpha_i} : i \in n$.

The induction step is performed easily with the help of Lemma 5.3.6 applied to the model M_n. In th end, consider the sequence $\langle z_\alpha : \alpha \in \omega_2^M \rangle$. Observe that it is generic over the model M for the finite iteration S of the Hechler forcing of length ω_2^M. To see this, note that S is a c.c.c. poset in M, therefore every maximal antichain $A \in M$ of S is already a subset of $S_0 * S_1 * \cdots * S_n$, and as such is met by the filter $g_0 * g_1 * \cdots * g_n \subset S_0 * S_1 \ldots S_n$, which is generic over M and a subset of g. Let $\langle \beta_n : n \in \omega \rangle$ be a Namba-generic sequence over the model M. The sequences $\langle z_{\beta_n} : n \in \omega \rangle$ and $\langle z_{\alpha_n} : n \in \omega \rangle$ are E-related. Now, the sequence $\langle z_{\alpha_i} : i \in \omega \rangle$ is a valid counterplay of Player I to the strategy σ. As σ was a winning strategy for Player II, Player I must have lost, and so $\langle z_{\alpha_n} : n \in \omega \rangle \notin A$. On the other hand, the sequence $\langle z_{\beta_n} : n \in \omega \rangle$ belongs to A by the forcing theorem applied in the model M. This contradicts the fact that the set A is E-invariant. □

The position of the mutual domination equivalence relation among the other analytic equivalence relations is unclear at this point. I include the following result:

Theorem 5.3.9. (Kechris, Macdonald) *E_{K_σ} is Borel reducible to the mutual domination equivalence relation.*

Proof. Write E for the mutual domination equivalence, with $\mathrm{dom}(E) = (\omega^\omega)^\omega = X$. Let $m \mapsto (m_0, m_1)$ be a bijection between ω and $\omega \times \omega$. Let $h : \omega^\omega \to X$ be the Borel function defined by $h(y)(n)(m) = 1$ if $y(m_0) + n > m_1$, and $h(y)(n)(m) = 0$ otherwise. I claim that this is a reduction of E_{K_σ} to E.

Indeed, if $y, z \in \omega^\omega$ are E_{K_σ}-equivalent, then there is a number n' such that $y + n'$ at all entries dominates the function z and $z + n'$ at all entries dominates the function y. For every number $n \in \omega$ then, $h(y)(n + n')$ dominates $h(z)(n)$ and $h(z)(n + n')$ dominates $h(y)$, and therefore $h(y)\ E\ h(z)$. On the other hand, if $y, z \in \omega^\omega$ are not E_{K_σ}-equivalent, then either there is no n such that $y + n$ dominates z, or there is no n such that $z + n$ dominates y. Suppose for definiteness that the former is the case, and observe that $h(z)(0)$ is not modulo finite dominated by any $h(y)(n)$ for any $n \in \omega$. □

Chapter 6

Trim equivalence relations

6.1 Turbulence revisited

In this chapter, I will introduce a class of analytic equivalence relations somewhat reminiscent of pinned relations. It is nevertheless more useful fo obtaining ergodicity results among very simple equivalence relations. To motivate the definitions, I will prove a restatement of Hjorth's concept of turbulence in terms of models of set theory. Recall:

Definition 6.1.1. [12, Section 13.1] Let $G \curvearrowright X$ be a continuous Polish group action on a Polish space.

1. If $U \subset G, O \subset X$ are sets and $x \in O$ then U, O-orbit of x is the set $\{y \in O :$ there are points $\{g_i : i \in n\}$ in U and $\{x_i : i \in n+1\}$ in O such that $x_0 = x$, $x_n = y$ and $x_{i+1} = g_i \cdot x_i$.
2. The action is *turbulent* at $x \in X$ if for all open sets $U \subset G$ and $O \subset X$ with $1 \in U$ and $x \in O$ the U, O-orbit of X is somewhere dense.
3. The action is *generically turbulent* if its orbits are meager and dense and the set of points of turbulence is comeager.

To restate turbulence in forcing terms, recall the poset P_X consisting of all nonempty open subsets of X ordered by inclusion–Definition 2.2.1. Write $\dot{x}_{gen}$ for the P_X-name for the unique point in all sets in the P_X-generic filter. It is clear that P_X is in the forcing sense isomorphic to Cohen forcing. A point $x \in X$ is P_X-generic over the ground model if and only if it belongs to all open dense subsets of X coded in the ground model. I will use this poset in the following situation. Let $G \curvearrowright X$ be a continuous Polish group action, consider the product $P_G \times P_X$ and its associated names $\dot{g}_{\mathrm{gen}}$ and $\dot{x}_{gen}$ for generic elements of G and X. By the usual Kuratowski–Ulam argument, a pair $\langle g, x\rangle$ is $P_G \times P_X$-generic over the ground model if and only if it belongs to every open dense subset of $G \times X$ coded in the ground model. In the generic extension, the group G as well as the space X and the action can be reinterpreted using the standard absoluteness arguments.

Theorem 6.1.2. *Let $G \curvearrowright X$ be a continous Polish group action whose orbits are meager and dense. The following are equivalent:*

1. *the action is generically turbulent;*
2. *$P_G \times P_X$ forces $V[\dot{x}_{gen}] \cap V[\dot{g}_{\text{gen}} \cdot \dot{x}_{gen}] = V$.*
3. *$P_G \times P_X$ forces $2^\omega \cap V[\dot{x}_{gen}] \cap V[\dot{g}_{\text{gen}} \cdot \dot{x}_{gen}] = 2^\omega \cap V$.*

Thus, if the action is generically turbulent, then the poset $P_G \times P_X$ adds two points $x_0 = \dot{x}_{gen}, x_1 = \dot{g}_{\text{gen}} \cdot \dot{x}_{gen} \in X$ such that $x_0\ E\ x_1$ (as one of them is obtained by an application of the generic group element from the other), $V[x_0] \cap V[x_1] = V$ (that is the contents of the theorem), and x_0 is not E-related to any ground model point of X (as the orbits are meager and x_0, being Cohen-generic over V, does not belong to any ground model coded meager subset of X). It will turn out later that this fine task is impossible to perform for many equivalence relations E which do not come from turbulent actions, and this will lead to a number of new ergodicity results.

Proof of Theorem 6.1.2. I first need to evaluate the genericity of the point $\dot{g}_{\text{gen}} \cdot \dot{x}_{gen} \in X$. This is the contents of the following easy claim.

Claim 6.1.3. *$P_G \times P_X$ forces the following.*

1. *$\dot{g}_{\text{gen}} \cdot \dot{x}_{gen}$ is P_X-generic over the ground model;*
2. *if $h, k \in G$ are ground model elements of the group then $\langle \dot{g}_{\text{gen}} h, k \cdot \dot{x}_{gen} \rangle$ is $P_G \times P_X$-generic pair over the ground model.*

Proof. For (1), let $D \subset X$ be an open dense set, and consider the set $C = \{\langle g, x\rangle \in G \times X : g \cdot x \in D\}$. It is not difficult to see that C is open dense in $G \times X$. Thus, $P_G \times P_X$ forces the generic pair $\langle \dot{g}_{\text{gen}}, \dot{x}_{gen}\rangle$ to belong to the set C, and consequently the point $\dot{g}_{\text{gen}} \cdot \dot{x}_{gen}$ to belong to D. As $D \subset X$ was an arbitrary open dense set, (1) follows.

For (2), consider the maps $U \mapsto Uh$ for $U \in P_G$ and $O \mapsto kO$ for $U \in P_X$. These are automorphisms of P_G and P_X respectively, and so carry generic filters to generic filters. (2) follows immediately. $\square$

I will also need the following well-known complexity calculation.

Claim 6.1.4. *The set $B = \{x \in X :$ the action is not turbulent at $x\} \subset X$ is Borel.*

Proof. Let $H \subset G$ be a countable dense subgroup, and choose also a countable basis both for the topology of X and G. By the continuity of the group action, a point $x \in X$ is in B iff there are basic open sets $O \subset X$ and $U \subset G$ containing x and 1_G respectively such that the countable $U \cap H, O$-orbit of x is nowhere dense. This is easily checked to be a Borel condition. $\square$

For the implication (1)$\rightarrow$(2), suppose that (1) holds. Thus, the Borel set of all turbulent points in X is comeager and so $P_X \Vdash$ the action is turbulent at $\dot{x}_{gen}$. To confirm (2), suppose that $q \in P_G$ and $p \in P_X$ are conditions τ, σ are P_X-names for sets of ordinals such that $\langle q, p\rangle \Vdash_{P_G \times P_X} \tau/\dot{g}_{\mathrm{gen}} \cdot \dot{x}_{gen} = \sigma/\dot{x}_{gen}$. I will find a condition $p' \leq p$ which decides the membership of every ordinal in σ. An obvious density argument then confirms (2). Let $q' \leq q$ be a condition and $U \subset G$ be an open neighborhood of 1_G such that $q' \cdot U^{-1} \subset q$. Write also $O = p$ and $\tilde{U}$ for the set of ground model elements of U in the P_X-extension.

Claim 6.1.5. *The condition p forces the following in P_X:*

1. *if $h \in U$ is a ground model element such that $h \cdot \dot{x}_{gen} \in p$, then $\sigma/\dot{x}_{gen} = \sigma/h \cdot \dot{x}_{gen}$;*
2. *the evaluation σ/y is the same for every point $y \in X$ on the $\tilde{U}, O$-orbit of $\dot{x}_{gen}$.*

Proof. Let $x \in X$ be a P_X-generic point over V, in p. For (1), let $h \in U$ be a ground model element such that $h \cdot x \in p$. To compare σ/x and $\sigma/h \cdot x$, let g be a P_G-generic element over $V[G]$, in q'. By the product forcing theorem, the pair $\langle g, x\rangle$ is $P_G \times P_X$-generic over V. The pair $\langle gh^{-1}, h \cdot x\rangle$ is also $P_G \times P_X$-generic over V by Claim 6.1.3(2). Both pairs meet the condition $\langle q, p\rangle$. Thus, by the forcing theorem applied to these two pairs, $\sigma/x = \tau/g \cdot x = \tau/(gh^{-1} \cdot h \cdot x) = \sigma/h \cdot x$. This completes the proof of (1).

(2) now follows from (1) applied repeatedly at each step of the walk leading from x to y. Note that all points on the $\tilde{U}, O$ orbit are P_X-generic over the ground model. □

Now, let $x \in X$ be a P_X-generic point over V. Since the point x is turbulent for the action by (1), the U, O-orbit of x is dense in some basic open set $p' \leq p$. I claim that the condition $p' \leq p$ decides the membership of every ordinal in σ as required. Indeed, suppose that α is an ordinal and $p'_0, p'_1 \leq p'$ are conditions deciding the statement $\check{\alpha} \in \sigma$ in different ways. The U, O-orbit of x is dense in p', so it visits both sets p'_0, p'_1. Since the group action is continuous and $\tilde{U} \subset U$ is dense, the $\tilde{U}, O$-orbit of x is dense in the U, O-orbit of x and so it also visits the sets p'_0, p'_1 in respective points y_0, y_1. These points are P_X-generic over V and $\sigma/y_0 = \sigma/y_1$ by Claim 6.1.5(2). By the forcing theorem though, the sets σ/y_0, σ/y_1 should differ in the membership of the ordinal α. This contradiction completes the proof of (2).

(2) implies (3) is trivial. To show that (3) implies (1), suppose that (1) fails. Then, the Borel set B of nonturbulent points is not meager, and so some condition $p \in P_X$ forces $\dot{x}_{gen} \in \dot{B}$. Strengthening the condition p if necessary, I can also find open sets $U \subset G$ and $O \subset X$ such that $1_G \in U$, $p \subset O$, and $p \Vdash$ the U, O-orbit of $\dot{x}_{gen}$ is nowhere dense. Strengthening p further if necessary, I can find a basic open set $q \subset U$ such that $q^{-1} \subset U$ and $q \cdot p \subset O$. I claim that $\langle q, p\rangle$ in the poset $P_G \times P_X$ forces $2^\omega \cap V[\dot{x}_{gen}] \cap V[\dot{g}_{\mathrm{gen}} \cdot \dot{x}_{gen}] \neq 2^\omega \cap V$, violating (3). Let $\dot{D}$ be the name for the set of all basic open sets disjoint from the U, O-local

orbit of $\dot{x}_{gen}$; I will show that $\langle p, q\rangle$ forces $\dot{D} \in V[\dot{x}_{gen}] \cap V[\dot{g}_{\mathrm{gen}} \cdot \dot{x}_{gen}]$ and $\dot{D} \notin V$. This violates (3) as D can be easily coded as an element of 2^ω. Indeed, the conditions p, q are chosen so that the U, O-orbits of $\dot{x}_{gen}$ and $\dot{g}_{\mathrm{gen}} \cdot \dot{x}_{gen}$ are the same, as each of the two points is on the U, O-orbit of the other; thus, the set D belongs to both $V[\dot{x}_{gen}]$ and $V[\dot{g}_{\mathrm{gen}} \cdot \dot{x}_{gen}]$. To see that $\dot{D} \notin V$, note that the U, O-orbit of $\dot{x}_{gen}$ is nowhere dense, so $\bigcup D$ is open dense, and it does not contain the point $\dot{x}_{gen}$. This would be impossible if $D \in V$ as $\dot{x}_{gen}$ is forced to belong to all open dense subsets of X coded in the ground model. In conclusion, the failure of (1) implies the failure of (3). $\square$

The forcing restatement of turbulence can be used to provide a conceptual and short proof of every consequence of turbulence known so far. Here, I will illustrate it on a theorem of Hjorth that equates turbulence with generic F_2-ergodicity.

Theorem 6.1.6. *Let $G \curvearrowright X$ be a continuous action of a Polish group on a Polish space with dense meager orbits. Let E be the resulting orbit equivalence relation on X. Then, the following are equivalent:*

1. *the action is generically turbulent;*
2. *E is generically F_2-ergodic.*

Here, E is generically F_2-ergodic if for every Borel homomorphism h of E to F_2, there is a single F_2-class with a comeager h-preimage.

Proof. Now, the implication (1)→(2) was proved by Hjorth [13, Theorem 12.5] and follows from Theorem 7.1.1 below. To prove the opposite implication, I will first construct a certain critical Borel homomorphism of E to F_2. Let M be a countable elementary submodel of a large structure containing the condition p and let $B \subset X$ be the set of all P_X-generic points over M. Let $\bar{h} : B \to [2^\omega]^{\aleph_0}$ be the map defined by $\bar{h}(x) = \{y \in 2^\omega \cap M[x] : P_G \Vdash \check{y} \in M[\dot{g}_{\mathrm{gen}} \cdot x]\}$. I will show that this is a homomorphism of E to F_2 in the sense that if $x_0, x_1 \in B$ are E-related points then $\bar{h}(x_0) = \bar{h}(x_1)$. This follows from the next two claims.

Claim 6.1.7. *If $x \in B$ and $y \in M[x] \cap 2^\omega$, then the statement $\check{y} \in M[\dot{g}_{\mathrm{gen}} \cdot x]$ is decided by the largest condition in P_G.*

Proof. Suppose that this fails and find $y \in M[x] \cap 2^\omega$ and conditions $q_0, q_1 \in P_G$ such that $q_0 \Vdash \check{y} \in M[\dot{g}_{\mathrm{gen}} \cdot x]$ and $q_1 \Vdash \check{y} \notin M[\dot{g}_{\mathrm{gen}} \cdot x]$. Strengthening the conditions q_0, q_1 if necessary, I may find an element $h \in M \cap G$ such that $hq_0 = q_1$. Let $g \in G$ be a point P_G-generic over the model $M[x]$ such that $g \in q_0$. Then $hg \in q_1$ is also a point P_G-generic over the model $M[x]$ and $M[g \cdot x] = M[hg \cdot x]$ since the points $g \cdot x, hg \cdot x \in X$ can be obtained from each other by acting by the element h or h^{-1}. At the same time, the forcing theorem applied in $M[x]$ implies that $y \in M[g \cdot x]$ and $y \notin M[hg \cdot x]$, which is a contradiction. $\square$

Claim 6.1.8. *If $x_0, x_1 \in B$ are E-related sets and $y \in M[x_0] \cap 2^\omega$ is such that $P_G \Vdash \check{y} \in M[\dot{g}_{\text{gen}} \cdot x_0]$, then $y \in M[x_1]$ and $P_G \Vdash \check{y} \in M[\dot{g}_{\text{gen}} \cdot x_1]$.*

Proof. Let $h \in G$ be an element such that $h \cdot x_0 = x_1$. Let N be a countable elementary submodel of a large structure containing the objects G, X, M, x_0, x_1, h. Let $g \in G$ be a P_G-generic point over the model N. Then, $gh \in G$ is a P_G-generic point over the model N as well. Let $x_2 = gh \cdot x_0 = g \cdot x_1$. By the claim assumption, $y \in M[x_2]$. Since g is generic over the model N which contains y, and $M[x_2] \subset M[x_1][g]$, it follows that $y \in M[x_1]$. Finally, $P_G \Vdash \check{y} \in M[\dot{g}_{\text{gen}} \cdot x_1]$ follows by the forcing theorem and Claim 6.1.7. □

Now, by Lemmas 2.4.2 and 2.4.3, find a Borel function $h : B \to (2^\omega)^\omega$ such that for every $x \in B$, the value $h(x)$ is an enumeration generic over the model $M[x]$ of the set $\bar{h}(x) \in M[x]$. The function h is a homomorphism of E to F_2 by Claim 6.1.8. Use Lemma 2.1.5 to extend the homomorphism h to the whole space X and by abuse of notation denote the extension by h as well. I will show that if (1) fails then every comeager set contains two points with F_2-unrelated h-images, confirming the failure of (2).

First, use the failure of (1) to find a condition $p \in P_X$ which forces that for some $y \in 2^\omega$ in the P_X-extension, $P_G \Vdash y \in V[\dot{g}_{\text{gen}} \cdot \dot{x}_{gen}]$. Let σ be a name for such an element, by the elementarity of the model M I can require $\sigma \in M$. Now, suppose that $C \subset X$ is a comeager set. Let N be a countable elementary submodel of a large structure containing h, M, C and let $\langle x_0, x_1 \rangle \in (C \cap p)^2$ be a $P_X \times P_X$-generic pair of points over the model N. I claim that $h(x_0)\ F_2\ h(x_1)$ fails as desired. To see this, note that $\langle x_0, x_1 \rangle \in C^2$ is a $P_X \times P_X$-generic pair of points over the model M as well. By the product forcing theorem applied in the model M, $\sigma/x_0 \notin M[x_1]$ and $\sigma/x_1 \notin M[x_0]$. Thus, $\sigma/x_0 \in \bar{h}(x_0) \setminus \bar{h}(1)$ and $\sigma/x_1 \in \bar{h}(x_1) \setminus \bar{h}(x_0)$, in particular $h(x_0)\ F_2\ h(x_1)$ fails. □

6.2 Definitions and basic concerns

In this section, I will define the notion of *trimness*, which is a strengthening of the pinned property of equivalence relations tailored to provide ergodicity results for turbulent action orbit equivalence relations.

Definition 6.2.1. Let E be an analytic equivalence relation on a Polish space X. E is *trim* if whenever $V[G_0]$ and $V[G_1]$ are two forcing extensions of the ground model and $x_0 \in V[G_0]$ and $x_1 \in V[G_1]$ are two E-related points of the space X, then either $V[G_0] \cap V[G_1] \neq V$, or there is $x \in V$ such that $x\ E\ x_0\ E\ x_1$.

To see the link between the pinned and trim equivalence relations more clearly, consider the following definitions:

Definition 6.2.2. Let E be an analytic equivalence relation on a Polish space X, and let τ_0, τ_1 be names for an element of X on the respective posets P_0, P_1.

1. $\langle P_0, \tau_0\rangle \ \tilde{E}\ \langle P_1, \tau_1\rangle$ if for every $p_0 \in P_0$ and $p_1 \in Q_1$, in some generic extension there are filters $G_0 \subset P_0, G_1 \subset P_1$, each generic over V, such that $\tau_0/G_0\ E\ \tau_1/G_1$ and $V[G_0] \cap V[G_1] = V$;

2. the name τ_0 is E-*trim* if $\langle P_0, \tau_0\rangle\ \tilde{E}\ \langle P_0, \tau_0\rangle$;

3. the name τ_0 is E-*trivial* if $P_0 \Vdash$ for some $x \in X$ in the ground model, $\tau_0\ E\ x$.

Theorem 6.2.3. *Let E be an analytic equivalence relation on a Polish space X.*

1. *A name is E-trim if and only if it belongs to the domain of $\tilde{E}$;*

2. *$\tilde{E}$ is an equivalence relation on trim names;*

3. *E is trim iff all E-trim names are E-trivial.*

Proof. For (1) and (2), it is enough to show that $\tilde{E}$ is transitive, because it is clearly symmetric, and every transitive and symmetric relation is an equivalence relation on its domain.

Thus, suppose that $\langle P_0, \tau_0\rangle\ \tilde{E}\ \langle P_1, \tau_1\rangle\ \tilde{E}\ \langle P_2, \tau_2\rangle$. I must conclude that $\langle P_0, \tau_0\rangle\ \tilde{E}\ \langle P_2, \tau_2\rangle$. Let $p_0 \in P_0$ and $p_2 \in P_2$ be conditions; in some generic extension, I must produce filters $G_0 \subset P_0$ and $G_2 \subset P_2$ separately generic over V such that $p_0 \in G_0$, $p_2 \in G_2$, $V[G_0] \cap V[G_2] = V$, and $\tau_0/G_0\ E\ \tau_2/G_2$. Towards the construction of these filters, use the definition of $\tilde{E}$ and the assumptions to conclude that

- $P_0 \Vdash$ in some further extension, there is a filter $G_1 \subset P_1$ generic over V such that $V[G_0] \cap V[G_1] = V$ and $\tau_0/G_0\ E\ \tau_1/G_1$, where G_0 is the P_0-generic filter;

- $P_1 \Vdash$ in some further extension, there is a filter $G_2 \subset P_2$ generic over V such that $p_2 \in G_2$ and $V[G_1] \cap V[G_2] = V$ and $\tau_1/G_1\ E\ \tau_2/G_2$, where G_1 is the P_1-generic filter.

Let $G_0 \subset P_0$ be a filter generic over V with $p_0 \in G_0$. Use the first item above to pass to some larger forcing extension $V[G_0][H_0]$ which contains some filter $G_1 \subset P_1$ such that $V[G_0] \cap V[G_1] = V$ and $\tau_0/G_0\ E\ \tau_1/G_1$. Let Q_1 be a poset in $V[G_1]$ such that $V[G_0][H_0] = V[G_1][H_1]$ for some filter $H_1 \subset Q_1$ generic over $V[G_1]$. Let Q_2 be a poset in $V[G_1]$ which forces the existence of a filter $G_2 \subset R$ as in the second item above. Let $H_2 \subset Q_2$ be a filter generic over the model $V[G_1][H_1]$ and let $G_2 \in V[G_1][H_1]$ be a filter as in the second item above. I claim that the filters G_0, G_2 work.

First of all, $\tau_0/G_0\ E\ \tau_1/G_1$ by the choice of G_1, and $\tau_1/G_1\ E\ \tau_2/G_2$ by the choice of G_2. By the transitivity of the relation E, $\tau_0/G_0\ E\ \tau_2/G_2$ as desired. Second, $V[G_1][H_1] \cap V[G_1][H_2] = V[G_1]$ by the product forcing theorem, Fact 2.2.6. Since $G_0 \in V[G_1][H_1]$ and $G_2 \in V[G_1][H_2]$, it follows that

$V[G_0] \cap V[G_2] \subset V[G_1]$. Since $V[G_0] \cap V[G_1] = V$ by the choice of G_2, it follows that $V[G_0] \cap V[G_2] = V$ and (1) has been verified.

For the left-to-right implication of (3), if τ is a nontrivial trim name on a poset P, then by the definition of a trim name and (1), in some generic extension there are filters $G_0 \subset P, G_1 \subset P$, each generic over V, such that $\tau/G_0 \; E \; \tau/G_1$ and $V[G_0] \cap V[G_1] = V$. Since the name τ is nontrivial, the class $[\tau/G_0]_E$ is not represented in the ground model. This shows that E is not trim.

The right-to-left implication of (3) is more difficult. Suppose that E is not trim; we must produce a nontrivial E-trim name. As E is not trim, there are posets P_0, P_1, Q, Q-names $\dot{G}_0, \dot{G}_1$ for filters on $\check{P}_0, \check{P}_1$ generic over V, and a P_0-name τ_0 and a P_1-name τ_1 such that $R \Vdash \tau_0/\dot{G}_0 \; E \; \tau_1/\dot{G}_1$, $V[\dot{G}_0] \cap V[\dot{G}_1] = V$, and the class $[\tau_0/\dot{G}_0]_E$ is not represented in V. I will find conditions $p_0 \in P_0$ and $p_1 \in P_1$ such that $\langle P_0 \restriction p_0, \tau_0\rangle \; \tilde{E} \; \langle P_1 \restriction p_1, \tau_1\rangle$; this will complete the proof.

Let $H \subset Q$ be a filter generic over V, $G_0 \subset P_0$ and $G_1 \subset P_1$ the associated filters generic over V in the model $V[H]$, and write $x_0 = \tau_0/G_0$ and $x_1 = \tau_1/G_1$. In particular, $V[G_0] \cap V[G_1] = V$ and $x_0 \; E \; x_1$. Let κ be a cardinal larger than $|Q|$, and in the respective models $V[G_0]$ and $V[G_1]$ define the sets $A_0 = \{p \in P_1 : \mathrm{Coll}(\omega, \kappa) \Vdash \exists h \subset P_1 \; h$ is generic over V, $p \in h$, and $x_0 \; E \; \tau/h\}$ and $A_1 = \{p \in P_1 : \mathrm{Coll}(\omega, \kappa) \Vdash \exists h \subset P_1 \; h$ is generic over V, $p \in h$, and $x_1 \; E \; \tau/h\}$. Both are subsets of P_1.

Claim 6.2.4. *$A_0 = A_1$.*

Proof. First, use the homogeneity of $\mathrm{Coll}(\omega, \kappa)$ to show that in the definitions of A_0, A_1, the $\mathrm{Coll}(\omega, \kappa) \Vdash$ sign can be equivalently replaced with "some condition in $\mathrm{Coll}(\omega, \kappa)$ forces". Let $K \subset \mathrm{Coll}(\omega, \kappa)$ be a filter generic over $V[H]$. By Fact 2.2.3, the model $V[H][K]$ is a $\mathrm{Coll}(\omega, \kappa)$-extension of both $V[G_0]$ and $V[G_1]$. Thus, by the forcing theorem in the models $V[G_0]$ and $V[G_1]$, in the model $V[H][K]$ the set A_0 is equal to $\{p \in P_1 : \exists h \subset P_1 \; h$ is generic over V, $p \in h$, and $x_0 \; E \; \tau/h\}$ and the set A_1 is equal to $\{p \in P_1 : \exists h \subset P_1 \; h$ is generic over V, $p \in h$, and $x_1 \; E \; \tau/h\}$. As $x_0 \; E \; x_1$, these two sets are equal. □

Since $A_0 \in V[G_0]$ and $A_1 \in V[G_1]$ and $V[G_0] \cap V[G_1] = V$, it follows that $A_0 = A_1 \in V$. By the definition of A_1, the set $A_1 \in V$ contains the filter $G_1 \supset P_1$ generic over V; this means that it has to be dense under some condition $p_1 \in P_1$. Let $q \in Q$ be some condition deciding the value of A_1 and $p_1 \in P_1$. Let $p_0 \in P_0$ be some condition such that every generic filter on P_0 containing p_0 can be extended to a generic filter on Q containing q. I claim that back in the ground model, $\langle P_0 \restriction p_0, \tau_0\rangle \; \tilde{E} \; \langle P_1 \restriction p_1, \tau_1\rangle$; this will complete the proof as the name σ is E-nontrivial by the initial assumptions.

Indeed, suppose that $p_0' \leq p_0$ and $p_1' \leq p_1$ are conditions in P_0, P_1 respectively. By the choice of p_0, q, there is a filter $H' \subset Q$ generic over V containing q such that the derived filter $G_0' \subset P_0$ contains the condition p_0'. Let $G_1' \subset P_1$ be the derived filter generic over V. The challenge is that the filter G_1' may not contain the condition p_1'.

Let $K \subset \mathrm{Coll}(\omega, \kappa)$ be a filter generic over $V[H']$. By the definition of the set A_1 in the model $V[G_1']$, in $V[G_1'][K]$ there is a filter $G_1'' \subset P_1$ generic over V, containing the condition p_1', such that $\tau_1/G_1' \mathrel{E} \tau_1/G_1''$. Clearly, $\tau_0/G_0' \mathrel{E} \tau_1/G_1''$ by the transitivity of the equivalence relation E. I will show that $V[G_0'] \cap V[G_1''] = V$. Indeed, $V[H'] \cap V[G_1'][K] \subset V[G_1']$ by the product forcing theorem. Since $G_0' \in V[H']$ and $G_1'' \in V[G_1'][K]$, it follows that $V[G_0'] \cap V[G_1''] \subset V[G_1']$. By the initial assumptions, $V[G_0'] \cap V[G_1'] = V$, so $V[G_0'] \cap V[G_1''] = V$ as desired. □

Theorem 6.2.5. *If E, F are analytic equivalence relations on Polish spaces X, Y, $E \leq_{\mathrm{wB}} F$, and F is trim, then E is trim.*

Proof. Let $f : X \to Y$ be a Borel function which is a reduction of E to F except on the set $[a]_E$ for some countable set $a \subset X$. Let $V[G_0], V[G_1]$ be two generic extensions such that $V[G_0] \cap V[G_1] = V$, and let x_0, x_1 be E-related points in the respective models $V[G_0], V[G_1]$. I must find a ground model element in their E-equivalence class. If $x \in [a]_E$ then this is clear, so assume that $x \notin [a]_E$.

Consider the points $y_0 = f(x_0) \in V[G_0] \cap Y$ and $y_1 = f(x_1) \in V[G_1] \cap Y$. These are F-related, as f is a reduction in the model $V[G_0, G_1]$, and by the trimness assumption on F there is a ground model point $y \in Y$ in their E-equivalence class. By the Mostowski absoluteness between V and $V[G_0]$, there is a ground model point $x \in X \setminus [a]_E$ such that $f(x) \mathrel{F} y$. Since f is a reduction of E to F outside $[a]_E$, the point x works as required. □

6.3 Variations: classes of forcings

The trim concept allows many useful variations.

Definition 6.3.1. Let $\mathfrak{P}$ be a class of forcing notions closed under restriction (if $P \in \mathfrak{P}$ then $P \restriction p \in \mathfrak{P}$ for all $p \in P$). Let E be an analytic equivalence relation on a Polish space X. Call E $\mathfrak{P}$-trim if all E-trim names on posets with property $\mathfrak{P}$ are trivial.

A brief perusal of the definition reveals that a more restrictive class of forcing notions will lead to a larger class of equivalence relations. While a priori any class $\mathfrak{P}$ may be useful, I will find use only for proper-trimness. The most prominent example of an equivalence relation which is proper-trim but not trim is F_2.

The properties of the class of trim equivalence relations proved above remain in force for $\mathfrak{P}$-trim equivalence relations, with a literal repetition of the arguments. Thus,

Theorem 6.3.2.

1. *An analytic equivalence relation E on a Polish space X is $\mathfrak{P}$-trim iff there are no nontrivial E-trim names on posets with property $\mathfrak{P}$.*

2. *If E, F are analytic equivalence relations on Polish spaces X, Y, $F \leq_{\mathrm{wB}} E$, and E is $\mathfrak{P}$-trim, then F is $\mathfrak{P}$-trim.*

The main result of this section is the following theorem, showing that some of these a priori interesting variations really boil down to the case of Cohen forcing.

Theorem 6.3.3. *Let E be an analytic equivalence relation on a Polish space X.*

1. *E is proper-trim iff E is Cohen-trim;*
2. *E is trim iff E is pinned and Cohen-trim;*
3. *if E is Borel reducible to an orbit equivalence relation generated by a continuous action of a Polish group, then E is $\aleph_1$-preserving trim iff E is Cohen-trim.*

Proof. For (1), since the Cohen forcing is proper, it is clear that every proper-trim equivalence relation is also Cohen-trim. For the opposite implication, suppose then that E is not proper-trim. Thus, there are posets P, Q, R, R-names $\dot{G}$, $\dot{H}$ for filters on P, Q respectively generic over V, and a P-name τ and a Q-name σ for elements of the space X such that $R \Vdash \tau/\dot{G}\ E\ \sigma/\dot{H}$, $\tau/\dot{G}$ has no E-equivalent in the ground model, and P is proper. Standard forcing manipulations show that I may assume that P, Q are in fact regular subposets of R and $\dot{G}, \dot{H}$ are R-names for the intersection of the R-generic filter with P, Q respectively.

Let M be a countable elementary submodel of a large structure containing all the abovementioned objects, and let $R' = R \cap M$, $P' = P \cap M$, $Q' = Q \cap M$, $\tau' = \tau \cap M$ and $\sigma' = \sigma \cap M$. Thus, P', Q' are regular subposets of R' and they are in fact independent by Lemma 2.2.9. Also, τ', σ' are P', Q'-names for elements of X, and by the Mostowski absoluteness between the R'-extensions of M and V, $R' \Vdash \tau'\ E\ \sigma'$. I must show that τ' is a nontrivial name on P'. The theorem will then follow as the poset P' is countable and therefore in forcing sense equivalent to Cohen forcing.

Suppose for contradiction that some condition in R' forces $\tau'\ E\ \check{x}$ for some $x \in X$. Since τ' is a P'-name, there must be a condition $p' \in P'$ forcing this. By the Mostowski absoluteness between the $P' \times P' \restriction \langle p', p' \rangle$-extensions of M and V, $M \models \tau'$ is an E-pinned name on $P' \restriction p'$. However, $M \models P'$ is proper by elementarity of the model M, and all E-pinned names on proper posets are trivial by Theorem 3.3.2. Thus, there is $y \in M \cap X$ such that $p' \Vdash_{P'} \tau'\ E\ \check{y}$. By elementarity of the model M, $p' \Vdash_P \tau\ E\ \check{y}$, and this contradicts the initial choice of the poset R.

For the left-to-right implication of (2), suppose that E is trim. Then certainly E is Cohen-trim. Also, every nontrivial E-pinned name is also a nontrivial E-trim name by the definitions and the product forcing theorem. For the opposite implication, suppose that E is pinned, and τ is a nontrivial E-trim name on some poset P; I must find a nontrivial E-trim name on the Cohen forcing. The proof is a literal repetition of the proof of (1), except the consideration that all E-pinned names on proper posets are trivial is replaced by the pinned assumption: all E-pinned names are trivial without any restriction on the poset.

The proof of (3) follows the same lines as (1) with the added piece of information that there are no nontrivial E-pinned names on $\aleph_1$-preserving posets if E is Borel reducible to an orbit equivalence relation, Theorem 5.2.9. $\square$

On one hand, Theorem 6.3.3 explains the preoccupation with category-type ergodicity results observed so frequently in the theory of analytic equivalence relations. On the other hand, it would be wrong to conclude on the basis of Theorem 6.3.3 that the only interesting trim names are those on Cohen forcing. In Section 7.2, I will study trim names on the random forcing and use them to prove measure theoretic ergodicity results that would be impossible to obtain otherwise.

6.4 Absoluteness

As it was the case with the pinned equivalence relations, the definition of trimness and its variations seems to take into account large objects in the universe. Thus, the absoluteness of the trim property between various models of set theory may be questionable. In this section, I will show that at least the basic variations of trimness are suitably absolute.

Theorem 6.4.1. *Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is proper-trim;*
2. *for every transitive model N of a large part of ZFC containing the code for E, $N \models E$ is proper-trim.*

Proof. Suppose first that E is not proper-trim; then (the transitive collapse of) any countable elementary submodel of a large enough structure witnesses the latter statement. For the other direction, suppose that N is a countable transitive model containing the code for E as an element, and E is not proper-trim in N. Thus, N contains posets P, Q, R and P-name τ and a Q-name τ such that P, Q are regular independent subposets of R, P is proper, $R \Vdash \tau/\dot{G} \cap P \ E \ \sigma/\dot{G} \cap Q$, where $\dot{G}$ is the usual R-name for the generic filter, and $P \Vdash \tau$ is not E-related to any element of the ground model. I will show that the objects P, Q, R, τ, σ maintain these properties even in V, showing that E is not proper-trim in V.

By Lemma 2.2.9, P, Q are regular independent posets of R. The poset P is countable in V (as it is a subset of the countable model N) and therefore proper. The statement $R \Vdash \tau/\dot{G} \cap P \ E \ \sigma/\dot{G} \cap Q$ holds in V by the Mostowski absoluteness between the R-generic extension of V and N. I still must prove that the name τ is E-nontrivial. Suppose for contradiction that there is a condition $p \in P$ and a point $x \in X$ (possibly not in N) such that $p \Vdash \tau \ E \ \check{x}$. By the Mostowski absoluteness between the $P \times P \restriction \langle p, p \rangle$ extensions of V and N, it follows that in N, the name τ on $P \restriction p$ is E-pinned. Since the poset P is proper in N, and there are no nontrivial pinned names on proper posets by

Theorem 3.3.2, it must be the case that there is $y \in X$ in the model N such that $p \Vdash \tau \; E \; \check{y}$. This contradicts the nontriviality assumption on the name τ in the model N. □

Corollary 6.4.2. *Let E be an analytic equivalence relation on a Polish space X. The truth value of the statement "E is proper-trim" is the same in all forcing extensions.*

Proof. The second item in Theorem 6.4.1 is equivalent to its version for countable models by an immediate downward Löwenheim–Skolem argument. That version is coanalytic and so absolute between forcing extensions by Mostowski absoluteness. □

For trimness itself, it implies the pinned property of the equivalence relation in question and so its absoluteness must be spelled out more carefully due to results of Section 3.2. I have the following:

Theorem 6.4.3. *Let E be a Borel equivalence relation on a Polish space X. The following are equivalent:*

1. *E is trim;*
2. *for every ω-model N of a large part of ZFC containing the code for E, $N \models E$ is trim.*

Proof. This is an immediate consequence of Theorems 6.3.3, 6.4.1, and 3.2.1. □

Corollary 6.4.4. *Let E be a Borel equivalence relation on a Polish space X. The truth value of the statement "E is trim" is the same in all forcing extensions.*

Proof. The second item in Theorem 6.4.3 is equivalent to its version for countable models by an immediate downward Löwenheim–Skolem argument. That version is coanalytic and so absolute between forcing extensions by Mostowski absoluteness. □

6.5 Operations

The collection of trim (or $\mathfrak{P}$-trim, for suitable forcing properties $\mathfrak{P}$) equivalence relations is closed under a good number of natural operations on analytic equivalence relations. This is the contents of this section.

Theorem 6.5.1. *If $\{E_n : n \in \omega\}$ is a countable collection of ($\mathfrak{P}$-) trim analytic equivalence relations on a Polish space X whose union $E = \bigcup_n E_n$ is an equivalence relation, then E is ($\mathfrak{P}$-) trim.*

Proof. Let $V[G_0], V[G_1]$ be two generic extensions such that $V[G_0] \cap V[G_1] = V$, containing respective E-related points $x_0, x_1 \in X$. I must find a point $x \in X$ in the ground model in the E-equivalence class of x_0, x_1. Just observe that there must be a number $n \in \omega$ such that $x_0 \mathrel{E_n} x_1$, and then there must be $x \in X$ in the ground model which is E_n-related to both x_0, x_1 by the trimness assumption on E_n. Since $E_n \subset E$, the point x is also E-related to both x_0, x_1 as required. $\square$

Theorem 6.5.2. *Whenever C is a countable set, J an analytic ideal on C such that $=_J^{2^\omega}$ is ($\mathfrak{P}$-)trim, and $\{E_c : c \in C\}$ is a collection of ($\mathfrak{P}$-) trim equivalence relations on respective Polish spaces $\{X_c : c \in C\}$, then $F = \prod E_c/J$ is a ($\mathfrak{P}$-) trim equivalence relation on $\prod_c X_c$.*

Proof. Suppose that $V[G_0], V[G_1]$ are two generic extensions such that $V[G_0] \cap V[G_1] = V$, and $x_0, x_1 \in \prod_c X_c$ are F-related points in the respective models. I must find a point $x \in X$ in the ground model in the F-equivalence class of x_0, x_1.

First, consider the set $a_0 = \{c \in C : x_0(c)$ is not E_c-related to any point in $V\}$. Note that for every $c \in a_0$ the points $x_0(c), x_1(c)$ cannot be E_c-related by the trimness assumption on E_c. Thus, $a_0 \in J$. Similarly, the set $a_1 = \{c \in C : x_1(c)$ is not E_c-related to any point in $V\}$ belongs to J.

Find injections $\langle f_c : c \in C\rangle$ in the ground model mapping the set of E_c-classes to 2^ω and consider the points $x_0', x_1' \in (2^\omega)^C$ defined by $x_0'(c) = f_c([x_0]_{E_c})$ if $x_0(c)$ is E_c-related to some ground model element, and $x_0'(c) = 0$ otherwise; similarly for x_1'. By the previous paragraph, x_0' and x_1' are $=_J$-related elements of the respective models N_0, N_1. By the trimness assumption on $=_J$, there is a point $y' \in V \cap \prod_c 2^\omega$ which is $=_J$-related to both x_0', x_1'. Any point $y \in \prod_c X_c$ in the ground model with the property that for every $c \in C$, if there is $z \in X_c$ with $f_c(z) = y'(c)$ then $f_c(y(c)) = y'(c)$, is F-related to both x_0, x_1. $\square$

Corollary 6.5.3. *Let E_n for $n \in \omega$ be ($\mathfrak{P}$-)trim analytic equivalence relations. Then $\prod_n E_n$ and $\prod_n E_n$ modulo finite are also ($\mathfrak{P}$-)trim equivalence relations.*

Proof. In view of Theorem 6.5.2, it is enough to check that $=_J^{2^\omega}$ and $=_K^{2^\omega}$ are trim equivalence relations, where J is the ideal containing just the empty set, and K is the ideal containing just the finite sets. Now, $=_J^{2^\omega}$ is smooth and therefore trim. The relation $=_K^{2^\omega}$ is just equal to E_1. To see that E_1 is trim, let $V[G_0], V[G_1]$ be generic extensions containing respective E_1-related points $x_0, x_1 \in (2^\omega)^\omega$. The points x_0, x_1 have the same tail. If this tail is not in the ground model, then $V[G_0] \cap V[G_1] \neq V$. If the tail is in the ground model, then the equivalence class of x_0, x_1 has a representative in the ground model, verifying the trimness. $\square$

Theorem 6.5.4. *Whenever E is a proper-trim analytic equivalence relation on a Polish space X, then E^+ is a proper-trim equivalence relation on X^ω.*

Proof. Let $V[G_0], V[G_1]$ be generic extensions such that $V[G_0] \cap V[G_1] = V$, $V[G_0]$ is an extension effected by a proper notion of forcing, and let $y_0, y_1 \in X^\omega$ be two E^+-related points in the respective models. I must find a point $y \in X^\omega$ in the ground model in their E^+-class.

First note that for every $n \in \omega$, the point $y_0(n) \in X \cap V[G_0]$ must be E-related to a point in the ground model. If this failed for some $n \in \omega$, then $y_0(n)$ is not E-related to any point in the model $V[G_1]$ either, by the trimness assumption on E. In particular, $y_0(n)$ could not be E-related to any $y_1(m)$ for $m \in \omega$, contradicting the assumption that $y_0 \ E^+ \ y_1$.

Thus, the set $a = \{[y_0(n)]_E : n \in \omega\} = \{[y_1(n)]_E : n \in \omega\}$ is a subset of the ground model, it belongs to both $V[G_0]$ and $V[G_1]$ and therefore to the ground model. It is countable in the proper extension $V[G_0]$ and therefore must be countable in the ground model. Let $y \in X^\omega$ be any point in the ground model visiting exactly the E-equivalence classes in the set a. The point y works as desired. $\square$

As a final remark in this section, I will compare the proper-trimness with a class of equivalence relations introduced by Kanovei.

Definition 6.5.5. (Kanovei, [12, Section 13.5]) $\mathfrak{K}$ is the smallest class of analytic equivalence relations containing the identity and closed under Borel reduction, countable union, modulo finite product, and Friedman–Stanely jump.

Thus,the class of proper-trim equivalence relations is closed under the generating operations of $\mathfrak{K}$ and so Kanovei's class $\mathfrak{K}$ is included in the class of proper-trim equivalence relations. There are trim equivalence relations which do not belong to $\mathfrak{K}$. Some of them are introduced in Definition 6.6.19; they have in fact strong ergodicity properties with respect to equivalence relations in kanovei's class by Theorem 7.1.11.

6.6 Examples

The purpose of this section is to explore a spectrum of trim or proper-trim equivalence relations.

Among the equivalence relations Borel reducible to an orbit equivalence relation, the landscape is not altered much by the introduction of trimness. By the following Theorem 6.6.1, all equivalence relations classifiable by countable structures are proper-trim. On the other hand, all orbit equivalence relations which are not classifiable by countable structures reduce an orbit equivalence relation of a turbulent action by a theorem of Hjorth [12, Lemma 13.3.4], which prevents them from being proper-trim by Theorem 6.1.2. Thus, the class of proper-trim equivalence relations Borel reducible to an orbit equivalence relation coincides with the class of relations classifiable by countable structures. By Theorem 6.3.3 then, the class of trim orbit equivalence relations coincides with the class of pinned equivalence relations classifiable by countable structures, for which I do not have a good characterization–Question 3.1.11.

Theorem 6.6.1. *If E is an equivalence relation classifiable by countable structures, then E is proper-trim.*

Proof. Such an equivalence relation is Borel-reducible to isomorphism of two-place relations on ω, so by Theorem 6.2.5 it is enough to deal with the equivalence relation F of isomorphism of two-place relations on ω; the underlying space of F is $X = \mathcal{P}(\omega^2)$. Let $V[G_0], V[G_1]$ be two generic extensions with $V[G_0]$ a proper forcing extension of V and $V[G_0] \cap V[G_1] = V$. Let $x_0 \in V[G_0]$ and $x_1 \in V[G_1]$ be F-related points. I must show that x_0, x_1 are F-related to a point in V.

Let α be an ordinal such that x_0 has Scott rank α. Since $V[G_0]$ is a model of ZFC, the ordinal α is countable there, and since it is a proper forcing extension of V, the ordinal α is countable in V. Consider the equivalence relation F restricted to the Borel set B of relations of rank α. This relation belongs to the ground model, it is Borel, classifiable by countable structures. Thus, by [12, Theorem 12.5.2] it is reducible to an equivalence relation obtained by a countable transfinite repetition of the operation of Friedman-Stanley jump and increasing union. In view of Theorems 6.5.4 and 6.5.1, $F \restriction B$ is proper-trim, and so x_0, x_1 are $F \restriction B$-related to a point $x_2 \in B$ in the ground model. Clearly, $x_2 \; F \; x_0$ as desired. □

Thus, most applications of the trim concept can be found on the "dark side"– among the equivalence relations that are not reducible to an orbit equivalence relation. Among those, I will explore the equivalence relations of the form $=_J$ or $=_J^{2^\omega}$ for various analytic ideals J on ω. The following concept will be used frequently to disprove trimness of such equivalence relations.

Definition 6.6.2. Let P be a partial ordering and G a graph on P. Say that G is a *graphing of* P if for every $p \in P$ and every $q_0, q_1 \leq p$ and every open dense set $D \subset P$ there is a finite sequence $\langle r_i : i \in n \rangle$ of conditions in the set D below p such that $r_0 \leq q_0$, $r_{n-1} \leq q_1$ and $r_i \; G \; r_{i+1}$ for all $i \in n-1$.

For example, consider the poset $P = 2^{<\omega}$ ordered with reverse inclusion and the relation G defined by $s \; G \; t$ if $\mathrm{dom}(s) = \mathrm{dom}(t)$ and $\{j \in \mathrm{dom}(s) : s(j) \neq t(j)\}$ has at most one element. It is not difficult to see that G is a graphing of P.Whenever $s \in P$ and $t_0, t_1 \leq P$ and D is an open dense set, first extend t_0, t_1 if necessary so that $\mathrm{dom}(t_0) = \mathrm{dom}(t_1)$, let u be a binary string such that for every $t \in 2^{<\omega}$ with $\mathrm{dom}(t) = \mathrm{dom}(t_0)$ $t^\frown u \in D$ holds, and consider any finite sequence of binary strings $\langle v_i : i \in j \rangle$, starting with $t_0^\frown u$, ending with $t_1^\frown u$, all of the same length, starting with s, ending with u, such that two successive strings on the sequence differ in exactly one entry. This sequence verifies that G is a graphing of P.

Lemma 6.6.3. *If P is a partial ordering with graphing G, $p \in P$ and $p \Vdash \tau$ is a set of ordinals not in the ground model, then there are conditions $p_0, p_1 \leq p$ and an ordinal α such that $p_0 \; G \; p_1$ and $p_0 \Vdash \check{\alpha} \in \tau$ and $p_1 \Vdash \check{\alpha} \notin \tau$.*

Proof. Since $p \Vdash \tau \notin V$, there must be an ordinal α and conditions $q_0, q_1 \leq p$ and an ordinal α such that $q_0 \Vdash \check{\alpha} \in \tau$ and $q_1 \Vdash \check{\alpha} \notin \tau$. Let D be the open dense poset of conditions deciding the statement $\check{\alpha} \in \tau$. Use the graphing property of G to find a finite sequence $\langle r_i : i \in n\rangle$ of conditions in D below p such that $r_i \; G \; r_{i+1}$ for every $i \in n-1$. Since the endpoints of this sequence decide the statement $\check{\alpha} \in \tau$ differently, there must be two successive points, say r_i and r_{i+1} on the sequence deciding it differently. The conditions $p_0 = r_i$ and $p_1 = r_{i+1}$ work as desired. $\square$

The situation is simplest in the case of analytic P-ideals J. Here, there is a complete characterization of trimness of $=_J$ as follows:

Theorem 6.6.4. *Let J be an analytic P-ideal on ω containing all singletons. Exactly one of the following occurs:*

1. *there is an infinite set $a \subset \omega$ such that $J = \{b \subset \omega : b \cap a$ is finite$\}$;*
2. *there are pairwise disjoint infinite sets $a_i \subset \omega$ for $i \in \omega$ such that $J = \{b \subset \omega : b \cap a_i$ is finite for all $i \in \omega\}$;*
3. *$=_J$ is not proper-trim.*

Observe that in item (1) the equivalence relation $=_J$ is bireducible with E_0 and $=_J^{2^\omega}$ is bireducible with E_1. In item (2), $=_J$ is bireducible with E_0^ω and $=_J^{2^\omega}$ is bireducible with E_1^ω. It follows that all of these equivalence relations are trim. As a corollary,

Corollary 6.6.5. *Let J be an analytic P-ideal on ω. Then $=_J$ is trim if and only if $=_J^{2^\omega}$ is trim.*

Proof of Theorem 6.6.4. First observe that in item (1) the equivalence relation $=_J$ is bireducible with E_0 and in item (2) $=_J$ is bireducible with E_0^ω. As E_0^ω is not reducible to E_0, and both E_0, E_0^ω are trim relations, the three items are indeed mutually exclusive. I must prove that one of the items does occur. Using a theorem of Solecki [23], find a lower semicontinuous submeasure ϕ on ω such that $J = \{a \subset \omega : \lim_n \phi(a \setminus n) = 0\}$.

Case 1. There is $\varepsilon > 0$ such that the set $a_\varepsilon = \{n \in \omega : \phi(n) < \varepsilon\}$ is in J. This immediately implies that (1) occurs with $a = \omega \setminus a_\varepsilon$.

Case 2. Case 1 fails and for every $i \in \omega$ greater than 0 there is $\varepsilon_i > 0$ such that $\limsup_n \phi(a_{\varepsilon_i} \setminus n) < 2^{-i}$. Write also $a_{\varepsilon_0} = \omega$. Consider the sets $a_i = a_{\varepsilon_i} \setminus a_{\varepsilon_{i+1}}$. For every set $b \subset \omega$, if b has infinite intersection with some a_i, then $b \notin J$, since b contains infinitely many singletons of ϕ-mass $\geq \varepsilon_{i+1}$. On the other hand, if b has finite intersection with every a_i then $\limsup_n \phi(b \setminus n) = 0$ and so $b \in J$. Thus, item (2) occurs as witnessed by the sets $a_i : i \in \omega$ (possibly removing the finite ones among them).

Case 3. Both Case 1 and Case 2 fail. I will show that $=_J$ is not proper-trim. Let P be the poset of all pairs $p = \langle t_p, \varepsilon_p\rangle$ where t_p is a finite binary string and $\varepsilon_p > 0$ is a real number. The ordering is defined by $q \leq p$ if $t_p \subset t_q$, $\varepsilon_q \leq \varepsilon_p$, and for all $n \in \mathrm{dom}(t_q \setminus t_p)$, if $t_q(n) = 1$ then $\phi(\{n\}) < \varepsilon_p$. The poset P is countable and therefore proper. Let $\dot{x}_{gen} \in 2^\omega$ be the P-name for the union of the first coordinates of the conditions in the generic filter. I will show that $\dot{x}_{gen}$ is a nontrivial $=_J$-trim P-name.

I will first show that $P \Vdash \dot{x}_{gen}$ is not $=_J$-related to any element of the ground model. Suppose for contradiction that there is $y \in 2^\omega$ in the ground model and a condition $p \in P$ forcing $\dot{x}_{gen} =_J \check{y}$. As Case 2 fails, there is $\delta > 0$ such that for every $\varepsilon > 0$, $\limsup_n \phi(a_\varepsilon \setminus n) > \delta$. Strengthening the condition p if necessary, I may find a number $n \in \omega$ such that $p \Vdash \phi(\{m > n : \dot{x}_{gen}(m) \neq \check{y}(m)\}) < \delta$. Since $\phi(a_{\varepsilon_p} \setminus \max(n, \mathrm{dom}(t_p))) > \delta$, there is a finite set $u \subset a_{\varepsilon_p}$ such that $\min(u) > n, \mathrm{dom}(t_p)$ such that $\phi(u) > \delta$. Let $q \leq p$ be any condition such that t_q is a binary string extending t_p with $u \subset \mathrm{dom}(t_q)$ and such that for all $k \in \mathrm{dom}(t_q \setminus t_p)$, if $k \notin u$ then $t_q(k) = 0$ and if $k \in u$ then $t_q(k) = 1 - y(k)$. Then q forces the set $\{m > n : \dot{x}_{gen}(m) \neq \check{y}(m)\}$ to contain u as a subset and therefore to have mass $> \delta$, contradicting the choice of p, n.

Now I have to show that $\dot{x}_{gen}$ is a $=_J$-trim name. If $p_0, p_1 \in P$ are conditions, I must produce a generic extension in which there are filters $G_0, G_1 \subset P$ separately generic over V such that $p_0 \in G_0$, $p_1 \in G_1$, $\dot{x}_{gen}/G_0 =_J \dot{x}_{gen}/G_1$, and $V[G_0] \cap V[G_1] = V$. To simplify the notation, assume $p_0 = p_1$ is the largest condition in the poset P. Let Q be the poset of all pairs $q = \langle t_q, s_q, \varepsilon_q\rangle$ where t_q, s_q are finite binary strings of the same length, and $\varepsilon_q > 0$ is a real number. The ordering is defined by $r \leq q$ if $t_q \subset t_r$, $s_q \subset s_r$, whenever $n \in \mathrm{dom}(s_r \setminus s_q)$ and $s_r(n) = 1$ or $t_r(n) = 1$ then $\phi(\{n\}) < \varepsilon_q$, and finally $\phi(\{n \in \mathrm{dom}(s_r \setminus s_q) : t_r(n) \neq s_r(n)\}) + \varepsilon_r < \varepsilon_q$. If $H \subset Q$ is a filter generic over V, let $G_0 = \{\langle t_q, \varepsilon_q\rangle : q \in H\} \subset P$ and $G_1 = \{\langle s_q, \varepsilon_q\rangle : q \in H\} \subset P$, and write $x_0 = \dot{x}_{gen}/G_0$ and $x_1 = \dot{x}_{gen}/G_1$. The following claims complete the proof.

Claim 6.6.6. *Q forces both G_0, G_1 to be filters on P generic over the ground model.*

Proof. Given $q \in Q$, the pair $\langle t_q, \varepsilon_q\rangle$ is a condition in P, and if $p \leq \langle t_q, \varepsilon_q\rangle$ is its strengthening in P, then $\langle t_p, (t_p \setminus t_q) \cup s_q, \varepsilon_p\rangle$ is a condition in Q stronger than q forcing $p \in G_0$. The obvious density argument then shows that $Q \Vdash \dot{G}_0 \subset P$ is a generic filter over V. The case of $\dot{G}_1$ is symmetric. ☐

Claim 6.6.7. *$Q \Vdash \dot{x}_0 =_J \dot{x}_1$.*

Proof. Suppose $\varepsilon > 0$ is a real number and $q \in Q$ is a condition. Shrinking ε_q if necessary, I may assume that $\varepsilon_q \leq \varepsilon$. The definition of the poset Q then shows that $q \Vdash \phi(\{n \in \omega : \dot{x}_0(n) \neq \dot{x}_1(n)\} \setminus \mathrm{dom}(t_q)) \leq \varepsilon_q$. As $\varepsilon > 0$ was arbitrary, this proves the claim. ☐

Claim 6.6.8. *$Q \Vdash V[G_0] \cap V[G_1] = V$.*

Proof. For contradiction let τ_0, τ_1 be P-names for sets of ordinals and $q \in Q$ a condition such that $q \Vdash \tau_0/G_0 = \tau_1/G_1 \notin V$. Let $p = \langle t_q, \varepsilon_q/2\rangle$. It must be the case that $p \Vdash_P \tau_0 \notin V$, otherwise an extension of P forcing $\tau_0 \in V$ would yield a condition in the poset Q stronger than q such that $\tau_0/G_0 \in V$. Now, the relation connecting conditions in $P \restriction p$ if their first coordinates have the same length and differ in at most one entry is a graphing of $P \restriction p$. By Lemma 6.6.3, there are an ordinal α, a real number $\varepsilon > 0$, and finite binary strings u_0, u_1 of the same length such that u_0, u_1 differ in exactly one entry, $\langle u_0, \varepsilon\rangle$ and $\langle u_1, \varepsilon\rangle$ are conditions in the poset P stronger than p, and $\langle u_0, \varepsilon\rangle \Vdash \check\alpha \notin \tau_0$ and $\langle u_1, \varepsilon\rangle \Vdash \check\alpha \in \tau_0$.

Let v be a finite binary string and $\delta > 0$ a real number such that $\langle v, \delta\rangle \leq \langle u_0 \text{ rew } s_q, \varepsilon\rangle$ is a condition in P deciding the statement $\check\alpha \in \tau_1$. Suppose for definiteness that it decides it in the negative. In such a case, let $t = v \text{ rew } u_1$, $s = v$, and check that $\langle t, s, \delta\rangle$ is a condition in the poset Q stronger than q. The construction of this condition implies that it forces $\check\alpha \in \tau_0/G_0 \setminus \tau_1/G_1$, contradicting the choice of the condition q.

□

□

The situation of F_σ-ideals is not as clear-cut. Note that if J is an F_σ-ideal, then the equivalence relation $=_J$ has F_σ classes, therefore is pinned by Fact 3.1.3, and so by Theorem 6.3.3 the notions of trimness, proper-trimness and Cohen-trimness coincide in this case. I will discuss two apparently typical cases. First, let $C = \{\langle m, n\rangle \in \omega \times \omega : n \leq m\}$ and let the *eventually different ideal* be the ideal of those subsets $a \subset C$ for which the cardinalities of vertical sections of a are bounded.

Theorem 6.6.9. *Let J be the eventually different ideal. Then $=_J$ is not proper-trim.*

Proof. Let $X = 2^C$, let P be the usual Cohen poset on X consisting of finite partial maps from C to 2 ordered by reverse inclusion, and let τ be the usual P-name for the generic element of X. I will show that τ is a nontrivial E-trim name.

The nontriviality of τ is easy. Suppose that $x \in 2^C$ is an arbitrary element, $p \in P$ and $n \in \omega$. I must show that there is $q \leq p$ such that q forces the set $\{c \in C : x(c) \neq \tau(c)\}$ to contain a vertical section of size at least n. To do this, just find a number $m > n$ such that $C_m \cap \text{dom}(p) = 0$ and let $q = p_m \cup (1 - x \restriction C_m)$.

The trimness of τ is slightly more difficult. For conditions $p, q \in P$ I must find a way to force filters $G, H \subset P$ which are separately generic over V, $\tau/G =_J \tau/H$, $V[G] \cap V[H] = V$, and $p \in G, q \in H$. For simplicity assume that $p, q = 0$. Let R be the poset of all pairs $r = \langle p_r, q_r\rangle$ where $p_r, q_r \in P$ are two conditions with the same domain such that no two elements of the set $\{c \in \text{dom}(p_r) : p_r(c) \neq q_r(c)\}$ lie in the same vertical section of the set C. The ordering on Q is coordinatewise extension. Let $\dot G, \dot H$ be the R-names for the filters on P generated by the first or second coordinates of conditions in the R-generic filter. I will now verify the requisite properties of the poset R one by one.

Claim 6.6.10. *$R \Vdash \dot G, \dot H \subset P$ are both filters generic over V.*

Proof. Given any condition $r \in R$ and a condition $p' \leq p_r$, the pair $r' = \langle p', q_r \cup p' \setminus p_r \rangle$ is a condition in R stronger than r. The genericity of the filter $\dot{G}$ then follows by a straightforward density argument. The genericity of the filter $\dot{H}$ uses a symmetric argument. □

Claim 6.6.11. *$R \Vdash \tau/\dot{G} =_J \tau/\dot{H}$.*

Proof. The set $\{c \in C : (\tau/\dot{G})(c) \neq (\tau/\dot{H})(c)\}$ can contain at most one element in each vertical section of the set C by the definition of the poset R, and therefore it is forced to belong to J. □

Claim 6.6.12. *$R \Vdash \tau/\dot{G}$ is not $=_J$-related to any element of the ground model.*

Proof. It is easy to check that the $=_J$-classes are meager in 2^C, and therefore the P-generic point τ cannot belong to any $=_J$-class represented in the ground model. □

Claim 6.6.13. *$R \Vdash V[\dot{G}] \cap V[\dot{H}] = V$.*

Proof. Suppose that $r \in R$ is a condition, σ and χ are P-names for sets of ordinals and $r \Vdash \sigma/\dot{G} = \chi/\dot{H}$. Strengthening r if necessary I may assume that every vertical section of the set C is either disjoint from or a subset of $\mathrm{dom}(p_r)$. It will be enough to show that the condition $p_r \in P$ decides the membership of any ordinal in σ and therefore forces $\sigma \in V$.

Suppose that this fails. By Lemma 6.6.3 applied to P and σ, there must be an ordinal α and conditions $p_0, p_1 \leq p_r$ such that the conditions p_0, p_1 have the same domain, differ in exactly one entry, and $p_0 \Vdash \check{\alpha} \in \sigma$ and $p_1 \Vdash \check{\alpha} \notin \sigma$.

Now, let $q = q_r \cup (p_0 \setminus p_r)$; so $q \in P$ is a condition strengthening q_r. Find a condition $q' \leq q$ in P deciding the statement $\check{\alpha} \in \chi$; for definiteness assume that the decision is negative. Let $p' = p_1 \cup q' \setminus q \leq p_1$. This is a condition in P forcing $\check{\alpha} \notin \sigma$. The pair $\langle p', q' \rangle$ is a condition in R smaller than r, and it forces $\alpha \in \sigma/\dot{G} \Delta \chi/\dot{H}$. This is a contradiction. □

□

Question 6.6.14. Is there a turbulent group action whose orbit equivalence is Borel reducible to $=_J$?

Second, let the *branch ideal* be the ideal on $2^{<\omega}$ generated by linearly ordered subsets of $2^{<\omega}$.

Theorem 6.6.15. *Let J be the branch ideal. The equivalence relation $=_J^{2^\omega}$ is trim.*

Proof. Let $X = (2^\omega)^{2^{<\omega}} = \mathrm{dom}(=_J^{2^\omega})$. For every node $t \in \omega^{<\omega}$ write $[t]$ for the set of all nodes in $2^{<\omega}$ extending t. Let $V[G_0]$, $V[G_1]$ be two generic extensions containing respective $=_J$-related points $x_0, x_1 \in X$. Assume that $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $x \in X$ which is $=_J$-related to both x_0, x_1.

Let $T = \{t \in 2^{<\omega} : x_0 \restriction [t]$ is not $=_J$-equivalent to any point in the ground model$\}$. This is a subtree of $\omega^{<\omega}$. If $0 \notin T$ then the proof is complete; thus, it is only necessary to derive a contradiction from the assumption $0 \in T$. First, observe that the tree T cannot have any terminal nodes: if t was a terminal node of T then one could combine the ground model witnesses for $t^\frown 0 \notin T$ and $t^\frown 1 \notin T$ to find a witness for $t \notin T$. Second, observe that the definition of the tree T depends only on the $=_J$-class of x_0 and so $T \in V[G_0] \cap V[G_1] = V$. Since T is a nonempty ground model tree without endnodes, it must have an infinite branch $y \in 2^\omega$ in the ground model. Since $x_0 =_J x_1$, there is a number $n \in \omega$ such that for every $t \in 2^{<\omega}$ such that $x_0(t) \neq x_1(t)$, either t is incompatible with $y \restriction n$ or else t is an initial segment of y. Let $e = [y \restriction n] \setminus y \restriction m : m \geq n\}$ and observe that $e \in V$ and x_0 and x_1 coincide on e, therefore $x_0 \restriction e \in V$. If $z \in V$ is any function in $2^{[y \restriction n]}$ extending $x_0 \restriction e$, then $z =_J x_0 \restriction [t]$, contradicting the assumption that $y \restriction n \in T$. □

Even though the F_σ-ideals seem to be easy to deal with, I still do not know a good answer to the following question.

Question 6.6.16. Characterize the collection of those F_σ-ideals J such that the equivalence relation $=_J$ is trim.

The only piece of general information for F_σ-ideals I have at this point that the investigation of $=_J$ and the more complicated $=_J^{2^\omega}$ yields the same result:

Theorem 6.6.17. *For F_σ-ideal J on ω, $=_J^2$ is trim if and only if $=_J^{2^\omega}$ is trim.*

Proof. By a result of Mazur [20], there is a lower semicontinuous submeasure ϕ on ω such that $J = \{a \subset \omega : \phi(a) < \infty\}$. Write E for the equivalence relation $=_J^\omega$ on the space $X = (2^\omega)^\omega$. Since $=_J^2$ is easily continuously reducible to E, if E is trim then so is $=_J^2$. For the opposite implication, assume that E is not trim. As E is a K_σ-equivalence relation, it is pinned by Fact 3.1.3, and so by Theorem 6.3.3 it must fail to be Cohen-trim. Let P be the Cohen forcing and $\dot{x}$ a nontrivial E-trim name on P.

Let $\{p_n : n \in \omega\}$ be an enumeration of P with infinitely many repetitions, and by induction on $n \in \omega$ build finite sets $a_n \subset \omega$ and functions $f_n : a_n \to \omega$ so that

- the sets a_n are pairwise disjoint, of respective ϕ-mass at least n;
- there are conditions $q, r \leq p_n$ and a function $g_n : a_n \to 2$ such that for every $i \in a_n$, $q \Vdash \dot{x}(i)(f_n(i)) = g_n(i)$ and $r \Vdash \dot{x}(i)(f_n(i)) = 1 - g_n(i)$.

Once this construction is performed, let $f \in \omega^\omega$ be any function extending $\bigcup_n f_n$ and let $h : (2^\omega)^\omega \to 2^\omega$ be the continuous map defined by $h(x) = x \circ f$. Since h is a homomorphism of $=_J^{2^\omega}$ to $=_J^2$, it follows that $\dot{y} = \dot{h}(\dot{x})$ is an $=_J^2$-trim P-name. A straightforward density argument using the construction of the function f shows that P forces $\dot{y}$ not to be $=_J^2$-equivalent to any element of the ground model. Thus, $=_J^2$ is not trim as required.

To perform the induction step, just let M be a countable elementary submodel of a large structure, let $g \subset P$ be a filter generic over M and let $h \subset P$ be a filter generic over $M[h]$, both containing p_n.Write $x_0 = \dot{x}/g$ and $x_1 = \dot{x}/h$. As P forces $\dot{x}_{\text{left}}$ not to be E-related to any ground model element of X, it must be the case that $x_0 E x_1$ fails, and so there is a set $a_n \subset \omega$ of ϕ-mass at least n, disjoint from all a_m for $m \in n$, such that for every $i \in a_n$ $x_0(i) \neq x_1(i)$ holds. For every $i \in a_n$, let $f_n(i) \in \omega$ be a number such that $x_0(i)(f(i)) \neq x_1(i)(f(i))$, and let $g_n(i) x_0(i)(f(i))$. The existence of the conditions $q \in g$ and $r \in h$ as required in the induction hypothesis follows from the forcing theorem easily. □

I will now move to analytic ideals which are neither P-ideals nor F_σ. For a set $a \subset 2^{<\omega}$ let $\operatorname{tr}(a) = \{x \in 2^\omega : \exists^\infty n\ \exists t \in a\ x \restriction n \subset t\}$. This is a closed subset of 2^ω. For an ordinal $\alpha \in \omega_1$ let $J_\alpha = \{a \subset \omega : \operatorname{tr}(a)$ is countable of Cantor–Bendixson rank $< \alpha\}$. It is not difficult to see that J_α is a Borel ideal.

Theorem 6.6.18. *The equivalence relation $=_{J_\alpha}^{2^\omega}$ is trim for every countable ordinal α.*

Proof. Write F_α for the equivalence relation $=_{J_\alpha}^{2^\omega}$ and $X = (2^\omega)^{2^{<\omega}} = \operatorname{dom}(=_{J_\alpha}^{2^\omega})$. The argument proceeds by induction on the ordinal α. For the basis step $\alpha = 1$, the equivalence $=_{J_0}^{2^\omega}$ is just E_1, and E_1 is trim by Corollary 6.5.3.

The limit stage of the transfinite induction follows from Theorem 6.5.1, as for limit α it is the case that $F_\alpha = \bigcup_{\beta \in \alpha} F_\beta$. For the successor stage, say $\alpha = \beta + 1$ and the theorem has been proved for β. Let $V[G_0], V[G_1]$ be generic extensions such that $V[G_0] \cap V[G_1] = V$, let $x_0, x_1 \in X$ be two F_α-related points in these respective models, and work to find a representative of their F_α-class in the ground model. Let $T = \{t \in 2^{<\omega} : x_0 \restriction [t]$ is not F_α-related to any ground model element of $X\}$. This definition depends only on the F_α-class of x_0, and therefore $T \in V[G_0] \cap V[G_1] = V$. Observe that T is closed under initial segment, and has no terminal nodes. Suppose for contradiction that there is no ground model point of X which is F_α-related to x_0. This means that $0 \in T$, and therefore T has an infinite branch z in the ground model. Consider the set $a = \{t \in 2^{<\omega} : x_0(t) \neq x_1(t)\}$ and the countable set $\operatorname{tr}(a)$ of Cantor–Bendixson rank $< \alpha$. There must be a number $n \in \omega$ such that for every $m \geq n$, writing s_m for the finite binary string $z \restriction m^\frown(1 - z(m))$, the set $\operatorname{tr}(a) \cap [t]$ has rank $< \beta$. In other words, for every $m \geq n$ the functions $x_0 \restriction [s_m]$ and $x_1 \restriction [s_m]$ are F_β-related and the tuples $\langle x_0 \restriction [s_m] : m > n \rangle$ and $\langle x_0 \restriction [s_m] : m > n \rangle$ are $\prod_m F_\beta \restriction [s_m]$-related. As F_β is trim, Corollary 6.5.3 shows that the equivalence relation $\prod_m F_\beta \restriction [s_m]$ is trim as well, and so there must be a point $y \in (2^\omega)^{[y \restriction n]}$ in the ground model such that for every $m \geq n$, $y \restriction s_m\ F_\beta\ x_0 \restriction s_m$. But then, $y\ F_\alpha\ x_0 \restriction [z \restriction n]$, contradicting the fact that $z \restriction n \in T$.

□

As a final class of examples, I will introduce an equivalence relation associated with analytic σ-ideals of compact subsets of compact separable spaces.

Definition 6.6.19. For a compact Polish space X and an analytic σ-ideal I of compact subsets of X containing all singletons, and $C \subset X$ a countable dense subset of X, let J_I be the ideal on C consisting of set $a \subset C$ such that $\overline{a} \in I$. Let also $=_I$ stand for $=_{J_I}$ and $=_I^{2^\omega}$ stand for $=_{J_I}^{2^\omega}$; these are equivalence relations on 2^C and $(2^\omega)^C$ respectively.

It is immediate that J_I is an analytic ideal on C and so $=_I$ and $=_I^{2^\omega}$ are analytic equivalence relations. The notation abstracts from the choice of the countable dense set $C \subset X$. The point is that for two such countable dense sets $C, D \subset X$, a simple back and forth argument produces a bijection $\pi : C \to D$ such that $\limsup_{c \in C} d(c, \pi(c)) = 0$ where d is any fixed compatible metric on X. The bijection transports the versions of the ideal J_I defined from C and D respectively to each other, and the function $h : x \mapsto x \circ \pi^{-1}$ is a homeomorphism of 2^C to 2^D which is also a Borel reduction between the two versions of $=_I$ on 2^C and 2^D.

Theorem 6.6.20. *The equivalence relation $=_I^{2^\omega}$ is trim for every analytic σ-ideal I of compact sets on a compact separable metric space.*

Proof. Let $C \subset X$ be a countable dense subset of X. Write E for the equivalence relation $=_I^{2^\omega}$ on $Y = 2^C$. Let $V[G_0], V[G_1]$ be generic extensions containing respective E_I-related points $y_0, y_1 \in Y$. Assume $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $y \in Y$ which is E-related to both y_0, y_1.

Let $\mathcal{O}$ be a countable basis for the space X in the ground model, closed under finite unions and intersections. Let $a \subset \mathcal{O}$ be the set of all $O \in \mathcal{O}$ such that $y_0 \restriction O$ is E-related to some ground model point of Y. As this definition depends only on the E-class of y_0, it belongs to both $V[G_0]$ and $V[G_1]$, and therefore to th.e ground model Note that $X \setminus \bigcup a \in I$, since it is a subset of the closure of the set $\{c \in C : y_0(c) \neq y_1(c)\}$. Now, consider the set $D = \{\langle O, z\rangle : O \in a, z \in 2^{C \cap O} \cap M \text{ and } z\ E\ y_0 \restriction O\}$. Again, the definition of this set depends only on the E-class of y_0, and so $D \in V[G_0]$ and $D \in V[G_1]$, and by the assumptions $D \in V$. Use the axiom of choice in the ground model to find a uniformization $g \subset D$. Still working in the ground model, find an enumeration $a = \{O_n : n \in \omega\}$ and consider any point $y \in 2^C$ such that for every $n \in \omega$, y agrees with $g(O_n)$ on the set $O_n \setminus \bigcup_{m \in n} O_m$. The proof will be complete once I show that $y_0\ E\ y$.

Let $B = \{c \in C : y_0(c) \neq y(c)\}$. Also, write $B_n = \{c \in O_n : y_0(c) \neq g(O_n)\}$ for every $n \in \omega$, and $A = \{c \in C : y_0(c) \neq y_1(c)\}$. I will show that $\overline{B} \subset \overline{A} \subset \bigcup_n \overline{B}_n$. This will prove that $\overline{B} \in I$, since the sets on the right hand side are all in the σ-ideal I by the assumptions. Suppose that $x \in \overline{B}$. If $x \in \bigcup a$, then find the smallest number $n \in \omega$ such that $x \in O_n$, and observe that $x \in \bigcup_{m \leq n} \overline{B}_m$. If $x \notin \bigcup a$ then $x \in \overline{A}$ by the remark right after the definition of the set a. In both cases, $x \in \overline{A} \subset \bigcup_n \overline{B}_n$ has been verified as desired. $\square$

The preceding list of trim equivalence relations hides the fact that for most equivalence relations the status of their trimness remains an open question.

Question 6.6.21. Are the treeable analytic equivalence relations trim or proper-trim?

In Section 6.7, I will show that the treeable equivalence relations have a property weaker than trimness, Theorem 6.7.3, which suffices for many ergodicity applications.

Question 6.6.22. Is there an analytic equivalence relation which is largest among the trim ones in the sense of weak Borel reducibility? How about the proper-trim relations?

At this point, there are not even good candidates for such universal trim or proper-trim equivalence relations.

Question 6.6.23. Is there a Borel equivalence relation which is minimal among the non-trim ones in the sense of weak Borel reducibility? How about the non-proper-trim relations?

It is known that there is no $\leq_{\mathrm{B}}$-smallest orbit equivalence relation of a turbulent group action [4]. Still, there could be minimal non-trim equivalence relations among those that are not reducible to orbit equivalences.

6.7 Variations: number of models

For the purposes of ergodicity results of Chapter 7, the trimness or proper-trimness may not suffice. The difficulty seems to be that they compare only two forcing extensions. Thus, one is lead to compare a greater finite number or an infinite number of forcing extensions that contain representatives of the same equivalence class. The finite case offers many possible variations. I know how to use at least one:

Definition 6.7.1. Let E be an analytic equivalence relation on a Polish space X. Let $\mathfrak{P}$ be a class of forcing notions and $m \in \omega$ a natural number larger than 1. Say that E is *m-$\mathfrak{P}$-trim* if in every forcing extension, whenever $V[G_n]$ are generic extensions of V using a poset in $\mathfrak{P}$ containing the respective pairwise E-related points $x_n \in X$ for $n \in m$, then either there are disjoint sets $a, b \subset m$ such that $V[G_n : n \in a] \cap V[G_n : n \in b] \neq V$, or there is $x \in V$ such that $x \mathrel{E} x_n$ for all $n \in m$. If $\mathfrak{P}$ is the class of all forcing notions then say that E is *m-trim.*

It is not difficult to see that increasing the number m leads to a larger class of equivalence relations. The m-trimness is also a reducibility invariant:

Theorem 6.7.2. *If $E \leq_{\mathrm{wB}} F$ are analytic equivalence relations, $m \in \omega$, and F is m-$\mathfrak{P}$-trim, then E is m-$\mathfrak{P}$-trim.*

Proof. Let $X = \mathrm{dom}(E)$, $Y = \mathrm{dom}(F)$, $a \subset X$ countable, and $h : X \to Y$ a Borel function which is a reduction of E to F on $X \setminus [a]_E$. In some forcing extension, let $V[G_n]$ for $n \in m$ be intermediate forcing extensions containing respective E-related points $x_n \in X$ for $n \in \omega$. If one of the x_n is E-related to some $x \in a$, then the conclusion of σ-$\mathfrak{P}$-trimness is verified. Otherwise, the Shoenfield absoluteness shows that $h(x_n) \in Y$ for $n \in \omega$ are F-related points in the respective models $V[G_n]$. By the m-trimness assumption about F, either there are disjoint sets $a, b \subset m$ such that $V[G_n : n \in a] \cap V[G_n : n \in b] \neq V$, or there is $y \in V$ such that $y \mathrel{F} h(x_n)$ for all $n \in \omega$. In the former case, the trimness conclusion for E has been verified. In the latter case, the Shoenfield absoluteness between V and $V[G_0]$ shows that there must be $x \in X$ in the ground model such that $x \notin [a]_E$ and $h(x) \mathrel{F} y$. Then, x is E-related to all x_n as required. □

One interesting case where I can verify 4-trimness but not trimness itself is the class of treeable equivalence relations.

Theorem 6.7.3. *If E is a treeable equivalence relation on a Polish space X, then E is 4-trim.*

Proof. Just like Theorem 3.1.6. Let $T \subset X^2$ be a cycle-free analytic graph such that the equivalence classes of E are exactly the connectedness components of T. Suppose that in some generic extension, $\{x_n : n \in 4\}$ are pairwise E-related points of the space X. For $i, j \in m$ let $w(x_i, x_j)$ be the unique injective walk from x_i to x_j along T. Then $w(x_i, x_j)$ is a finite sequence of points in X and all of its entries belong to the model $V[x_i, x_j]$ by the Shoenfield absoluteness. Reviewing all the finitely many possible configurations of the points $\{x_n : n \in 4\}$ with respect to T, it becomes obvious that there must be disjoint two-element sets $a = \{n_0, n_1\}, b = \{n_2, n_3\} \subset 4$ such that $w(x_{n_0}, x_{n_1}) \cap w(x_{n_2}, x_{n_3}) \neq 0$. Either the point in the intersection does not belong to V, in which case $V[x_{n_0}, x_{n_1}] \cap V[x_{n_2}, x_{n_3}] \neq V$, or the point belongs to V, in which case it is a representative of the E-class of the x_n's in V. In both cases, the criterion for 4-trimness has been verified. □

The following question remains open.

Question 6.7.4. If E is an equivalence relation and $m \in \omega$ is a natural number greater than 1, is m-trimness equivalent to trimness? Similar for m-$\mathfrak{P}$-trimness for various classes $\mathfrak{P}$ of posets.

Comparing infinitely many generic extensions at once also offers a number of variations, of which I consider only the most natural one:

Definition 6.7.5. Let E be an analytic equivalence relation on a Polish space X. Let $\mathfrak{P}$ be a class of forcing notions. Say that E is *σ-$\mathfrak{P}$-trim* if in every forcing extension, whenever $V[G_n]$ are forcing extensions of the ground model for every $n \in \omega$, $V[G_n]$ is a forcing extension using a poset in $\mathfrak{P}$, and $x_n \in V[G_n]$ are pairwise E-related points, either $\bigcap_n V[G_n] \neq V$ or there is $x \in V$ such that $x \mathrel{E} x_n$ for all $n \in \omega$. If $\mathfrak{P}$ is the class of all forcing notions then say that E is *σ-trim.*

The notion of σ-trimness is stronger than that of trimness. If E fails to be trim and, in some generic extension, $x_0, x_1 \in X$ are E-related points such that $V[x_0] \cap V[x_1] = V$ and x_0, x_1 are not E-related to any element of the ground model, then the points $x_n = x_0$ if n is even and $x_n = x_1$ if n is odd, for

$n \in \omega$, violate the σ-trimness. Another immediate observation is the fact that σ-trimness is a reducibility invariant:

Theorem 6.7.6. *If $E \leq_{\mathrm{wB}} F$ are analytic equivalence relations and F is σ-$\mathfrak{P}$-trim, then E is σ-$\mathfrak{P}$-trim.*

Proof. Let $X = \mathrm{dom}(E)$, $Y = \mathrm{dom}(F)$, $a \subset X$ countable, and $h : X \to Y$ a Borel function which is a reduction of E to F on $X \setminus [a]_E$. In some forcing extension, let $V[G_n]$ for $n \in \omega$ be intermediate forcing extensions containing respective E-related points $x_n \in X$ for $n \in \omega$. If one of the x_n is E-related to some $x \in a$, then the conclusion of σ-$\mathfrak{P}$-trimness is verified. Otherwise, the Shoenfield absoluteness shows that $h(x_n) \in Y$ for $n \in \omega$ are F-related points in the respective models $V[G_n]$. By the trimness assumption about F, either $\bigcap_n V[G_n] \neq V$, or there is $y \in V$ such that $y \mathrel{F} h(x_n)$ for all $n \in \omega$. In the former case, the trimness conclusion for E has been verified. In the latter case, the Shoenfield absoluteness between V and $V[G_0]$ shows that there must be $x \in X$ in the ground model such that $x \notin [a]_E$ and $h(x) \mathrel{F} y$. Then, x is E-related to all x_n as required. $\square$

Most trim equivalences in Section 6.6 fail to be σ-trim and this feature is used to show that they have ergodicity properties with respect to the class of equivalence relations classifiable by countable structures, which are σ-trim:

Theorem 6.7.7. *Every equivalence relation classifiable by countable structures is σ-proper-trim.*

Proof. The same proof as for proper-trimness. I will first show that σ-proper-trimness is preserved under the Friedman-Stanley jump and countable unions.

Claim 6.7.8. *Let E be an analytic equivalence relation on a Polish space X. If E is σ-proper-trim then so is E^+.*

Proof. Suppose that $V[G_m]$ for $m \in \omega$ are generic extensions of V, containing respective E^+-related poits $y_m \in X^\omega$, and $V[G_0]$ is a proper forcing extension of V. Suppose that $V = \bigcap_m V[G_m]$. For every $k \in \omega$, each model $V[G_m]$ contains an E-equivalent of the point $y_0(k)$ (it must appear on the sequence y_m). By the σ-proper-trimness, the equivalence class $[y_0(k)]_E$ has a representative in the ground model. The set $a = \{[z]_E : z \in V \text{ and } \exists k\; z \mathrel{E} y_0(k)\}$ belongs to all models $V[G_m]$ for all $m \in \omega$, and therefore to the ground model. The set a countable in the model $V[G_0]$, which is a proper forcing extension of V, and so a must be countable already in the ground model. Let $y \in V$ be a point which visits exactly all the E-equivalence classes in the set a. It is easy to see that $y \mathrel{E^+} y_0$, completing the proof of the claim. $\square$

Claim 6.7.9. *Let E_n be analytic equivalence relations on respective pairwise disjoint Polish spaces X_n for $n \in \omega$. If every E_n is σ-proper-trim then so is $\bigcup_n E_n$.*

Proof. Write $X = \bigcup_n X_n$ and $E = \bigcup_n E_n$. Suppose that $V[G_m]$ for $m \in \omega$ are generic extensions of V, containing respective E-related poits $x_m \in X$, and $V[G_0]$ is a proper forcing extension of V. Suppose that $V = \bigcap_m V[G_m]$. There must be a number $n \in \omega$ such that all x_m's are in the space X_n and are pairwise E_n-related. Applying the σ-proper-trimness to E_n, conclude that there is a point $x \in X_n$ in the ground model which is E_n-related to all x_m. Then x is also E-related to all x_m and the σ-proper-trimness of E follows. □

Now, towards the proof of the theorem, it is enough to argue that $E = E_{S_\infty}$, the isomorphism of binary relations on ω, is σ-proper-trim, since it is the $\leq_B$-largest equivalence relation classifiable by countable structures. Let $X = 2^{\omega\times\omega} = \mathrm{dom}(E)$. Let $V[G_n]$ for $n \in \omega$ be generic extensions of V containing respective pairwise E-related points $x_n \in X$. Assume also that $V[G_0]$ is a proper extension of V and $\bigcap_n V[G_n] = V$. I must show that the class $[x_0]_E$ has a representative in the ground model. Let α be the Scott rank of x_0. The ordinal α is countable in $V[G_0]$, and as $V[G_0]$ is a proper forcing extension of V, it is also countable in V. Let $B \subset X$ be the set of all relations of Scott rank $\leq \alpha$. Thus, B is a Borel set coded in the ground model, and $E \restriction B$ is a Borel relation. As $E \restriction B$ is a Borel equivalence relation classifiable by countable structures, it is obtained by a repeated use of Friedman–Stanley jump, disjoint union, and Borel reduction by [12, Theorem 12.5.2], and by the previous claim it is σ-proper-trim. Applying its σ-proper-trimness, there must be a point $x \in B$ in the ground model such that $x \mathrel{E} x_0$. This point confirms the σ-proper-trimness of E as well. □

The following question remains open:

Question 6.7.10. Are the following equivalent for an analytic equivalence relation E?

1. E is σ-proper-trim;
2. E is classifiable by countable structures.

Changing the class of partial orders in question may change the answer. The following related theorem will be used in Chapter 7.

Theorem 6.7.11. *Let J be the branch ideal on $2^{<\omega}$. Let $\mathfrak{P}$ be the class of all posets of the form P_I where I is a c.c.c. $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal of analytic subsets of some Polish space X and P_I is the collection of all Borel I-positive sets ordered by inclusion. The equivalence relation $=_J$ is σ-$\mathfrak{P}$-trim.*

The class $\mathfrak{P}$ includes posets as the Cohen poset (associated with I =meager ideal), the random poset (associated with I =the Lebesgue null ideal), the Maharam algebras, the eventually different real forcing [26, Proposition 3.8.12] and some other posets. The posets in $\mathfrak{P}$ do not add dominating reals by [26, Proposition 3.8.15], and so the theorem does not resolve the case of P =Hechler forcing.

The proof uses a lemma of independent pure forcing interest.

Lemma 6.7.12. *Suppose that $P \in \mathfrak{P}$, $p \in P$ is a condition and $p \Vdash \langle \dot{y}_n : n \in \omega\rangle$ is a sequence of points in 2^ω. Then one of the following holds:*

1. *there is $q \leq p$ and $n \in \omega$ such that $q \Vdash \dot{y}_n$ is in the ground model;*
2. *there is $q \leq p$ and a function $g \in \omega^\omega$ such that $q \Vdash \langle \dot{y}_n \restriction g(n) : n \in \omega\rangle$ is not in the ground model.*

Proof. Let I be a σ-ideal on a Polish space X such that $P = P_I$. Strengthening the condition $p \in P$ if necessary, I may assume that there are Borel functions $f_n : p \to 2^\omega$ such that $p \Vdash \forall n\ \dot{y}_n = \dot{f}_n(\dot{x}_{gen})$. Let $A \subset \omega^\omega \times 2^{\omega\times\omega}$ be the set $A = \{\langle z, w\rangle : \forall n \forall m > z(n)\ w(n,m) = 0$ and the set $\{x \in p : \forall n \forall m \leq z(n)\ f_n(x)(m) = w(n,m)\} \subset X$ is I-positive$\}$. As the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the set A is analytic. As the poset P_I is c.c.c., the set A has countable vertical sections. By the first reflection theorem [14, Theorem 35.10], it is covered by a Borel set with countable vertical sections, which in turn is covered by graphs of countably many Borel functions $\{h_k : k \in \omega\}$ from ω^ω to $2^{\omega\times\omega}$ by the Lusin-Novikov theorem [14, Theorem 18.10]. The usual Miller forcing type fusion arguments then yield a superperfect tree $T \subset \omega^{<\omega}$ such that all functions $h_k \restriction [T]$ are continuous, and in fact whenever $k \in \omega$, $t \in T$ is a splitnode of length $> k$, $j \in \omega$ is such that $t^\frown j \in T$, and $m \in j$, then the value of $h_k(z)(|t|, m)$ is the same for all $z \in [T \restriction t^\frown j]$. Further thinning out the infinite branchings of the tree T if necessary, I can also assume that whenever $k \in \omega$, $t \in T$ is a splitnode of length $> k$ and $m \in \omega$, there is a bit $y_{k,t}(m) \in 2$ such that for all but finitely many j such that $t^\frown j \in T$, $h_k(z)(|t|, m) = y_{k,t}(m)$ for all $z \in [T \restriction t^\frown j]$. Thus, $y_{k,t} \in 2^\omega$ for every such k, t.

For every $k \in \omega$ and a splitnode $t \in T$ longer than k, consider the set $q_{k,t} = \{x \in p : \exists^\infty j\ \exists z \in [T \restriction t^\frown j]\ \forall n\ \forall m \leq n\ f_n(x)(m) = h_k(z)(n,m)\}$. This is a Borel subset of p, since the existential quantification over z can be replaced with universal by the choice of the tree T. The treatment splits into two cases.

Case 1. Suppose first that there is k, t such that the set $q = q_{k,t}$ is I-positive. Then write $n = |t|$, and observe that for every point $x \in q_{k,t}$, $f_n(x) = y_{k,t}$. Thus, $q \Vdash \dot{y}_{|t|}$ is in the ground model, equal to $\check{y}_{k,t}$.

Case 2. Suppose that all sets $q_{k,t}$ are in I. In such a case, consider the set $C = \{\langle x, z\rangle : x \in p \setminus \bigcup_{k,t} q_{k,t},\ z \in [T]$, and there is $k \in \omega$ such that $\forall n\ \forall m \leq z(n)\ f_n(x)(m) = h_k(z)(m)\}$.

Claim 6.7.13. *C is a Borel set with σ-bounded vertical sections.*

Proof. If $x \in p$ was such that C_x is not σ-bounded, there would have to be $k \in \omega$ such that the set $\{z \in [T] : \forall n\ \forall m \leq z(n)\ f_n(x)(m) = h_k(z)(m)\}$ is not σ-bounded, and so there would have to be $t \in \omega^{<\omega}$ longer than k such that for infinitely many $j \in \omega$ there is $y \in [T \restriction t^\frown j]$ such that $\forall m \leq z(n)\ f_n(x)(m) = h_k(z)(m)$. This would, however, put x into the set $q_{k,t}$, an impossibility. □

Now, since I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the poset P does not add dominating reals by [26, Proposition 3.8.15], and so it has the Fubini property with the σ-bounded ideal on ω^ω [26, Definition 3.2.1]. As the Borel set $C \subset p \times [T]$ has σ-bounded vertical sections, this means that the complement $p \times [T] \setminus C$ must have an I-positive horizontal section corresponding to some $z \in \omega^\omega$. Let $q = q' \setminus \bigcup_{k,t} q_{k,t}$, and review the definition of the set A to conclude that for every $w \in 2^{\omega\times\omega}$, it must be the case that the set $\{x \in q : \forall n \forall m \leq z(n)\ f_n(x)(m) = w(n,m)\} \subset X$ is in I. This is to say that $q \Vdash \langle \dot{y}_n \restriction g(n) : n \in \omega\rangle$ is not in the ground model. The lemma follows. $\square$

Proof of Theorem 6.7.11. Let $P \in \mathfrak{P}$ and assume that $V[G_n]$ for $n \in \omega$ are generic extensions of the ground model $V[G_0]$ is a P-extension, containing the respective pairwise $=_J$ equivalent points $x_n \in X$. Assume that $\bigcap_n V[G_n] = V$ and work to find a ground model point $x \in X$ which is $=_J$-related to all x_n's.

Let $T = \{t \in 2^{<\omega} : x_0 \restriction [t]$ is not $=_J$-equivalent to any point in the ground model$\}$. Clearly, T is a tree. Its definition depends only on the $=_J$-class of x_0, therefore T belongs to all $V[G_n]$ for $n \in \omega$ and so to the ground model. Since the witnesses for the failure of membership of a binary sequence in the tree T can be combined, T has no terminal nodes. Assume for contradiction that $0 \in T$. Since T is a nonempty ground model tree without endnodes, it has a branch in the ground model. To simplify the notation, assume that this branch has only 0 entries along it. For every $m \in \omega$ write $t_m = (0^m)^\frown 1$.

Let $y_m = x_0 \restriction [t_m]$ for $m \in \omega$. Note that for every $n \in \omega$, for all but finitely many $m \in \omega$, $y_m = x_n \restriction [t_m]$ as $x_0 =_J x_n$. Thus, the function $z \in 2^\omega$ defined by $z(m) = 0$ if $y_m \in V$ is in all models $V[G_n]$ for $n \in \omega$, and therefore in V. There are two cases, both of which end in contradiction:

Case 1. For all but finitely many $m \in \omega$, $z(m) = 0$. Then, for some $m_0 \in \omega$, the sequence $\langle y_m : m > m_0\rangle$ consists of ground model points only. Since for every $n \in \omega$ the model $V[G_n]$ contains a finite modification of this sequence, the sequence must belong to the ground model. Then, working in the ground model, let $x : [0^{m_0}] \to 2$ be any function extending all y_m for $m > m_0$. The definitions show that the values of x and $x_0 \restriction [0^{m_0}]$ can differ only on the branch y and therefore $x =_J x_0 \restriction [0^{m_0}]$. This contradicts the definition of the tree T and the assumption that $0^{m_0} \in T$.

Case 2. The set $a = \{m \in \omega : z(m) = 1\}$ is infinite. As $z \in V$, $a \in V$ as well. Apply Lemma 6.7.12 to find a function $g : a \to \omega$ such that the sequence $y = \langle y_m \restriction 2^{\leq g(m)} \cap [t_m] : m \in a\rangle$ does not belong to the ground model. For every number $n \in \omega$, the model $V[G_n]$ contains a finite modification of y, namely the sequence $\langle x_n \restriction 2^{\leq g(m)} \cap [t_m] : m \in a\rangle$. Therefore, $y \in \bigcap_n V[G_n]$. This contradicts the assumption that $\bigcap_n V[G_n] = V$. $\square$

Chapter 7

Ergodicity results

The purpose of this chapter is to use variations of the previously obtained technologies to prove ergodicity results in the following sense.

Definition 7.0.14. Suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y and I is a σ-ideal on X. Say that E is F-I-*ergodic* if for every Borel homomorphism $h : X \to Y$ of E to F there is an F-equivalence class $C \subset Y$ such that $X \setminus h^{-1}C \in I$. If I is the ideal of meager sets on X, this property will be called I-*generically ergodic*. If I is the ideal of null sets with respect to some Borel probability measure μ on X, this property will be called F-μ-*ergodic*.

Clearly, ergodicity leads to Borel nonreducibility results: if E is F-I-ergodic and the equivalence classes of E are in I, then $E \leq_{\mathrm{B}} F$ must fail. Any Borel reduction h of E to F would be also a homomorphism, so there would have to be an F-class whose h-preimage contains more than one E-class, contradicting the properties of a reduction. Ergodicity has a great advantage to other methods of proving nonreducibility results in that it persists under supersets on the E-side as explained in the following lemma. The lemma also says that it is of interest to prove ergodicity results for equivalence relations E as small with respect to inclusion as possible.

Lemma 7.0.15. *Suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y and I is a σ-ideal on X. If E', F' are analytic equivalence relations on respective Polish spaces $X' = X, Y'$ such that $E \subset E'$ and $F' \leq_{\mathrm{B}} F$, then E' is F'-I-ergodic.*

Proof. Fix a Borel reduction $k : Y' \to Y$ of F' to F. Whenever $h : X \to Y'$ is a Borel homomorphism of E' to F', then $k \circ h$ is a Borel homomorphism of E to F. Therefore, there is an F-class whose $k \circ h$-preimage is I-large. The k-preimage of this F-class is an F'-class whose h-preimage is I-large. $\square$

The following technical lemma will be the main tool for obtaining ergodicity results:

Lemma 7.0.16. *Let E be an analytic equivalence relation on a Polish space X. Let σ be a nontrivial E-proper-trim name on a poset P, and let I be the σ-ideal of coanalytic sets $C \subset X$ such that $P \Vdash \sigma \notin \dot{C}$. For every proper-trim equivalence relation F, E is F-I-ergodic.*

Proof. Let F be a proper-trim equivalence relation on a Polish space Y and let $h : X \to Y$ be a Borel homomorphism of E to F. I will show that there is a point $y \in Y$ such that $P \Vdash \dot{h}(\tau)\ F\ \check{y}$. In other words, $X \setminus h^{-1}[y]_F \in I$ as desired.

Consider the name $\dot{h}(\tau)$ for an element of the Polish space Y. Since the function h remains a Borel homomorphism of E to F in the P-extension, $\dot{h}(\tau)$ is an F-trim name. Since F is proper-trim, $\dot{h}(\tau)$ must be an F-trivial F-trim name, and there must be a condition $p \in P$ and a point $y \in Y$ such that $p \Vdash \dot{h}(\tau)\ F\ \check{y}$. I will prove that in fact the largest condition forces $\dot{h}(\tau)\ F\ \check{y}$, and that will complete the proof.

Suppose for contradiction that some condition $q \in P$ forces $\dot{h}(\tau)\ F\ \check{y}$ to fail. Use the E-trimness of the name τ to find in some generic extension, filters $G \subset P$ and $H \subset P$ so that $p \in G$, $q \in H$, and $\tau/G\ E\ \tau/H$. Since h is a homomorphism in that extension, $y\ F\ h(\tau/G)\ F\ h(\tau/H)$ and so $y\ F\ h(\tau)/H$, which contradicts the choice of the condition q. □

7.1 The category case

I will start with the following strengthening of a seminal result of Greg Hjorth:

Theorem 7.1.1. *Let $G \curvearrowright X$ be a generically turbulent action of a Polish group, E its orbit equivalence relation, and F a proper-trim equivalence relation. Then E is F-generically ergodic.*

In particular, E is F-generically ergodic for every equivalence relation F classifiable by countable structures, since all such F's are proper-trim by Theorem 6.6.1.

Proof. Let P_X be the Cohen forcing on X, i.e. the poset of nonempty open subsets of X ordered by inclusion. The poset has countable density and adds a canonical single point $\dot{x}_{gen} \in X$ which falls out of all ground model meager subsets of X. By Lemma 7.0.16, it is enough to show that $\dot{x}_{gen}$ is a nontrivial E-trim name on P_X.

Indeed, $\dot{x}_{gen}$ must be nontrivial since the orbits of the action are meager by assumption and so no ground model orbit can contain the Cohen point $\dot{x}_{gen}$. The trimness follows from Theorem 6.1.2. Suppose that $p, q \in P_X$. Let $x \in p$ be a P_X-generic point. The orbit of x is dense, in particular it intersects q, and by the continuity of the action there is a nonempty open neighborhood $r \subset G$ such that $r \cdot x \subset q$. Let $g \in r$ be a P_G-generic point over $V[x]$. By the product forcing theorem, the pair $\langle g, x\rangle \in G \times X$ is $P_G \times P_X$-generic over V. By Theorem 6.1.2, $V[x] \cap V[g \cdot x] = V$ and so the generic filters on P_X obtained from the points x and $g \cdot x$ witness that $\dot{x}_{gen}$ is indeed an E-trim name. □

Theorem 7.1.2. *Let $G \curvearrowright X$ be a generically turbulent action of a Polish group, E its orbit equivalence relation, and F a σ-treeable analytic equivalence relation. Then E is F-generically ergodic.*

Proof. Since I do not know if σ-treeable equivalence relations are trim, I cannot use Theorem 7.1.1 and I have to attack the 4-trimness of treeable equivalence relations instead. Start with a general claim of independent interest. Let P_X be the usual Cohen poset on X, and P_G the usual Cohen poset on G. Let $x \in X, g_n \in G : n \in a$, and $g_n : n \in b$ be points generic over V for the finite support product of P_X with $a \cup b$ many copies of P_G, where a, b are pairwise disjoint index sets.

Claim 7.1.3. $V[x, g_n \cdot x : n \in a] \cap V[g_n \cdot x : n \in b] = V$.

Proof. Choose an arbitrary element $n_0 \in b$. Then $V[x] \cap V[h_{n_0} \cdot x] = V$ by Theorem 6.1.2.

As the next step, verify $V[x][g_n : n \in a] \cap V[g_{n_0} \cdot x] = V$. To see this, let Q_a be the product of a many copies of P_G. The sequence $\langle g_n : n \in a \rangle$ is Q_a-generic over the model $V[x, g_{n_0}]$ by the product forcing theorem. Thus, if c is a set of ordinals both in $V[x][g_n : n \in a]$ and in $V[g_{n_0} x]$, by the forcing theorem applied in the model $V[x, g_{n_0}]$ to the poset Q_a there must be a Q_a-name $\sigma \in V[x]$ for a set of ordinals and a condition $q \in Q$ in the compatible with $\langle g_n : n \in a \rangle$ such that $q \Vdash \sigma = \check{c}$. Then, q decides the membership of every ordinal in the set σ. This is a sentence of the model $V[x]$ and so $c = \{\alpha : q \Vdash \check{\alpha} \in \sigma\}$ is in the model $V[x]$. Since $V[x] \cap V[g_{n_0} \cdot x] = V$, I conclude that $c \in V$ as desired.

Further, verify $V[x][g_n : n \in a] \cap V[g_{n_0} \cdot x][g_n g_{n_0}^{-1} : n \in b, n \neq n_0] = V$. To see this, let Q_b be the product of $b \setminus \{n_0\}$-many copies of P_G. The sequence $\langle g_n : n \in b, n \neq n_0 \rangle$ is Q_b-generic over the model $V[x][g_n : n \in a][g_{n_0}]$ by the product forcing theorem. The sequence $\langle g_n g_{n_0}^{-1} : n \in b, n \neq n_0 \rangle$ is Q_b-generic over the same model, since the multiplication by $g_{n_0}^{-1}$ on the right induces an automorphism of the poset Q_b. Then, the equality $V[x][g_n : n \in a] \cap V[g_{n_0} \cdot x][g_n g_{n_0}^{-1} : n \in b, n \neq n_0] = V$ follows just like in the previous paragraph.

Lastly, observe that $V[x, g_n \cdot x : n \in a] \subset V[x][g_n : n \in a]$ and $V[g_n \cdot x : n \in b] \subset V[g_{n_0} \cdot x][g_n g_{n_0}^{-1} : n \in b, n \neq n_0]$. Thus, $V[x, g_n \cdot x : n \in a] \cap V[g_n \cdot x : n \in b] = V$ as desired. □

Let F be any σ-treeable equivalence relation on a Polish space Y, and $h : X \to Y$ any Borel homomorphism of E to F. Let $F = \bigcup_n F_n$, where F_n are treeable equivalence relations. In some large forcing extension let $x_0 \in X$ be a point P_X-generic over V and $\{g_z : z \in 2^\omega\}$ be a perfect collection of P_G-generic points over the model V obtained through Lemma 2.2.8. The points $x_z = g_z \cdot x_0$ for $z \in 2^\omega$ are pairwise E-equivalent, and so their values $h(x_z)$ for $z \in 2^\omega$ are pairwise F-related. Let $y_0 \in Y$ be an arbitrary fixed point in their common F-equivalence class. A counting argument shows that there is a number $n \in \omega$ such that the set $\{z \in 2^\omega : h(g_z \cdot x_0) \ F_n \ y_0\}$ is uncountable, containing four distinct elements $\{z_n : n \in 4\}$. By the transitivity of the relation F_n, the points $\{h(x_{z_n}) : n \in 4\}$ are pairwise F_n-related. By Claim 7.1.3, for any two disjoint sets $a, b \subset 4$, $V[x_{z_n} : n \in a] \cap V[x_{z_n} : n \in b] = V$. By the 4-trimness of the treeable equivalence relation F_n, (Theorem 6.7.3), I can conclude that there is an element $y \in Y$ in the ground model which is F_n-related to all points $h(x_{z_n})$ for $n \in 4$. The analytic set $h^{-1}[y]_F$ contains the P_X-generic point $g_z \cdot x_0$ and so it is nonmeager. It is also E-invariant and so it has to be in fact co-meager. This completes the proof of the theorem.

□

The conclusion of Theorem 7.1.1 is much more common than one may expect. In fact, the non-trim relations are generically ergodic for all trim relations with the correct choice of topology on the underlying space:

Theorem 7.1.4. *Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is not proper-trim;*
2. *there is an alternative Polish topology t on X yielding the same Borel structure, in which all E-classes are meager, and such that for every proper-trim equivalence relation F, E is F-t-meager ergodic.*

Proof. First, assume that (1) fails. Then, the identity on X is a homomorphism of E to a proper-trim equivalence relation which certainly violates (2).

Second, assume that (1) holds. Proper-trimness is equivalent to Cohen trimness by Theorem 6.3.3. Let P be the Cohen forcing. Thus, there is a nontrivial E-trim P-name τ for an element of X. Passing to the complete subalgebra of $RO(P)$ completely generated by τ if necessary, I may assume that $RO(P)$ is completely generated by τ. Let M be a countable elementary submodel of a large structure containing E as well as P, τ. Let Y be the space of filters on P generic over M, with the topology generated by sets $\{g : p \in g\}$ for all $p \in P \cap M$. This is a Polish space. The function $f : Y \to X$ defined by $f(g) = \tau/g$ is continuous, since for any basic open set $O \subset X$ $\tau/g \in O$ if and only if there is a condition $p \in g$ forcing $\tau \in \dot{O}$. The function f is also one-to-one as the name τ completely generated the algebra $RO(P)$. Adjusting f on a closed nowhere dense set if necessary (losing continuity but maintaining Borelness), I can find a Borel bijection $\bar{f} : Y \to X$ such that $f = \bar{f}$ on an open dense set. I claim that the topology t on X generated by $\bar{f}$-images of open subsets of Y has the desired properties.

First of all, it is clear that t is a Polish topology as it is homeomorphic to the Polish space Y via $\bar{f}$. It also generates the same Borel structure as the original topology as one-to-one Borel images of Borel sets are Borel, in particular $\bar{f}$ and $\bar{f}^{-1}$-images of Borel sets are Borel. The E-classes must be t-meager: they are analytic, therefore have the Baire property, and if one of them, say $[x]_E$ for some $x \in X$ were t-nonmeager, there would be a condition $p \in P$ such that $p \Vdash \tau \mathrel{E} \check{x}$. This contradicts the assumption that τ is a nontrivial name. Finally, for every proper-trim equivalence relation F, E is F-t-meager ergodic by Lemma 7.0.16, since the t-meager ideal is exactly the collection $\{B \subset X : B \text{ is Borel and } P \Vdash \tau \notin \dot{B}\}$ by the definitions. □

Even in quite natural situations, one cannot expect the topology of Theorem 7.1.4 to be the "natural" topology, and different topologies may have different generic ergodicity features. This is nicely documented by the following example.

Theorem 7.1.5. *Let J be the ideal on $2^{<\omega}$ generated by branches and antichains.*

1. *$=_J$ is not proper-trim;*
2. *there is a Borel homomorphism to a trim equivalence relation F such that preimages of F-classes are meager;*
3. *$=_J$ is F-generically ergodic for every equivalence F classifiable by countable structures as well as $=_K$-ergodic where K is the branch ideal on $2^{<\omega}$.*

Thus, the usual product topology on $2^{2^{<\omega}} = \mathrm{dom}(=_J)$ in this case does have generic ergodicity features for many trim equivalence relations, but not for all. There is a Polish topology on $\mathrm{dom}(=_J)$ which has generic ergodicity for all trim equivalence relations by Theorem 7.1.4, but it is not the usual product topology.

Proof. Write $X = 2^{2^{<\omega}} = \mathrm{dom}(=_J)$. For (1), let $C = \{\langle n, m\rangle : m \leq n\}$ and let K be the eventually different ideal on C. Let $\{t_n : n \in \omega\} \subset 2^{<\omega}$ be an antichain and let $\pi : C \to 2^{<\omega}$ be the injection defined by $\pi(n, m) = t_n^\frown 0^m$. It is easy to see that a subset of C is in K if and only if its π-image is in J. Thus, π naturally extends to a continuous injective reduction $h : 2^C \to X$ of $=_K$ to $=_J$ defined by $h(y)(\pi(n, m)) = y(n, m)$ and $h(y)(t) = 0$ if $t \notin \mathrm{rng}(\pi)$. As $=_K$ is not trim, $=_J$ cannot be trim either by Theorem 6.2.5.

For (2), let $Y = 2^\omega$ and let I be the σ-ideal of nowhere dense compact subsets of Y, and construct the desired homomorphism from $=_J$ to $=_I$. As $=_I$ is trim by Theorem 6.6.20, this will complete the proof of (2). Indeed, let $C \subset 2^\omega$ be a countable dense set so that $2^C = \mathrm{dom}(=_I)$. Let $\pi : 2^{<\omega} \to C$ be a bijection such that $\pi(t) \in [t]$. It is easy to see that if $a \subset 2^{<\omega}$ is a chain or an antichain, then the closure of $\pi''a \subset Y$ is nowhere dense. Thus, π naturally extends to a homomorphism $h : X \to 2^C$ defined by $h(x)(\pi(t)) = h(t)$ and $h(x)(c) = 0$ if $c \notin \mathrm{rng}(\pi)$. This homomorphism has the requested properties.

For (3), I will show that E is not σ-Cohen trim. More specifically, let P for the poset of finite partial functions from $2^{<\omega}$ to 2. In some generic extension, I will construct points $x_n \in X$ for $n \in \omega$ so that

- the points x_n are pairwise $=_J$-related;
- each x_n is P-generic over the ground model;
- the set $\{x_n : n \in \omega\}$ is dense in X;
- $\bigcap_n V[x_n] = V$.

The theorem then swiftly follows. If F is an equivalence relation classifiable by countable structures or the equivalence $=_J^{2^\omega}$ and $h : X \to \mathrm{dom}(F)$ is a Borel homomorphism of $=_J$ to F, then the values $h(x_n) \in V[x_n]$ for $n \in \omega$ are pairwise F-related by the first item. By Theorem 6.7.7 or 6.7.11, their F-class is represented in the ground model by some $y \in \omega^\omega$. By the second and third item, the set $h^{-1}[y]_F$ is comeager, as it is a Borel set which contains a dense set of P-generic reals. This will complete the proof of the theorem.

For the construction of the points x_n, it is enough to perform the following task. Given a P-generic point $x_0 \in X$, a condition $p \in P$, and a set $c \in V[x_0] \setminus V$ of ordinals, in some further forcing extension I will find a point x_1 extending p, $=_J$-related to x_0 and P-generic over the ground model, such that $c \notin V[x_1]$.

Work in the model $V[x_0]$. Consider the index set $2^{<\omega}$ with the extension ordering and the usual order topology. Let $\mathfrak{F}$ be the collection of clopen subsets $O \subset 2^{<\omega}$ such that $c \in V[x_0 \restriction O]$.

Claim 7.1.6. *$\mathfrak{F}$ is a filter of nonempty clopen sets.*

Proof. The empty set does not belong to $\mathfrak{F}$ by the choice of c. If $O_0, O_1 \in \mathfrak{F}$, then observe that the functions $x_0 \restriction (O_1 \setminus O_0)$ and $x_0 \restriction (O_0 \setminus O_1)$ are product generic over the model $V[x_0 \restriction O_0 \cap O_1]$. Now, $c \in V[x_0 \restriction O_0]$ and $c \in V[x_0 \restriction O_1]$, by the product forcing theorem 2.2.6 applied in the model $V[x_0 \restriction O_0 \cap O_1]$ it follows that $c \in V[x_0 \restriction O_0 \cap O_1]$, and so $O_0 \cap O_1 \in \mathfrak{F}$ as well. □

Use the claim and a compactness argument to find a branch $z \in 2^\omega$ such that each clopen set in F contains all but finitely many initial segments of z. Consider the poset Q of all pairs $q = \langle p_q, a_q \rangle$ where $p_q \in P$ and $a_q \subset \mathrm{dom}(p_q)$ is a finite antichain of binary strings which are not an initial segment of z such that for every $t \in \mathrm{dom}(p_q)$, if $p_q(t) \neq x_0(t)$ then $t \in a_q$ or t is an initial segment of z. The poset Q is ordered by $r \leq q$ if $p_q \subset p_r$ and $a_q \subset a_r$. Let $\dot{x}_1 \in X$ be the Q-name for the union of the first coordinates of the conditions in the generic filter. I will verify the required properties of the name $\dot{x}_1$ one by one.

Claim 7.1.7. *$Q \Vdash \dot{x}_1$ is P-generic over the ground model.*

Proof. Let $q \in Q$ and let $D \subset P$ be an open dense subset in the ground model. By the P-genericity of the point x_0, there is a condition $p \in P$ such that $p \subset x_0$ and p rew $p_q \in D$. Then, the condition $r = \langle p \text{ rew } p_q, a_q \rangle$ is stronger than q in the poset Q and it forces $\dot{x}_1$ to meet the open dense set D in the condition p rew p_q. □

Claim 7.1.8. *$Q \Vdash \check{x}_0 =_J \dot{x}_1$. The branch z is the only limit point of the set $\{t \in 2^{<\omega} : \check{x}_0(t) \neq \dot{x}_1(t)\}$.*

Proof. The definition of the poset Q immediately guarantees that the set $\{t \in 2^{<\omega} : x_0(t) \neq x_1(t)\}$ is covered by the union of the set of initial segments of z (a branch) and the union of the second coordinates of conditions in the generic filter (an antichain).

For the second sentence, suppose that $q \in Q$ is a condition and $n \in \omega$. It is easy to strengthen the condition q and find a number $m \geq n$ so that $a_q \cup \{z \restriction m\}$ is a maximal antichain in $2^{\leq m}$. Then q forces that all elements of the set $\{t \in 2^{<\omega} : x_0(t) \neq x_1(t)\}$ longer than m already extend the string $z \restriction m$. The second sentence now follows by a straightforward density argument. $\square$

Claim 7.1.9. $Q \Vdash \check{c} \notin V[\dot{x}_1]$.

Proof. If this failed, there would be a condition $q \in Q$, a P-name τ in the ground model such that $q \Vdash \tau/x_1 = \check{c}$. Find an initial segment $t \subset z$ such that all elements of a_q are incompatible with t in $2^{<\omega}$. Consider the clopen set $O \subset 2^{<\omega}$ consisting of all binary strings which do not extend t.

Use the genericity of x_0 to find a finite fragment $p \in P$ of x_0 such that p rew p_q decides in P the statement $\tau \in V[\dot{x}_{gen} \restriction O]$. The decision must be in the negative. Otherwise, the condition $r = \langle p \text{ rew } p_q, a_q\rangle \leq q$ in Q forces $\tau/\dot{x}_1 \in V[\dot{x}_1 \restriction O]$. Since $Q \Vdash \dot{x}_1 \restriction O = \check{x}_0 \restriction O$ up to finitely many exceptions by Claim 7.1.8, r also forces $\check{c} = \tau/\dot{x}_1 \in V[x_0 \restriction O]$, which contradicts the definition of O and the choice of the guiding branch $z \in 2^\omega$.

Now, by Lemma 6.6.3 applied in the model $V[x_0 \restriction O]$ and the P-genericity of x_0 over the ground model, there must be two conditions $p_0, p_1 \in P$ and an ordinal α such that

- p_0 is a finite fragment of x_0;
- $\mathrm{dom}(p_0) = \mathrm{dom}(p_1)$ and there is unique $s \in \mathrm{dom}(p_0)$ such that $p_0(s) \neq p_1(s)$. This s moreover does not belong to the set O;
- the conditions p_0 rew p_q and p_1 rew p_q in the poset P decide the statement $\check{\alpha} \in \tau$ differently.

Suppose for definiteness that p_1 rew $p_q \Vdash_P \check{\alpha} \in \tau$ while $\alpha \notin c$. Let $s \in \mathrm{dom}(p_1)$ be the unique string such that $p_0(s) \neq p_1(s)$. Note that s extends t. Suppose for definiteness that s is not an initial segment of z. Then $r = \langle p_1 \text{ rew } p_q, a_q \cup \{s\}\rangle$ is a condition in the poset Q forcing that α belongs to the symmetric difference of c and τ/x_1. This contradicts the choice of q and τ. $\square$

$\square$

There are many ergodicity results among various trim equivalence relations as well. I will show that the trim equivalence relations $=_I$ for a σ-ideal I of compact sets possess ergodicity properties with respect to great many other trim or not trim equivalence relations. In particular, this shows that they are not included in Kanovei's class $\mathfrak{K}$ of Definition 6.5.5.

Definition 7.1.10. $\mathfrak{E}$ is the smallest class of analytic equivalence relations containing the identity and closed under the operations of Borel reduction, countable union, Friedman-Stanley jump, and infinite product modulo any F_σ-ideal.

Note that the generating operations of the class $\mathfrak{E}$ include the generating operations of $\mathfrak{K}$, and therefore $\mathfrak{K} \subset \mathfrak{E}$. The inclusion between these two classes is proper. For example E_2 belongs to $\mathfrak{E}$, as the summable ideal is F_σ. On the other hand, E_2 has ergodicity properties with respect to every equivalence relation in $\mathfrak{K}$ by [12, Theorem 13.5.3]; in particular, $E_2 \notin \mathfrak{K}$.

Theorem 7.1.11. *Let X be a zero-dimensional compact Polish space without isolated point. Let I be an analytic σ-ideal of compact sets on X containing all singletons. Then $=_I$ is F-generically ergodic for every equivalence relation $F \in \mathfrak{E}$.*

The proof uses the following abstract technical lemma.

Lemma 7.1.12. *Let E be an analytic equivalence relation on a Polish space X. If G_0, G_1 are mutually Cohen-generic filters over V and $x_0 \in V[G_0]$, $x_1 \in V[G_1]$ are E-related points, then they are E-related to some point in the ground model.*

Proof. It is possible to use a standard Kuratowski–Ulam argument. I will provide a more general proof using the technologies developed earlier in the book. Write P for the Cohen forcing. Suppose that $p_0, p_1 \in P$ are conditions and $\dot{x}_0, \dot{x}_1$ are P-names such that $\langle p_0, p_1\rangle \Vdash_{P\times P} \dot{x}_0/\dot{G}_{\text{left}}\ E\ \dot{x}_1/\dot{G}_{\text{right}}$. Then $\langle P \restriction p_0, \dot{x}_0\rangle\ \bar{E}\ \langle P \restriction p_1, \dot{x}_1\rangle$, and so $\dot{x}_0$ is an E-pinned name on $P \restriction p_0$. However, the Cohen forcing is c.c.c., therefore reasonable, and so all E-pinned names on it must be trivial by Theorem 3.3.2. Thus, there must be $x \in X$ in the ground model such that $\langle p_0, p_1\rangle \Vdash_{P\times P} \dot{x}_0/\dot{G}_{\text{left}}\ E\ \check{x}$. This completes the proof. □

Proof of Theorem 7.1.11. Let $C \subset X$ and $D \subset Y$ be countable sets so that $2^C = \text{dom}(=_I)$. Let P be the poset of finite partial functions from C to 2 ordered by inclusion. In some generic extension, I will construct points $x_n \in 2^C$ for $n \in \omega$ such that

- each x_n is P-generic over the ground model;
- the set $\{x_n : n \in \omega\} \subset 2^C$ is dense;
- the points x_n for $n \in \omega$ are pairwise $=_I$-related;
- if $F \in \mathfrak{E}$ is a ground model coded analytic equivalence relation and an F-class is represented in all models $V[x_n]$, then it is represented already in the ground model.

This immediately implies the theorem. If $F \in \mathfrak{E}$ is an analytic equivalence relation on a Polish space Y and $h : 2^C \to Y$ is a Borel homomorphism of E to F, then the values $h(x_n) \in V[x_n]$ for $n \in \omega$ come from the same F-class. By the last item, there is a point $y \in Y$ in the ground model which belongs to this F-class as well. As a result, the preimage $h^{-1}[y]_F \subset 2^C$ is a ground model coded analytic set which in some extension contains a dense set of Cohen reals, therefore must be comeager as required.

Let $x_0 \in 2^C$ be a P-generic point over V; work in the model $V[x_0]$. I will describe a poset Q adding a point $x_1 \in 2^C$ which is also P-generic over V, $=_I$-related to x_0, and such that for every equivalence relation $F \in \mathfrak{E}$ on some Polish space Y, every point $y \in Y$ in the model $V[x_0]$ which has no F-equivalent in V, there is a condition $q \in Q$ which forces that $V[\dot{x}_1]$ contains no F-equivalent of y. Then, force with a finite support product of countably many copies of the poset Q over the model $V[x_0]$. Denoting these copies with Q_n for $n > 0$ and their respective generic points with $x_n \in 2^C$, elementary density arguments will show that the points $\{x_n : n \in \omega\}$ have the required properties.

Work in the model $V[x_0]$, let Q be the poset of all triples $q = \langle p_q, O_q, D_q \rangle$ where $p_q \in P$, $O_q \in \mathcal{O}$, and $D_q \subset X$ is a finite set disjoint from O_q. The ordering is defined by $r \leq q$ if $p_q \subset p_r$, $O_q \subset O_r$, $D_q \subset D_r$, and $(p_r \setminus p_q) \restriction O_q \subset x_0$. Let $\dot{x}_1$ be the Q-name for the union of the first coordinates in the generic filter.

Claim 7.1.13. *$Q \Vdash \dot{x}_1 \in 2^C$ is a P-generic point over the ground model.*

Proof. Let $q \in Q$ be a condition and $B \subset P$ an open dense subset in the ground model. The genericity of the point $x_0 \in 2^C$ implies that there is a finite fraction p of x_0 such that p rew $p_q \in B$. Then, the condition $\langle p \text{ rew } p_q, O_q, D_q \rangle$ is stronger than q in the poset Q and forces the point $\dot{x}_1$ to meet the open dense set $B \subset P$ in the condition p rew p_q. □

Claim 7.1.14. $Q \Vdash \check{x}_0 =_I \dot{x}_1$.

Proof. Let $\dot{O}_{gen}$ be the Q-name for the union of the second coordinates of conditions in the generic filter. I will prove that $Q \Vdash X \setminus \dot{O}_{gen} \in I$. This immediately implies the claim.

Since I is an analytic σ-ideal of compact sets, it is in fact G_δ in the hyperspace $K(X)$ by [14, Theorem 33.3]. Thus, $I = \bigcap_n U_n$ for some open sets $U_n \subset K(X)$. The sets U_n may be selected downwards closed, and as is the case for every downwards closed open subset of $K(X)$, there are collections $\mathcal{O}_n$ of open subsets of X such that $K \in U_n$ if and only if there is $O \in \mathcal{O}_n$ such that $K \subset O$.

Now, let $q \in Q$ be a condition and $n \in \omega$. It will be enough to find a condition $r \leq q$ and an open set $O \in \mathcal{O}_n$ such that $O_r \cup O = X$. Such a condition r forces $X \setminus \dot{O}_{gen} \subset O$, and a straighforward density argument leads to the conclusion that $X \setminus \dot{O}_{gen} \in \bigcap_n U_n = I$.

To find the condition r and the open set $O \in \mathcal{O}_n$, just observe that the finite set $D_q \subset X$ is in the ideal I by the assumptions, and so there is $O \in \mathcal{O}_n$ such that $D_q \subset O$. Since the points in the set D_q are not isolated, a compactness argument yields a clopen set O' such that $O' \cup O = X$ and $D_q \cap O' = 0$. The condition $r = \langle p_q, O_q \cup O', D_q \rangle \leq q$ together with the set O works. □

Let Y be a Polish space and F and equivalence relation on Y, both in the ground model. Let $y \in Y$ be a point in $V[x_0]$ which has no F-equivalent in the ground model. Let $\mathfrak{F}_{y,F}$ be the collection of those sets $O \in \mathcal{O}$ such that y has an F-equivalent in the model $V[x_0 \restriction O]$.

Claim 7.1.15. *$\mathfrak{F}_{y,F}$ is a filter of nonempty clopen subsets of the space X.*

Proof. The empty set does not belong to $\mathfrak{F}$ by the choice of the point $y \in Y$. $\mathfrak{F}$ is closed under taking supersets essentially by its definition. To show that it is closed under intersections, suppose that $O_0, O_1 \subset 2^{<\omega}$ are clopen sets such that the F-class of y is represented both in $V[x_0 \restriction O_0]$ and $V[x_0 \restriction O_1]$. Observe that $V[x_0 \restriction O_0]$ and $V[x_0 \restriction O_1]$ are product Cohen-generic extensions of the model $V[x_0 \restriction (O_0 \cap O_1)]$. Apply Lemma 7.1.12 in the model $V[x_0 \restriction (O_0 \cap O_1)]$ to conclude that the F-class of y is represented already there, and so $O_0 \cap O_1 \in \mathfrak{F}$ as desired. $\square$

A compactness argument shows that the filter $\mathfrak{F}_{y,F}$ has a nonempty intersection.

Claim 7.1.16. *Suppose that $F \in \mathfrak{E}$. Any condition $q \in Q$ with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$ forces y to have no F-equivalent in the model $V[\dot{x}_1]$.*

Proof. I will show that the class of all equivalence relations in the ground model satisfying the statement of the claim is closed under the generating operations of the class $\mathfrak{E}$.

The argument divides into a rather long list of cases, and one specific trick will be used in many of them. Let $q \in Q$ be an condition, and work in the model $V[x_0 \restriction O_q]$. Define a poset P_q of all finite partial functions from $C \cap O_q$ to 2 extending $p_q \restriction (C \cap O_q)$, ordered by reverse inclusion. Let $\dot{x}$ be the P_q-name for the point in 2^C obtained as the union of $x_0 \restriction O_q$ with all functions in the P_q-generic filter. As $q \Vdash_Q \dot{x}_1$ is a Cohen-generic point over V extending $(x_0 \restriction O_q)$ rew p_q, the usual Cohen forcing factorization arguments show that $q \Vdash_Q$ the equation $\dot{x} = \dot{x}_1$ yields a P_q-generic filter over $V[x_0 \restriction O_q]$. This implies the following:

(*) if $\bar{p} \in P_q$ is a condition and $\phi(\cdot)$ is an analytic formula with parameters in the model $V[x_0 \restriction O_q]$, and $\bar{p} \Vdash_{P_q} \phi(\dot{x})$, then $\langle p_q \cup \bar{p}, O_q, D_q \rangle \Vdash_Q \phi(\dot{x}_1)$.

Now I am ready to tackle the discussion of the generating operations of the class $\mathfrak{E}$.

Case 1. Regarding the Friedman–Stanley jump, assume that F is an analytic equivalence relation on a Polish space Y for which the statement has been verified. Consider the equivalence relation F^+ on the space Y^ω. Let $y \in Y^\omega$ be a point in $V[x_0]$ which has no F^+-equivalent in the ground model. let $q \in Q$ be a condition with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, and let τ be a ground model P-name for an element of the space Y^ω. I will produce a condition $r \leq p$ and a number $i \in \omega$ such that either $r \Vdash (\tau/\dot{x})(i) \notin [\mathrm{rng}(y)]_F$ or $r \Vdash \check{y}(i) \notin [\mathrm{rng}(\tau/\dot{x}_1)]_F$. This will complete the proof.

Case 1a. There is an number $i \in \omega$ such that $y(i)$ has no F-equivalent in the model $V[x_0 \restriction O_q]$. Consider the set $\mathfrak{F}_{y(i),F}$. Just as in Claim 7.1.15, the collection $\{O \setminus O_q : O \in \mathfrak{F}_{y(i),F}\}$ consists of nonempty clopen sets and has the finite intersection property. A compactness argument then yields a point $z' \in X$ in its intersection. Consider the condition $r = \langle p_q, O_q, D_q \cup \{z'\}\rangle \leq q$. By the assumption on the equivalence relation F, $r \Vdash y(i)$ has no F-equivalent in the model $V[x_1]$, in particular no F-equivalent in the set $\mathrm{rng}(\tau/\dot{x}_1)$.

Case 1b. If Case 1a fails, work in the model $V[x_0 \restriction O_q]$ and consider the poset P_q and its name $\dot{x} \in 2^C$. There are three subcases:

Case 1ba. There is a condition $\bar{p} \in P_q$ forcing $\tau/\dot{x}$ to have an F^+-equivalent in the model $V[x_0 \restriction O_q]$. Strengthening the condition $\bar{p}$ if necessary, I may identify this equivalent $y' \in Y^\omega$. Since $y'\ F^+\ y$ fails, $[\mathrm{rng}(y)]_E \neq [\mathrm{rng}(y')]_E$ and so there must be $i \in \omega$ such that either $y(i) \notin [\mathrm{rng}(y')]_E$ or $y'(i) \notin [\mathrm{rng}(y)]_E$. By (*) above, the condition $r = \langle p_q \cup \bar{p}, O_q, D_q\rangle \leq q \in Q$ and the number $i \in \omega$ clearly work as desired.

Case 1bb. There is a condition $\bar{p} \in P_q$ forcing that there is some $i \in \omega$ such that $(\tau/\dot{x})(i)$ has no F-equivalent in the model $V[x_0 \restriction O_q]$. Strengthening $\bar{p}$ if necessary, I may find a specific $i \in \omega$ satisfying this. In view of Case 1b assumption and (*) above, the condition $r = \langle p_q \cup \bar{p}, O_q, D_q\rangle \leq q \in Q$ and the number $i \in \omega$ work as required.

Case 1bc. If both Cases 1ba and 1bb fail, then there must be a point $y' \in Y$ in the model $V[x_0 \restriction O_q]$ and conditions $\bar{p}_0, \bar{p}_1 \in P_q$ such that $\bar{p}_0 \Vdash \check{y}' \in [\tau/\dot{x}_1]_F$ and $\bar{p}_1 \Vdash \check{y}' \notin [\tau/\dot{x}_1]_F$. (Otherwise, the set $\{y' \in Y : P_q \Vdash y' \in [\tau/\dot{x}_1]_F\}$ contains only countably many F-equivalence classes by the c.c.c. of P_q, and if $y'' \in Y^\omega$ visits exactly these F-classes then $P_q \Vdash \check{y}''\ F^+\ \tau/\dot{x}$ by the failure of Case 1bb. This directs us to Case 1ba.) The treatment now divides into two further subcases depending on whether $y' \in [\mathrm{rng}(y)]_F$ or not. Assume for definiteness that the latter is the case. Strengthen the condition $\bar{p}_0 \in P_q$ if necessary to find a specific number i such that $\bar{p}_0 \Vdash (\tau/\dot{x})(i)\ F\ y'$. In view of (*), the condition $r = \langle p_q \cup \bar{p}_0, O_q, D_q\rangle \leq q \in Q$ with the number $i \in \omega$ work as required.

Case 2. Now move to the case in which the equivalence relation F is obtained by a product modulo an F_σ-ideal of equivalence relations on which the claim has been already verified. Suppose that J is an F_σ-ideal and use a theorem of Mazur [20] to find a lower semicontinuous submeasure μ on ω such that $J = \{a \subset \omega : \mu(a) < \infty\}$. Let $\{F_i : i \in \omega\}$ be a collection of equivalence relations on the respective Polish spaces Y_i for $i \in \omega$ in the ground model such that the statement of the claim holds for each of them. Let $Y = \prod_i Y_i$, let $F = \prod_i F_i$ modulo J, let $y \in Y$ be a point in $V[x_0]$ which has no F-equivalent in the ground model. Let $q \in Q$ be a condition with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, let τ be a ground model P-name for an element of the space Y, and let $n \in \omega$. I must produce a condition $r \leq q$ and a finite set $b \subset \omega$ such that $\mu(b) > n$ and $r \Vdash \forall i \in \check{b}\ \neg(\tau/\dot{x}_1)(i)\ F_i\ y(i)$. The treatment divides into two cases.

Case 2a. The set $a = \{i \in \omega : y(i)$ has no F_i-equivalent in the model $V[x_0 \restriction O_q]\}$ is J-positive. Find a subset b of it of μ-mass $> n$. For every $i \in b$ consider the set $\mathfrak{F}_{y(i),F_i}$. Just as in Claim 7.1.15, the collection $\{O \setminus O_q : O \in \mathfrak{F}_{y(i),F_i}\}$ consists of nonempty clopen sets and has the finite intersection property. A compactness argument then yields a point $z_i \in X$ in its intersection. Then, the condition $r = \langle p_q, O_q, D_q \cup \{z_i : i \in b\}\rangle \leq q \in Q$ has the desired properties. For every $i \in b$, $D_r \cap \bigcap \mathfrak{F}_{y,F_i} \neq 0$ and so by the assumption on the equivalence relation F_i, $r \Vdash y_i$ has no F_i-equivalent in the model $V[x_1]$, in

particular $(\tau/\dot{x}_1)(i)$ is not equivalent to $y(i)$.
Case 2b. If Case 1 fails, then $a \in J$ and so $\mu(a) < m$ for some number $m \in \omega$. Work in the model $V[x_0 \restriction O_q]$. Consider the poset P_q with its name $\dot{x} \in 2^C$. There are three subcases:
Case 2ba. There is a condition $\bar{p} \in P_q$ forcing $\tau/\dot{x}$ to have an F-equivalent in the model $V[x_0 \restriction O_q]$. Strengthening the condition $\bar{p}$ if necessary, I may identify this equivalent y' as well as the number $k \in \omega$ such that $\bar{p} \Vdash \mu(\{i \in \omega : \neg(\tau/\dot{x})(i)\ F_i\ y'(i)\}) < k$. Since $y'\ F\ y$ fails, there is a finite set $b' \subset \omega$ of such that $\mu(b') > n + k$ and $\forall i \in a\ \neg y(i)\ F_i\ y'(i)$. Strengthening the condition $\bar{p}$ further I can identify the set $b = \{i \in b' : (\tau/\dot{x})(i)\ F_i\ y'(i)\}$; by the subadditivity of the submeasure μ, it has to be the case that $\mu(b) > n$. Now (*) above shows that the condition $r = \langle p_q \cup \bar{p}, O_q, D_q\rangle \leq q \in Q$ and the set b have the required properties.
Case 2bb. There is a condition $\bar{p} \in P_q$ forcing the set $\{i \in \omega : (\tau/\dot{x})(i)$ has no F_i-equivalent in the model $V[x_0 \restriction O_q]\}$ is in the ideal J. Strengthening the condition $\bar{p}$ if necessary, I can identify a finite subset b' of this set such that $\mu(b') > n + m$. The set $b = b' \setminus a$ must have $\mu(b) > n$. Again, (*) above shows that the condition $r = \langle p_q \cup p', O_q, D_q\rangle \leq q \in Q$ and the set b have the required properties.
Case 2bc. If both Cases 2ba and 2bb fail, then there must be conditions $\bar{p}_0, \bar{p}_1 \in P_q$, a finite set $b' \subset \omega$ with $\mu(b') > 2n$, and functions y_0, y_1 with domain b' such that for every $i \in b'$, $\bar{p}_0 \Vdash (\tau/\dot{x}_i)\ F_i\ y_0(i)$, $\bar{p}_1 \Vdash (\tau/\dot{x}_i)\ F_i\ y_1(i)$, and $\neg y_0(i)\ F_i\ y_1(i)$. One of the sets $\{i \in b' : \neg y(i)\ F_i\ y_0(i)\}$, $\{i \in b' : \neg y(i)\ F_i\ y_1(i)\}$ must have μ-mass greater than n as they together cover the set b'. Suppose for definiteness it is the former, and call it b. Then (*) above implies that the condition $r = \langle p_q \cup p', O_q, D_q\rangle \leq q \in Q$ and the set b have the required properties.
Case 3. Now, move to the case where F is obtained as a countable union $F = \bigcup_n F_n$ of analytic equivalence relations on some Polish space Y for which the statement of the claim has already been verified. Here, observe that $\mathfrak{F}_{y,F_n} \subset \mathfrak{F}_{y,F}$: if $O \subset X$ is a clopen set and $y \in Y$ has an F_n-equivalent in the model $V[x_0 \restriction O]$, then this same point is in fact an F-equivalent of y as $F_n \subset F$. Thus, if $q \in Q$ is a condition with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, then for every $n \in \omega$, $D_q \cap \bigcap \mathfrak{F}_{y,F_n} \neq 0$. By the assumption on the equivalence relations F_n, the condition $q = \langle 0, 0, \{z\}\rangle$ forces that y has no F_n-equivalent in the model $V[\dot{x}_1]$, and since $F = \bigcup_n F_n$, it also cannot have an F-equivalent there.
Case 4. Finally, move to the case where the equivalence relation F is Borel reducible to an equivalence relation F' on a Polish space Y' via some Borel reduction $h : Y \to Y'$, and the statement of the claim holds for F'. Observe that $\mathfrak{F}_{y,F} = \mathfrak{F}_{h(y),F'}$: if $O \subset X$ is a clopen set and $y \in Y$ has an F-equivalent $y_0 \in V[x_0 \restriction O]$ then $h(y_0)$ is an F'-equivalent of $h(y)$ in the same model. On the other hand, if $h(y)$ has an F'-equivalent $y'_0 \in Y'$ in the model $V[x_0 \restriction O]$, then by the Mostowski absoluteness between the models $V[x_0 \restriction O]$ and $V[x_0]$, the model $V[x_0 \restriction O]$ must contain a point $y_0 \in Y$ such that $h(y_0)\ F'\ y'_0$, and then y_0 is an F-equivalent of y in the model $V[x_0 \restriction O]$ as h is a reduction. Thus, if $q \in Q$ is a condition such that $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, then also $D_q \cap \bigcap \mathfrak{F}_{y,F'} \neq 0$. By

the claim applied to the equivalence relation F', the condition $q \in Q$ then forces the model $V[\dot{x}_1]$ to contain no F'-equivalent of $h(y)$. Thus, the model cannot contain any F-equivalent of y, since its h-image would be an F'-equivalent of $h(y)$. □

This completes the verification of the desired properties of the poset Q and the proof of the theorem. □

Corollary 7.1.17. *Let J be the asymptotic density zero ideal on ω. Then $=_J$ is F-generically ergodic for every equivalence relation $F \in \mathfrak{E}$.*

This greatly strengthens the standard result: $=_J$ is not Borel reducible to E_{K_σ} .

Proof. Let $\pi : \omega \to 2^{<\omega}$ be any bijection sending each interval $[2^n, 2^{n+1}) \subset \omega$ to the set $2^n \subset 2^{<\omega}$. Let $\chi : 2^{<\omega} \to 2^\omega$ be any injection such that $\chi(t) \in [t]$ holds for every $t \in 2^{<\omega}$. Let ϕ be the usual product measure on 2^ω, let I be the σ-ideal of ϕ-null subsets of 2^ω, let $C = \mathrm{rng}(\chi)$, and let K be the ideal on C consisting of the sets $a \subset C$ such that the closure of a belongs to I. In view of Theorem 7.1.11, it is enough to show that if $b \subset C$ is any set then $\pi^{-1}\chi^{-1}b \in J$. In other words, the ideal K becomes a subset of J under the identification given by $\pi^{-1}\chi^{-1}$.

Suppose that $a \subset 2^{<\omega}$ is a set such that $\pi^{-1}a \notin J$. This means that there is a positive real $\varepsilon > 0$ and infinitely many numbers $n_i \in \omega$ for $i \in \omega$ such that $a \cap 2^{n_i}$ has cardinality at least $\varepsilon 2^{n_i}$. A standard measure theoretic argument shows that the set $B = \{x \in 2^\omega : \exists^\infty i\ x \restriction n_i \in a\}$ has product mass at least ε, and it is a subset of the closure of $\chi''b$. This completes the proof. □

Corollary 7.1.18. *Suppose that I is a nonprincipal analytic σ-ideal of compact sets on a zero-dimensional compact space without isolated points. Then $=_I$ does not belong to Kanovei's class $\mathfrak{K}$.*

Proof. The generating operations of Kanovei's class $\mathfrak{K}$ are included in the generating operations of the class $\mathfrak{E}$, so $\mathfrak{K} \subset \mathfrak{E}$. Thus, not only $=_I$ does not belong to $\mathfrak{K}$, it is even F-generically ergodic for every equivalence relation $F \in \mathfrak{K}$ by Theorem 7.1.11. □

Thus, the equivalence relations of the form $=_I$ for a σ-ideal I of compact sets on a compact space seem to be enormously complicated, and quite high in the Borel reducibility order. The last theorem of this section shows that among equivalence relations of this form, there are still further distinctions.

Theorem 7.1.19. *Let X, Y be compact Polish spaces without isolated points and μ a Borel probability measure on Y. Let I be the σ-ideal of meager sets on X and J the σ-ideal of μ-null sets. Then $=_I$ is $=_J$-generically ergodic.*

Note that the opposite ergodicity does not hold. If, for example, $X = Y = 2^\omega$ and μ is the usual product measure on 2^ω, then $J \subset I$ on compact subsets of X and therefore the identity is a continuous homomorphism of $=_J$ to $=_I$ in which preimages of $=_I$-classes are meager. This raises an obvious question.

Question 7.1.20. Let I be the σ-ideal of meager sets and J the σ-ideal of product null sets on 2^ω. Is $=_J$ Borel reducible to $=_I$?

Proof of Theorem 7.1.19. Let $C \subset X$ and $D \subset Y$ be countable sets so that $2^C = \mathrm{dom}(=_I)$ and $2^D = \mathrm{dom}(=_J)$. Let P be the poset of finite partial functions from C to 2 ordered by inclusion. In some generic extension, I will construct points $x_n \in 2^C$ for $n \in \omega$ such that

- each x_n is P-generic over the ground model;
- the set $\{x_n : n \in \omega\} \subset 2^C$ is dense;
- the points x_n for $n \in \omega$ are pairwise $=_I$-related;
- if a $=_J$ class is represented in all models $V[x_n]$, then it is represented already in the ground model.

This immediately implies the theorem. If $h : 2^C \to 2^D$ is a Borel homomorphism of $=_I$ to $=_J$, then the values $h(x_n) \in V[x_n]$ for $n \in \omega$ come from the same $=_J$-class. By the last item, there is a point $y \in 2^D$ in the ground model which belongs to this $=_J$-class as well. As a result, the preimage $h^{-1}[y]_{=_J} \subset 2^C$ is a ground model coded analytic set which in some extension cotains a dense set of Cohen reals, therefore must be comeager as required.

Let $x_0 \in 2^C$ be a P-generic point over the ground model, and $y \in 2^D \cap V[x_0]$ be a point which is not $=_J$-related to any element in the ground model. It will be enough to produce a forcing adding a point $x_1 \in 2^C$ which is $=_I$-related to x_0, P-generic over the ground model, and such that y has no $=_J$-equivalent in the model $V[x_1]$.

Let $\mathcal{O}_X, \mathcal{O}_Y$ be countable bases of the topologies on X, Y closed under finite unions and intersections such that the boundaries of sets in them do not intersect the countable sets C, D. The treatment divides into two cases.

Case 1. There is $\varepsilon > 0$ such that for no set $U \in \mathcal{O}_Y$ of μ-mass $> 1-\varepsilon$ there is a ground model point $z \in 2^D$ such that $y \restriction U =_J z \restriction U$. In this case, let Q be the poset of all pairs $q = \langle p_q, O_q \rangle$ such that $p_q \in P, O_q \in \mathcal{O}_X$, and there is $\delta_q > \varepsilon/2$ such that there is no set $U \in \mathcal{O}_Y$ of μ-mass $> 1 - \delta_q$ and no point $z \in 2^D$ in the model $V[x_0 \restriction O_q]$ such that $y \restriction U =_J z \restriction U$. The ordering is defined by $r \leq q$ if $p_q \subset p_r$ and $O_q \subset O_r$ and $(p_r \setminus p_q) \restriction O_q \subset x_0$. Let $\dot{x}_1$ be the Q-name for the union of the first coordinates of conditions in the generic filter. I will verify the requisite properties of the name $\dot{x}_1$ one by one.

Claim 7.1.21. $Q \Vdash \dot{x}_1 \in 2^C$ *is a P-generic point over the ground model.*

Proof. Let $q \in Q$ be a condition and $B \subset P$ an open dense subset in the ground model. The genericity of the point $x_0 \in 2^C$ implies that there is a finite fraction p of x_0 such that p rew $p_q \in B$. Then, the condition $\langle p \text{ rew } p_q, O_q \rangle$ is stronger than q in the poset Q and forces the point $\dot{x}_1$ to meet the open dense set $B \subset P$ in the condition p rew p_q. $\square$

Claim 7.1.22. $Q \Vdash \check{x}_0 =_I \dot{x}_1$.

Proof. Let $\dot{O}_{gen}$ be the Q-name for the union of the second coordinates of conditions in the generic filter. I will prove that Q forces $\dot{O}_{gen} \subset X$ to be open dense. This immediately implies the claim. Look at the closure of the set $\{c \in C : x_0(c) \neq x_1(c)\}$ in the space X. To show that this is nowhere dense, it is enough to show that it is a subset of the union of the nowhere dense set $X \setminus \dot{O}_{gen}$ and the countable set C. Indeed, suppose that $z \in \dot{O}_{gen}$ is a point in this closure. Then there is a condition $q \in Q$ in the generic filter such that $z \in O_q$. This condition forces the set $\{c \in O_q \cap C : x_0(c) \neq \dot{x}_1(c)\}$ to be finite, included in the set $\mathrm{dom}(p_q)$. Thus, the point z must be an element of this finite set and therefore belongs to C.

Towards the proof of density of $\dot{O}_{gen}$, let $q \in Q$ and $O \in \mathcal{O}_X$ be a nonempty open set. I must produce a condition $r \leq q$ such that $O_r \cap O \neq 0$. Clearly, I may assume that $O_q \cap O = 0$. Let δ be some real number such that $\varepsilon/2 < \delta < \delta_q$ and let $n \in \omega$ be so large that $\delta_q(1 - 1/n) > \delta$. The purpose of the choice of such number n is to make sure that whenever $\langle Z, \eta\rangle$ is a probability measure space and $B \subset n \times Z$ is a Borel set whose vertical sections have ν-mass at least $1 - \delta$, then at least ν-mass $1 - \delta_q$ many elements of Z belong to at least two vertical sections of the set B. This is proved by a straightforward Fubini argument. Now, let $\{O_i : i \in n\}$ be pairwise disjoint nonempty subsets of O in $\mathcal{O}_X$; I will show that for some $i \in n$, the pair $r_i = \langle p_q, O_q \cup O_i\rangle$ is a condition in Q as witnessed by δ. Then $r \leq q$ will be the desired condition.

Suppose that r_i is not a condition in the poset Q as witnessed by δ for any $i \in n$. This means that there are open sets $U_i \in \mathcal{O}_Y$ and points $y_i \in 2^D$ such that $\mu(U_i) > \delta$, $y_i \in V[x_0 \restriction O_q \cup O_i]$, and $y \restriction U_i =_J y_i \restriction U_i$. Now, if $i \neq j$ then the functions $y_i \restriction U_i \cap U_j$ and $y_j \restriction U_i \cap U_j$ are $=_J$-equivalent to $y \restriction U_i \cap U_j$, and by Lemma 7.1.12 they must be $=_J$-equivalent to some function $y_{i,j} : D \cap U_i \cap U_j \to 2$ in the model $V[x_0 \restriction O_q]$. Working in the model $V[x_0 \restriction O_q]$, the choice of the number n shows that the set $U = \bigcup_{i \neq j \in n}(U_i \cap U_j)$ has μ-mass $> 1 - \delta_q$, and $y \restriction U$ is $=_J$-related to some boolean combination of the functions $y_{i,j} : i \neq j \in n$. This, however, contradicts the definition of the number δ_q and concludes the proof of density of the open set $\dot{O}_{gen}$.

$\square$

Claim 7.1.23. *$Q \Vdash \check{y}$ has no $=_J$-equivalent in the model $V[\dot{x}_1]$.*

Proof. Towards a contradiction, suppose that $q \in Q$ is a condition and τ is a P-name such that $q \Vdash \check{y} =_J \tau/\dot{x}_1$. Strengthening the condition q if necessary, I may find an open set $U \in \mathcal{O}_Y$ with $\mu(U) > 1 - \varepsilon/4$ such that $q \Vdash (\tau/\dot{x}_1) \restriction U = \check{y} \restriction U$.
Case 1a. If there are conditions $p \in P$ consistent with $(x_0 \restriction O_q)$ rew p_q and $d \in D \cap U$ such that $p \Vdash_P \tau(d) = 1 - y(d)$, then the condition $r = \langle p \cup p_q, O_q\rangle \leq q$ forces in Q that $(\tau/\dot{x}_1)(\check{d}) \neq \check{y}(\check{d})$. This contradicts the choice of q, τ, and U.
Case 1b. If Case 1a fails, then for every $d \in D \cap U$ there is exactly one value $b \in 2$ such that there is a condition $p \in P$ consistent with $(x_0 \restriction O_q)$ rew p_q such that $p \Vdash_P \tau(\check{d}) = \check{b}$, and this value is equal to $y(d)$. Thus, the function $y \restriction U$ can be reconstructed in the model $V[x_0 \restriction O_q]$, which contradicts the definition of the poset Q.

□

Case 2. If Case 1 fails, consider the collection $\mathfrak{F}$ of sets $O \in \mathcal{O}_X$ such that the point $y \in 2^D$ has $=_J$-equivalent in the model $V[x_0 \restriction O]$.

Claim 7.1.24. *$\mathfrak{F}$ is a filter of nonempty sets in $\mathcal{O}_X$.*

This is proved exactly like Claim 7.1.15. Use a compactness argument to find a point $z \in X$ such that $z \in \bigcap_{O \in \mathfrak{F}} \bar{O}$. Let Q be the poset of all pairs $q = \langle p_q, O_q \rangle$ such that $p_q \in P$, $O_q \in \mathcal{O}_X$, and $z \notin \bar{O}_q$. The ordering is defined by $r \leq q$ if $p_q \subset p_r$, $O_q \subset O_r$ and $(p_r \setminus p_q) \restriction O_q \subset x_0$. Let $\dot{x}_1$ be the Q-name for the union of the first coordinates of the conditions in the generic filter. I will verify the requisite properties of the name $\dot{x}_1$ one by one.

Claim 7.1.25. *$Q \Vdash \dot{x}_1 \in 2^C$ is a P-generic point over the ground model.*

Proof. Let $q \in Q$ be a condition and $B \subset P$ an open dense subset in the ground model. The genericity of the point $x_0 \in 2^C$ implies that there is a finite fraction p of x_0 such that p rew $p_q \in B$. Then, the condition $\langle p \text{ rew } p_q, O_q \rangle$ is stronger than q in the poset Q and forces the point $\dot{x}_1$ to meet the open dense set $B \subset P$ in the condition p rew p_q. □

Claim 7.1.26. *$Q \Vdash \dot{x}_1 =_I \check{x}_0$.*

Proof. First argue that Q forces the union of the second coordinates of the conditions in the generic filter to be equal to $X \setminus \{z\}$. To see this, note that whenever $q \in Q$ is a condition and $O \in \mathcal{O}_X$ is an open set whose closure does not contain the guiding point $z \in X$, then $\langle p_q, O_q \cup O \rangle \leq q$ is again a condition, and use a straightforward density argument. It follows that z is forced to be the only accumulation point of the set $B = \{c \in C : x_0(c) \neq \dot{x}_1(c)\}$: if $\bar{z} \in X$ is another point, then there is a basis set O containing $\bar{z}$ such that $z \notin \bar{O}$, there is a condition $q \in Q$ in the generic filter such that $O \subset O_q$, and then the set $B \cap O$ is a subset of the finite set $\{c \in \text{dom}(p_q) : p_q(c) \neq x_0(c)\}$, therefore finite and has no accumulation points. In particular, $\bar{z}$ cannot be an accumulation point of the set B.

All in all, the set B must be a sequence converging to z and so its closure is nowhere dense in the space X. □

Claim 7.1.27. *$Q \Vdash \check{y}$ has no $=_J$-equivalent in the model $V[\dot{x}_1]$.*

Proof. Towards a contradiction, suppose that $q \in Q$ is a condition and τ is a P-name such that $q \Vdash \check{y} =_J \tau/\dot{x}_1$. Work in the model $V[x_0 \restriction O_q]$. Consider the poset P' of finite partial functions from the set $C \setminus O_q$. Let $\dot{x} \in 2^C$ be the P'-name for the union of $(x_0 \restriction O_q)$ rew p_q and all conditions in the P'-generic filter. Thus, $q \in Q$ forces that the equation $\dot{x} = \dot{x}_1$ defines a point generic for P' over the model $V[x_0 \restriction O_q]$. It will be enough to show that $P' \Vdash \tau/\dot{x}_1 \neq_J \check{y}$.

Observe that for every $m \in \omega$ there is an open set $U_m \in \mathcal{O}_Y$ and $y_m \in 2^D$ in the ground model and a condition $p_m \in P'$ such that $\mu(U_m) > 1 - 2^{-m}$ and $p_m \Vdash_{P'} (\tau/\dot{x}) \restriction U_m =_J \check{y}_m \restriction U_m$. To find U_m, y_m and p_m, just let $x_1 \in 2^C$

be a Q-generic point over the model $V[x_0]$ consistent with the condition q, use the case assumption to find a set $U \in \mathcal{O}_Y$ of mass $> 1 - 2^{-m}$ such that $y \restriction U$ has a $=_J$-equivalent in the ground model, find an open set $U_m \subset U$ in $\mathcal{O}_Y$ of mass $1 - 2^m$ such that $y \restriction U_m = (\tau/x_1) \restriction U_m$ (this is possible by the initial contradictory assumption of this proof), let $y_m \in V$ be point such that such that $y \restriction U_m =_J y_m \restriction U_m$, and let p_m be the finite fraction of x_1 which forces in P' that $y_U \restriction U = (\tau/\dot{x}) \restriction U_m$.

The treatment now breaks into two subcases.

Case 2a. There is $m \in \omega$ as above and a condition $p \in P'$ such that $p \Vdash \neg\tau =_J y_m \restriction U_m$. Then either of the conditions $\langle p_m \text{ rew } p_q, O_q\rangle$ or $\langle p \text{ rew } p_q, O_q\rangle$ below $q \in Q$ forces $\neg\tau =_J \check{y}$ depending on whether $y_m \restriction U_m =_J y$ or not. This is a contradiction to the choice of q and τ.

Case 2b. If Case 2a fails, then for each U as above, then $P' \Vdash (\tau/\dot{x}) \restriction U_m =_J y_m \restriction U_m$. Let $y' \in 2^D$ be any point in the model $V[x_0 \restriction O_q]$ such that for every $m \in \omega$, $y' \restriction U_m \setminus \bigcup_{k \in m} U_k = y_m$. The assumptions immediately imply that $P' \Vdash \tau/\dot{x} =_J \check{y}'$. Therefore, $q \in Q$ forces $\tau/\dot{x}_1 =_J \check{y}'$ and so it must be the case that $y' =_J y$. Since $y' \in V[x_0 \restriction O_q]$, this contradicts the assumption that y has no $=_J$-equivalent in the model $V[x_0 \restriction O_q]$.

□

□

7.2 The measure case

In this section, I will prove a number of results regarding μ-ergodicity for various natural Borel probability measures. It appears that great many of such results depend on the concentration of measure phenomenon explained for example in [21]. I will start with this group of theorems.

Definition 7.2.1. A collection $\{\langle X_n, d_n, \mu_n, \delta_n, \varepsilon_n\rangle : n \in \omega\}$ has *concentration of measure* if

1. X_n is a finite set, d_n is a metric on X_n, μ_n is a probability measure on X_n, $\delta_n, \varepsilon_n > 0$ are real numbers;
2. $\sum_n \varepsilon_n < \infty$ and for every $n \in \omega$, $7\delta_{n+1} < \delta_n$;
3. for every set $A \subset X_n$ of μ-mass at least δ_n, the ε_n-neighborhood of A in the metric d_n has μ-mass at least $1/2$.

As a matter of notation, if $\{\langle X_n, d_n, \mu_n, \delta_n, \varepsilon_n\rangle : n \in \omega\}$ is a collection with concentration of measure, I will write $X = \prod_n X_n$ (so X is a compact space with the product topology), $\mu = \prod_n \mu_n$ (so μ is a Borel probability measure on X), d for the sum of the metrics d_n (so d is an extended value metric), and E for the equivalence relation on X connecting points with finite d-distance (so E is a K_σ equivalence relation on X).

Theorem 7.2.2. *If $\{\langle X_n, d_n, \mu_n, \delta_n, \varepsilon_n\rangle : n \in \omega\}$ is a collection with concentration of measure, then E is F-μ-ergodic for every proper-trim equivalence relation F.*

The main ingredient of the proof is the following purely measure-theoretic consequence of concentration of measure:

Lemma 7.2.3. *For every Borel μ-positive set $B \subset X$ and every $\varepsilon > 0$ there is a μ-positive Borel set $B' \subset B$ such that whenever $C_0, C_1 \subset B'$ are Borel μ-positive sets with $\mu(B' \setminus (C_0 \cup C_1)) = 0$, then there are points $x \in C_0$ and $y \in C_1$ with $d(x, y) < \varepsilon$.*

Proof. Given a number $n \in \omega$, call a set $A \subset X_n$ *connected* if the graph on A relating points of d_n-distance $\leq 2\varepsilon_n$ has a single connectedness component. The following claim is central:

Claim 7.2.4. *Whenever $n \in \omega$ and $A \subset X_n$ is a set of μ_n-mass $> 4\delta_n$ then there is a connected set $A' \subset A$ with $\mu(A') > \mu(A) - \delta_n$.*

Proof. Let H be the graph on the set A connecting points of d_n-distance $< 2\varepsilon_n$. I claim that H has a connected component of μ_n-mass at least δ.

If this failed, it would be possible to divide the set A into two pieces A_0, A_1 respecting the connectedness equivalence such that both have mass at least δ. Then, the ε_n-neighborhoods of these two pieces have μ_n-mass greater than $1/2$ and so they intersect. It follows that there are points $k \in A_0$ and $l \in A_1$ whose d_n-distance is at most $2\varepsilon_n$, contradicting the assumption that the partition of the set A respects the connectedness classes of the graph H.

Now, let $A' \subset A$ be a connected component of μ_n-mass at least δ. I claim that in fact $\mu(A') > \mu(A) - \delta$. To see this, use the concentration of measure assumption again to see that the $2\varepsilon_n$-neighborhood of A' contains all points of X_n with an exception of μ_n-mass δ_n. Also, by the definition of connectedness, A' contains all points of the set A which belong to the $2\varepsilon_n$-neighborhood of A'. this completes the proof. ☐

Let T_{ini} be the tree of all finite sequences t such that for every $n \in \mathrm{dom}(t)$, $t(n) \in X_n$. For a node $t \in T_{\mathrm{ini}}$, write $[t]$ for the set of those infinite branches of T which extend t. For a tree $T \subset T_{\mathrm{ini}}$, its *trunk* is the longest common initial segment of all its infinite branches. Say that a tree $T \subset T_{\mathrm{ini}}$ is *good* if for all nodes $t \in T$ extending the trunk,

- the relative μ-mass of $[T]$ in $[t]$ is $> 4\delta_n$;
- the set $\{u \in X_{|t|} : t^\frown u \in T\} \subset X_{|t|}$ is connected.

I will show that every Borel set $B \subset X$ contains all branches of some good tree, and if T is a good tree with trunk of length m such that $\varepsilon < 2\sum_{m \leq n} \varepsilon_n$, then the set $B' = [T]$ satisfies the conclusion of the proposition. This will complete the proof.

To find the good tree inside the Borel set $B \subset X$, first find a tree $U \subset T_{\mathrm{ini}}$ such that $[U]$ is a μ-positive subset of B. Use the Lebesgue density theorem to find a node $t \in T_{\mathrm{ini}}$ such that $[U]$ has relative mass $> 1/2$ in $[t]$, and $1/2 > 7\delta_{|t|}$. Remove all nodes $u$from U which are either incompatible with t, or such that $t \subseteq u$ and $[U]$ has relative mass $\leq 7\delta_{|u|}$ in $[u]$. In the resulting tree U', whenever $u \in U'$ is a node extending t, the relative mass of $[U']$ in $[u]$ is greater than $6\delta_{|u|}$. By Claim 7.2.4, for each node $u \in U'$ extending t it is possible to remove fewer than μ_n-mass δ_n many immediate successors of u in such a way that the remaining set of immediate successors is connected. The tree U'' obtained in this way is good.

Now suppose that T is a good tree such that writing m for the length of its trunk, $\varepsilon < 2\sum_{m\leq n} \varepsilon_n$. I claim that the set $B' = [T]$ works as required in the theorem.

Claim 7.2.5. *For every number $n \geq m$, whenever $u, v \in T$ are nodes of length n, then there is a sequence $\langle u_i : i \in j\rangle$ such that $u_0 = u, u_{j-1} = v$, and any two successive elements of the sequence have d-distance $< 2\sum_{m\leq l<n} \varepsilon_l$.*

Proof. By induction on n. For $n = m$ the statement is trivial. Suppose it is known for some $n \in \omega$, and u, v are some nodes of T of length $n + 1$. Let $u' = u \restriction n$, $v' = v \restriction n$, and $u'_i : i \in j$ is a walk of nodes of length n obtained from the induction hypothesis. For each $i \in j$ let $a_i = \{z \in X_n : (u'_i)^\frown z \in T\} \subset X_n$. These are sets of μ_n-mass $> \delta_n$, and so by the concentration of measure assumption, for each $i \in j - 1$ the sets a_i and a_{i+1} contain points which are at d_n-distance $< 2\varepsilon_n$. The sets $a_i \subset X_n$ are also connected, and so there is a number $k \in \omega$, successive nonempty intervals $b_i \subset k$ for $i \in j$ exhausting all of k, and a sequence $\langle z_l : l \in k\rangle$ of points in X_n so that

- if $l \in b_i$ then $z_l \in a_i$;
- $(u')^\frown z_0 = u$, $(v')^\frown z_{k-1} = v$;
- successive points on the sequence are at d_n-distance $< 2\varepsilon_n$.

The sequence $\langle u_l : l \in k\rangle$ defined by $u_l = (u'_i)^\frown z_l$ whenever $l \in b_i$ then is as required in the induction step. □

Now, suppose that $C_0, C_1 \subset [T]$ are Borel μ-positive sets such that $\mu([T] \setminus (C_0 \cup C_1)) = 0$. I must produce points $x_0 \in C_0$ and $x_1 \in C_1$ of d-distance $< \varepsilon$. First, use the Lebesgue density theorem to find a number $n \in \omega$ and nodes $u_0, u_1 \in T$ of length n such that C_0 has relative mass $> 2\delta_n$ in $[u_0]$ and C_1 has relative mass $> 2\delta_n$ in $[u_1]$. A finite walk argument from Claim 7.2.5 shows that that such nodes u_0, u_1 can be found within d-distance $< 2\sum_{m\leq l<n} \varepsilon_l$ from each other. Find trees $U_0, U_1 \subset T$ such that the sets $[U_0] \subset C_0$, $[U_1] \subset C_1$, $[U_0]$ have relative μ-mass $> 2\delta_n$ in $[u_0]$ and $[u_1]$ respectively. By induction on $k \geq n$ build nodes $v_0^k \in U_0$ and $v_1^k \in U_1$ so that

- $u_0 = v_0^n$, $u_1 = v_1^n$, v_0^{k+1} is an immediate successor of v_0^k, and v_1^{k+1} is an immediate successor of v_1^k;

- the respective relative mass of $[U_0], [U_1]$ in $[v_0^k], [v_0^k]$ is $> 2\delta_k$;
- $d(v_0^k, v_1^k) < 2\sum_{m\leq l<k} \varepsilon_l$.

Once this is done, then the points $x_0 = \bigcup_k v_0^k \in C_0$ and $x_1 = \bigcup_k v_1^k \in C_1$ are as required by the third item above.

The induction itself again uses the concentration of measure assumptions. Suppose that the nodes $v_0^k \in U_0, v_1^k \in U_1$ have been constructed. The sets $a_0 = \{i \in X_k : U_0$ has relative μ-mass $> 2\delta_{k+1}$ in $[(v_0^k)^\frown i]\} \subset X_k$ and $a_1 = \{i \in X_k : U_1$ has relative μ-mass $> 2\delta_{k+1}$ in $[(v_1^k)^\frown i]\} \subset X_k$ have both μ_k-mass $> \delta_k$ by a Fubini argument. By the concentration of measure assumption, there are points $i_0 \in a_0$ and $i_1 \in a_1$ such that $d_k(i_0, i_1) < 2\varepsilon_k$. The nodes $v_0^{k+1} = (v_0^k)^\frown i_0$ and $v_1^{k+1} = (v_1^k)^\frown i_1$ complete the induction step.

□

Proof of Theorem 7.2.2. Let P be the usual random forcing associated with the Borel probability measure μ, i.e. the poset of Borel μ-positive sets ordered by inclusion. Let $\dot{x}_{gen}$ be the usual P-name for a generic element of the space X. In view of Lemma 7.0.16 it is enough to show that $\dot{x}_{gen}$ is a nontrivial E-trim name. For conditions $p_0, p_1 \in P$ I must produce filters $G_0, G_1 \subset P$ separately generic over V so that $p_0 \in G_0$, $p_1 \in G_1$, $\dot{x}_{gen}/G_0 \; E \; \dot{x}_{gen}/G_1$, and $V[G_0] \cap V[G_1] = V$. To simplify the notation, assume $p_0 = p_1 =$the largest element of P.

Let T_{ini} be the tree of all finite sequences t such that for every $n \in \mathrm{dom}(t)$, $t(n) \in X_n$. For $t, u \in T_{\mathrm{ini}}$ with $|t| \geq |u|$ write t rew u for the sequence obtained from t by rewriting its initial segment of length $|u|$ with u. Let Q be the poset of all pairs $q = \langle B_q, C_q\rangle$ where $B, C \in P$ are μ-positive Borel sets and there are finite sequences $t, u \in T_{\mathrm{ini}}$ such that $d(t, u) < 1$, all elements of B_q contain t as an initial segment, all elements of C_q contain u as an initial segment, and $B = \{x \text{ rew } t : x \in C\}$, or equivalently $C = \{x \text{ rew } u : x \in B\}$. The ordering is coordinatewise inclusion. If $H \subset Q$ is a generic filter, just let $G_0 = \{B_q : q \in H\}$ and $G_1 = \{B_q \oplus s_q : q \in H\}$. I will now verify the requisite properties of this set-up one by one.

Claim 7.2.6. *Q forces both G_0, G_1 to be P-generic filters over the ground model.*

Proof. Below any condition $q \in Q$, B_q can be strengthened arbitrarily within the poset P; transporting the result to the C_q coordinate, one again obtains a condition in Q which is stronger than q. A straightforward density argument then shows that G_0 is forced to be a generic filter over V. The case of G_1 is symmetric. □

Claim 7.2.7. *Q forces $\dot{x}_{gen}/G_0$ is E-related to $\dot{x}_{gen}/G_1$ and not E-related to any ground model point of X.*

Proof. From the definition of Q it is clear that for every $n \in \omega$, the d_n-distance of $\dot{x}_{gen}/G_0(n)$ and $\dot{x}_{gen}/G_1(n))$ is not greater than ε_n. This proves the first sentence. The second sentence follows from the simple fact that E-classes are

μ-null; therefore, no ground model E-class can contain the random generic point $\dot{x}_{gen}/G_0$ by Claim 7.2.6. $\square$

Finally, the most difficult part, which uses the concentration of measure assumptions:

Claim 7.2.8. *$Q \Vdash V[G_0] \cap V[G_1] = V$.*

Proof. Suppose that $q \in Q, q = \langle B, C\rangle$ is a condition and σ, τ are P-names for sets of ordinals such that $q \Vdash \sigma/G_0 = \tau/G_1$. I have to find a condition below B_q in the poset Q which forces σ to belong to the ground model. Find finite sequences $t, u \in T_{\mathrm{ini}}$ witnessing that $q \in Q$, and let $\varepsilon = \frac{1-d(t,u)}{2}$. Find a condition $B' \subset B$ in the poset P which exemplifies Lemma 7.2.3 for ε. I claim that this condition works.

Suppose for contradiction that there is an ordinal α such that B' does not decide the statement $\check{\alpha} \in \sigma$. By the c.c.c. of the measure algebra, this means that there is a partition of B' into Borel μ-positive sets $D_0, D_1 \subset B'$ such that $D_0 \Vdash \check{\alpha} \in \sigma$ and $D_1 \Vdash \check{\alpha} \notin \sigma$. Let $C_0 \subset D_0$ and $C_1 \subset D_1$ be the respective sets of Lebesgue density points. By the choice of the set B'_q, there are points $x_0 \in C_0$ and $x_1 \in C_1$ such that $d(x_0, x_1) < \varepsilon$. Let $t_0 \subset x_0$ and $t_1 \subset x_1$ be finite sequences of the same length say $m > |t|$, such that D_0 has relative density $> 1/2$ in $[t_0]$ and D_1 has relative density $> 1/2$ in $[t_1]$. Let $C' = \{y \in C : y \text{ rew } t_0 \in D_0$ and $y \text{ rew } t_1 \in D_1\}$; this is a Borel subset of C of positive μ-mass. Find a condition $C'' \subset C'$ such that all of its elements start with some fixed sequence $u'' \in 2^m$ and such that it decides the statement $\alpha \in \tau$; for definiteness say that it decides it in the affirmative. Let $B'' = \{x \in D_1 : t_1 \subset x$ and $x \text{ rew } u'' \in C''\}$. The definitions show that the condition $\langle B'', C''\rangle$ is in Q, it is stronger than q, and it forces $\check{\alpha} \in \sigma/G_0 \Delta \tau/G_1$. This is a contradiction. $\square$

$\square$

Corollary 7.2.9. *Let J be the summable ideal on ω: $a \in J$ if $\sum_{n\in a} \frac{1}{n+1} < \infty$. Then $=^2_J$ is F-ν-ergodic for every proper-trim equivalence F where ν is the usual product probability measure on 2^ω.*

Proof. Choose positive real numbers ε_m and δ_m for $m \in \omega$ so that $\sum_m \varepsilon_m < \infty$ and $\delta_m > 16\delta_{m+1}$. Let I_m for $m \in \omega$ be a sequence of consecutive intervals of natural numbers so that for all $m > 0$, and $2\exp(\frac{-\varepsilon^2_{m+1}}{8\sum_{n>\max(I_m)} \frac{1}{n^2}}) < \delta_{m+1}$. This is possible as $\sum_n \frac{1}{n^2} < \infty$. The sequence $\langle X_m, d_m, \mu_m, \varepsilon_m, \delta_m : n \in \omega\rangle$ where $X_m = 2^{I_m}$, μ_m is the usual product measure on X_n, and $d_m(x, y) = \sum\{\frac{1}{n} : n \in I_m$ and $x(n) \neq y(n)\}$ has the concentration of measure by the computation in [21, Theorem 4.3.19]. The corollary immediately follows from Theorem 7.2.2 and the observation that $\nu = \prod_m \mu_m$ under the natural identifiaction of $\prod_m X_m$ and 2^ω.

$\square$

The generic ergodicity results for equivalence relations $=_I$ for a σ-ideal I of compact sets on a compact Polish space X have also ν-ergodic counterparts, where ν is the usual product probability measure on the domain of $=_I$. Compare the following with Theorem 7.1.11. Recall the class $\mathfrak{E}$ of equivalence relations of Definition 7.1.10.

Theorem 7.2.10. *Let X be a zero-dimensional compact Polish space without isolated points. Let I be an analytic σ-ideal of compact sets on X containing all singletons. Then $=_I$ is ν-generically ergodic for every equivalence relation $F \in \mathfrak{E}$.*

The proof uses repeatedly the following variation of Lemma 7.1.12.

Lemma 7.2.11. *Let E be an analytic equivalence relation on a Polish space X. Let $z_0, z_1 \in 2^\omega$ be mutually random-generic filters over V. If $x_0 \in V[z_0]$ and $x_1 \in V[z_1]$ are E-related points in the space X, then they are E-related to some point in the ground model.*

Proof. The difficulty compared to Lemma 7.1.12 is that the points $z_0, z_1 \in 2^\omega$ are not product-generic. I have to resort to a Fubini-style argument. First, I will specify the version of random forcing I will use. Let μ be the usual product Borel probability measure on 2^ω, and ν its Fubini product with itself, a Borel probability measure on $2^\omega \times 2^\omega$. Let I be the σ-ideal of μ-null sets on 2^ω, and let J be the σ-ideal on ν-null sets on $2^\omega \times 2^\omega$. Thus, the poset P_J of all Borel J-positive subsets of $2^\omega \times 2^\omega$ adds a pair $\langle \dot z_0, \dot z_1 \rangle$ of mutually random generic points in 2^ω.

Suppose that some Borel set $B \in P_J$ forces $\dot x_0 \in V[\dot z_0]$, $\dot x_1 \in V[\dot z_1]$ are E-related points. By the usual Borel reading of names for c.c.c. posets, thinning out the condition B if necessary, I may assume that there are Borel functions $f_0 : 2^\omega \to X$ and $f_1 : 2^\omega \to X$ such that $B \Vdash \dot x_0 = \dot f_0(\dot z_0)$ and $\dot x_1 = \dot f_1(\dot z_1)$. The analytic set $C \subset B$ of all pairs $\langle y_0, y_1 \rangle \in B$ such that $f_0(y_0) \mathrel{E} f_1(y_1)$ is J-positive since B forces the generic pair to belong into C. Removing a ν-null set if necessary I may assume that the set C is Borel and all its vertical sections are either empty or non-μ-null. Use the Fubini theorem to find a point $y \in 2^\omega$ such that the horizontal section C^y has positive μ-mass. The definition of the set C shows that $f_1(y) \mathrel{E} f_0(x)$ for every $x \in C^y$. Let $D = \{\langle x, z \rangle \in C : x \in C^y\}$; use the Fubini theorem again to argue that the Borel set $D \subset C$ is ν-positive. Finally, $D \Vdash \dot f_0(\dot z_0) \mathrel{E} f_1(y)$ and so $D \Vdash \dot x_0$ has an E-equivalent in the ground model. $\square$

Proof of Theorem 7.2.10. Let $C \subset X$ be a countable dense set so that $2^C = \mathrm{dom}(=_I)$. Let ν be the usual product measure on the space 2^C. Let P be the partial ordering of Borel subset of 2^C of positive ν-mass. In some forcing extension, I will find points $x_n \in 2^C$ for $n \in \omega$ so that

- each x_n is P-generic over the ground model;
- every condition in P coded in the ground model contains a point x_n for some $n \in \omega$;

- the points x_n are pairwise $=_I$-related;
- if $F \in \mathfrak{E}$ is an equivalence relation in the ground model and an F-class has representatives in each model $V[x_n]$ for $n \in \omega$, then the class has a representative in the ground model.

This immediately implies the theorem. If $F \in \mathfrak{E}$ is an equivalence relation on some Polish space Y and $h : 2^C \to Y$ is a Borel homomorphism, then the F-equivalence class of $h(x_n)$ is represented in every model $V[x_n]$. By the last item above, it has a representative $y \in Y$ in the ground model. The preimage $h^{-1}[y]_F \subset 2^C$ must then be a set of full ν-mass as otherwise there would be a number $n \in \omega$ with $x_n \notin h^{-1}[y]_F$ by the second item, contradicting the choice of y.

Let $x_0 \in 2^C$ be a P-generic point over V; work in the model $V[x_0]$. I will describe a poset Q adding a point $x_1 \in 2^C$ which is also P-generic over V, $=_I$-related to x_0, and such that for every equivalence relation $F \in \mathfrak{E}$ on some Polish space Y, every point $y \in Y$ in the model $V[x_0]$ which has no F-equivalent in V, there is a condition $q \in Q$ which forces that $V[\dot{x}_1]$ contains no F-equivalent of y. Then, force with a finite support product of countably many copies of the poset Q over the model $V[x_0]$. Denoting these copies with Q_n for $n > 0$ and their respective generic points with $x_n \in 2^C$, elementary density arguments will show that the points $\{x_n : n \in \omega\}$ have the required properties.

Let Q be the partial order of certain tuples $q = \langle t_q, O_q, p_q, D_q\rangle$ such that $t_q : C \cap O_q \to 2$ is a finite partial function, O_q is a clopen set of X, and $D_q \subset X$ is a finite set disjoint from O_q. The nature of p_q will be specified in a moment. For each such triple, I will need a good amount of notation. Let $X_q = 2^{C\cap(O_q\cup D_q)}$, let ν_q be the usual product Borel probability measure on X_q, let P_q be the poset of Borel ν_q-positive subsets of X_q coded in the ground model, ordered by inclusion, and let $x_q = (x_0 \restriction C \cap (O_q \cup D_q))$ rew $t_q \in X_q$. The point x_q, as a finite alteration of $x_0 \restriction C \cap (O_q \cup D_q)$ is P_q-generic over the ground model. Let $X^q = 2^{C\setminus(O_q\cup D_q)}$, let ν^q be the usual Borel probability measure on X^q, let P^q be the poset of Borel ν^q-positive subsets of X^q coded in $V[x_q]$, ordered by inclusion, and let $x^q = x_0 \restriction C \setminus (O_q \cup D_q) \in X^q$. Thus, x^q is P^q-generic point over the model $V[x_q]$. The final requirement on the condition q is that $p_q \in P^q$. For each $q \in Q$, write $[q] = \{x_q\} \times p_q \subset 2^C$. The ordering on the poset Q is defined by $r \leq q$ if $t_q \subset t_r$, $O_q \subset O_r$, $D_q \subset D_r$, $t_r \restriction O_q = t_q$, and $[r] \subset [q]$.

I will start the analysis of the poset Q with a technical claim. Call a condition $q \in Q$ *regular* if x_0 rew $t_q \in [q]$.

Claim 7.2.12. *The set of regular conditions is dense in Q. Whenever $q \in Q$ is a regular condition and $D \subset X$ is a finite set disjoint from O_q, there is a condition $r \leq q$ with $D \subset D_r$.*

Proof. For the first sentence, let $q \in Q$ be an arbitrary condition; I must produce a regular condition $r \leq q$. The closure of the condition p_q under finite changes is a Borel subset of X^q of full ν^q-mass, coded in the model $V[x_q]$ and therefore contains x^q. It follows that there is a finite partial function $t_r : C \to 2$ such that

$t_r \restriction O_q \cup D_q = t_q$ and x_0 rew $t_r \in [q]$. Let $O_r \subset X$ be any clopen set containing O_q as as well as the domain of t_r as a subset, disjoint from the set D_q. The condition $r \leq q$ will be of the form $r = \langle t_r, O_r, p_r, D_q \rangle$ for suitable $p_r \in P_r$. Just let $p_r = \{x \in X_r : (x_0 \restriction (O_r \cup D_q))$ rew $t_r \cup x \in p_q\}$. The set $p_r \subset X_r$ is Borel, in the model $V[x_0 \restriction O_r]$. Also, it contains the sequence $x_0 \restriction C \setminus O_r \cup D_q$, which is P_r-generic over the model $V[x_0 \restriction O_r]$. It follows that $\nu_r(p_r) > 0$. The condition $r = \langle t_r, D_r, p_r, D_q \rangle$ works as required.

For the second sentence, let $q \in Q$ be a regular condition and let $D \subset X$ be a finite set disjoint from O_q. To produce the required condition $r \leq q$ with $D \subset D_r$, let $t_r = t_q, O_r = O_q$, $D_r = D_q \cup D$, and $p_r = \{x \in X^r : x_r^\frown x \in p_q\}$. I claim that $r = \langle t_r, O_r, p_r, D_r \rangle \leq q$ is a condition in the poset Q. The only nontrivial point is to check $p_r \in P^r$. Now, $p_r \subset X^r$ is a Borel set coded in $V[x_r]$. To verify that $\nu^r(p_r) > 0$, just observe that the regularity of the condition q implies that the point $x_0 \restriction (C \setminus (O_r \cup D_r))$ belongs to p_r. This point is P^r-generic over the model $V[x_r]$, and therefore $\nu^r(p_r) > 0$ follows. □

Let $\dot{x}_1$ by the Q-name for the element of 2^C obtained as the unique element of the intersection of all sets $[q] \subset 2^C$ for q in the generic filter.

Claim 7.2.13. *$Q \Vdash \dot{x}_1$ is P-generic over the ground model. For every condition $q \in Q$, $q \Vdash \dot{x}_1 \restriction C \setminus (O_q \cup D_q) \in X_q$ is P^q-generic over the model $V[x_q]$.*

Proof. For the first sentence, suppose $B \subset 2^C$ is a Borel set of full ν-mass coded in the ground model and $q \in Q$ is a condition; I will produce a condition $r \leq q$ such that $[r] \subset B$. To this end, use the Fubini theorem to show that the Borel set $B_q = \{x \in X_q : \{x' \in X^q : x \cup x' \in B\}$ has full ν^q-mass$\}$ has full ν_q-mass. As B_q coded in the ground model, $x_q \in B_q$. Let $B^q = \{x \in X^q : x_q^\frown x \in B\}$, let $r = \langle t_q, O_q, p_q \cap B^q, D_q \rangle$ and observe that $r \leq q$ and $[r] \subset B$ as required.

The second sentence is proved in the same way. □

Claim 7.2.14. $Q \Vdash \check{x}_0 =_I \dot{x}_1$.

Proof. Let $\dot{O}_{gen}$ be the Q-name for the union of the second coordinates of conditions in the generic filter. I will prove that $Q \Vdash X \setminus \dot{O}_{gen} \in I$. This immediately implies the claim.

Since I is an analytic σ-ideal of compact sets, it is in fact G_δ in the hyperspace $K(X)$ by [14, Theorem 33.3]. Thus, $I = \bigcap_n U_n$ for some open sets $U_n \subset K(X)$. The sets U_n may be selected downwards closed, and as is the case for every downwards closed open subset of $K(X)$, there are collections $\mathcal{O}_n$ of open subsets of X such that $K \in U_n$ if and only if there is $O \in \mathcal{O}_n$ such that $K \subset O$.

Now, let $q \in Q$ be a regular condition and $n \in \omega$. It will be enough to find a condition $r \leq q$ and an open set $O \in \mathcal{O}_n$ such that $O_r \cup O = X$. Such a condition r forces $X \setminus \dot{O}_{gen} \subset O$, and a straighforward density argument leads to the conclusion that $X \setminus \dot{O}_{gen} \in \bigcap_n U_n = I$.

To find the condition r and the open set $O \in \mathcal{O}_n$, just observe that the finite set $D_q \subset X$ is in the ideal I by the assumptions, and so there is $O \in \mathcal{O}_n$ such that $D_q \subset O$. Since the points in the set D_q are not isolated, a compactness argument yields a clopen set O' such that $O' \cup O = X$ and $D_q \cap O' = 0$. Set $t_r = t_q$, $D_r = D_q$, $O_r = O' \cup O_q$, $x_r = x_0 \restriction (O_r \cup D_r)$ rew t_q, and $p_r = \{x \in X^r : x_r^\frown x \in p_q\}$. The condition $r = \langle t_r, O_r, p_r, D_r\rangle \leq q$ together with the set O works. □

Let Y be a Polish space and F and equivalence relation on Y, both in the ground model. Let $y \in Y$ be a point in $V[x_0]$ which has no F-equivalent in the ground model. Let $\mathfrak{F}_{y,F}$ be the collection of those clopen subsets $O \subset X$ such that y has an F-equivalent in the model $V[x_0 \restriction O]$.

Claim 7.2.15. *$\mathfrak{F}_{y,F}$ is a filter of nonempty clopen subsets of the space X.*

Proof. The same as Claim 7.1.15 with Lemma 7.2.11 replacing Lemma 7.1.12. The main point is the standard observation in the ground model that if $C_0, C_1 \subset C$ are any sets then $P \Vdash \dot{x}_{gen} \restriction C_0 \setminus D$ and $\dot{x}_{gen} \restriction C_1 \setminus D$ are mutually random points over the model $V[\dot{x}_{gen} \restriction D]$, where $D = C_0 \cap C_1$ and $\dot{x}_{gen} \in 2^C$ is the usual P-name for the generic point. □

A compactness argument shows that the filter $\mathfrak{F}_{y,F}$ has a nonempty intersection.

Claim 7.2.16. *Suppose that $F \in \mathfrak{E}$. Any condition $q \in Q$ with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$ forces y to have no F-equivalent in the model $V[\dot{x}_1]$.*

Proof. I will show that the class of all equivalence relations in the ground model satisfying the statement of the claim is closed under the generating operations of the class $\mathfrak{E}$. The following observations and notation will be useful in all cases. Let $q \in Q$ be a condition. In the model $V[x_q]$, the poset P^q adds an element $\dot{x}^q_{gen} \in X^q$. The point $x_q \cup \dot{x}^q_{gen} \in 2^C$ is P-generic over V, and so whenever τ is a P-name in V, it makes sense to write τ^q for the P^q-name $\tau/(x_q \cup \dot{x}^q_{gen})$. Now, in the model $V[x_0]$, the condition $q \in Q$ forces $\dot{x}_1 \restriction C \setminus (O_q \cup D_q)$ to be P^q-generic over $V[x_q]$ by Claim 7.2.13, meeting the condition p_q. It follows that if ϕ is an analytic formula with parameters in $V[x_q]$ and τ is a P-name in V for an element of a Polish space and $p_q \Vdash_{P^q} \phi(\tau^q)$, then $q \Vdash_Q \phi(\tau/\dot{x}_1)$.

Case 1. Regarding the Friedman–Stanley jump, assume that F is an analytic equivalence relation on a Polish space Y for which the statement has been verified. Consider the equivalence relation F^+ on the space Y^ω. Let $y \in Y^\omega$ be a point in $V[x_0]$ which has no F^+-equivalent in the ground model. Let q be a regular condition such that $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, and let τ be a ground model P-name for an element of the space Y^ω. I will produce a condition $r \leq q$ and a number $i \in \omega$ such that either $r \Vdash (\tau/\dot{x}_1)(i) \notin [\mathrm{rng}(y)]_F$ or $r \Vdash \check{y}(i) \notin [\mathrm{rng}(\tau/\dot{x}_1)]_F$. This will complete the proof.

Case 1a. There is a number $i \in \omega$ such that $y(i)$ has no F-equivalent in the model $V[x_q]$. Consider the set $\mathfrak{F}_{y(i),F}$. Just as in Claim 7.1.15, the collection $\{O \setminus O_q : O \in \mathfrak{F}_{y(i),F}\}$ consists of nonempty clopen sets and has the finite intersection property. A compactness argument then yields a point $z \in X$ in its intersection. Use Claim 7.2.12 to find a condition $r \leq q$ with $z \in D_r$. By the

assumption on the equivalence relation F, $r \Vdash y(i)$ has no F-equivalent in the model $V[x_1]$, in particular no F-equivalent in the set $\mathrm{rng}(\tau/\dot{x}_1)$.
Case 1b. If Case 1a fails, work in the model $V[x_q]$. There are three subcases:
Case 1ba. There is a condition $p \leq p_q$ in the poset P^q forcing τ^q to have an F^+-equivalent in the model $V[x_0 \restriction O_q]$. Strengthening the condition p if necessary, I may identify this equivalent $y' \in Y^\omega$. Since $y'\ F^+\ y$ fails, $[\mathrm{rng}(y)]_E \neq [\mathrm{rng}(y')]_E$ and so there must be $i \in \omega$ such that either $y(i) \notin [\mathrm{rng}(y')]_E$ or $y'(i) \notin [\mathrm{rng}(y)]_E$. The condition $r = \langle t_q, O_q, p, D_q\rangle \leq q$ and the number $i \in \omega$ clearly work as desired.
Case 1bb. There is a condition $p \leq p_q$ in P^q forcing that there is some $i \in \omega$ such that $\tau^q(i)$ has no F-equivalent in the model $V[x_q]$. Strengthening p if necessary, I may find a specific $i \in \omega$ satisfying this. In view of Case 1b assumption, the condition $r = \langle t_q, O_q, p, D_q\rangle \leq q$ and the number $i \in \omega$ work as required.
Case 1bc. If both Cases 1ba and 1bb fail, then there must be a point $y' \in Y$ in the model $V[x_q]$ and conditions $p', p'' \leq p_q$ in P^q such that $p' \Vdash \check{y}' \in [\tau^q]_F$ and $p'' \Vdash \check{y}' \notin [\tau^q]_F$. (Otherwise, the set $\{y' \in Y : p_q \Vdash_{P^q} \tau^q\ F\ y'\}$ contains only countably many F-equivalence classes by the c.c.c. of P^q, and if $y'' \in Y^\omega$ visits exactly these F-classes then $p_q \Vdash_{P^q} \check{y}''\ F^+\ \tau^q$ by the failure of Case 1bb. This directs us to Case 1ba.) The treatment now divides into two further subcases depending on whether $y' \in [\mathrm{rng}(y)]_F$ or not. Assume for definiteness that the latter is the case. Strengthen the condition p if necessary to find a specific number i such that $p \Vdash_{P^q} (\tau^q)(i)\ F\ y'$. The condition $r = \langle t_q, O_q, p, D_q\rangle \leq q$ with the number $i \in \omega$ work as required.
Case 2. Now move to the case in which the equivalence relation F is obtained by a product modulo an F_σ-ideal of equivalence relations on which the claim has been already verified. Suppose that J is an F_σ-ideal and use a theorem of Mazur [20] to find a lower semicontinuous submeasure μ on ω such that $J = \{a \subset \omega : \mu(a) < \infty\}$. Let $\{F_i : i \in \omega\}$ be a collection of equivalence relations on the respective Polish spaces Y_i for $i \in \omega$ in the ground model such that the statement of the claim holds for each of them. Let $Y = \prod_i Y_i$, let $F = \prod_i F_i$ modulo J, let $y \in Y$ be a point in $V[x_0]$ which has no F-equivalent in the ground model. Let $q \in Q$ be a regular condition with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, let τ be a ground model P-name for an element of the space Y, and let $n \in \omega$. I must produce a condition $r \leq q$ and a finite set $b \subset \omega$ such that $\mu(b) > n$ and $r \Vdash \forall i \in \check{b}\ \neg(\tau/\dot{x}_1)(i)\ F_i\ y(i)$. The treatment divides into two cases.
Case 2a. The set $a = \{i \in \omega : y(i)$ has no F_i-equivalent in the model $V[x_q]\}$ is J-positive. Find a subset b of it of μ-mass $> n$. For every $i \in b$ consider the set $\mathfrak{F}_{y(i),F_i}$. Just as in Claim 7.1.15, the collection $\{O \setminus O_q : O \in \mathfrak{F}_{y(i),F_i}\}$ consists of nonempty clopen sets and has the finite intersection property. A compactness argument then yields a point $z_i \in X$ in its intersection. Use Claim 7.2.12 to find a condition $r \leq q$ with $\{z_i : i \in b\} \subset D_r$; this condition works as required. For every $i \in b$, $D_r \cap \bigcap \mathfrak{F}_{y(i),F_i}$ is nonempty, containing at least the point z_i. By the assumption on the equivalence relation F_i, $r \Vdash y_i$ has no F_i-equivalent in the model $V[x_1]$, in particular $(\tau/\dot{x}_1)(i)$ is not equivalent to $y(i)$.
Case 2b. If Case 2a fails, then $a \in J$ and so $\mu(a) < m$ for some number $m \in \omega$.

Work in the model $V[x_q]$. There are three subcases:

Case 2ba. There is a condition $p \leq p_q$ in P^q forcing τ^q to have an F-equivalent in the model $V[x_q]$. Strengthening the condition p if necessary, I may identify this equivalent y' as well as the number $k \in \omega$ such that $p \Vdash \mu(\{i \in \omega : \neg\tau^q(i)\ F_i\ y'(i)\}) < k$. Since $y'\ F\ y$ fails, there is a finite set $b' \subset \omega$ of such that $\mu(b') > n + k$ and $\forall i \in a\ \neg y(i)\ F_i\ y'(i)$. Strengthening the condition p further I can identify the set $b = \{i \in b' : \tau^q(i)\ F_i\ y'(i)\}$; by the subadditivity of the submeasure μ, it has to be the case that $\mu(b) > n$. It is not difficult to see that the condition $r = \langle t_q, O_q, p, D_q\rangle$ and the set b have the required properties.

Case 2bb. There is a condition $p \leq p_q$ in P^q forcing the set $\{i \in \omega : \tau^q(i)$ has no F_i-equivalent in the model $V[x_q]\}$ to be in the ideal J. Strengthening the condition p if necessary, I can identify a finite subset b' of this set such that $\mu(b') > n + m$. The set $b = b' \setminus a$ must have $\mu(b) > n$. It is not difficult to see that the condition $r = \langle t_q, O_q, p, D_q\rangle \leq q$ and the set b have the required properties.

Case 2bc. If both Cases 2ba and 2bb fail, then there must be conditions $p', p'' \in P^q$, a finite set $b' \subset \omega$ with $\mu(b') > 2n$, and functions y', y'' with domain b' such that for every $i \in b'$, $p' \Vdash \tau^q(i)\ F_i\ y'(i)$, $p'' \Vdash \tau^q(i)\ F_i\ y''(i)$, and $\neg y'(i)\ F_i\ y''(i)$. One of the sets $\{i \in b' : \neg y(i)\ F_i\ y'(i)\}$, $\{i \in b' : \neg y(i)\ F_i\ y''(i)\}$ must have μ-mass greater than n as they together cover the set b'. Suppose for definiteness it is the former, and call it b. It is not difficult to see that the condition $r = \langle t_q, O_q, p, D_q\rangle \leq q$ and the set b have the required properties.

Case 3. Now, move to the case where F is obtained as a countable union $F = \bigcup_n F_n$ of analytic equivalence relations on some Polish space Y for which the statement of the claim has already been verified. Here, observe that $\mathfrak{F}_{y,F_n} \subset \mathfrak{F}_{y,F}$: if $O \subset X$ is a clopen set and $y \in Y$ has an F_n-equivalent in the model $V[x_q]$, then this same point is in fact an F-equivalent of y as $F_n \subset F$. Thus, if $q \in Q$ is any condition with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, then for every $n \in \omega$, $D_q \cap \bigcap \mathfrak{F}_{y,F_n} \neq 0$. By the assumption on the equivalence relations F_n, the condition $q \in Q$ forces that y has no F_n-equivalent in the model $V[\dot{x}_1]$, and since $F = \bigcup_n F_n$, it also cannot have an F-equivalent there.

Case 4. Finally, move to the case where the equivalence relation F is Borel reducible to an equivalence relation F' on a Polish space Y' via some Borel reduction $h : Y \to Y'$, and the statement of the claim holds for F'. Observe that $\mathfrak{F}_{y,F} = \mathfrak{F}_{h(y),F'}$: if $O \subset X$ is a clopen set and $y \in Y$ has an F-equivalent $y_0 \in V[x_0 \restriction O]$ then $h(y_0)$ is an F'-equivalent of $h(y)$ in the same model. On the other hand, if $h(y)$ has an F'-equivalent $y_0' \in Y'$ in the model $V[x_0 \restriction O]$, then by the Mostowski absoluteness between the models $V[x_0 \restriction O]$ and $V[x_0]$, the model $V[x_0 \restriction O]$ must contain a point $y_0 \in Y$ such that $h(y_0)\ F'\ y_0'$, and then y_0 is an F-equivalent of y in the model $V[x_0 \restriction O]$ as h is a reduction. Thus, if $q \in Q$ is any condition with $D_q \cap \bigcap \mathfrak{F}_{y,F} \neq 0$, then $D_q \cap \bigcap \mathfrak{F}_{y,F'} \neq 0$. By the assumption on the equivalence relation F', the condition $q \in Q$ then forces the model $V[\dot{x}_1]$ to contain no F'-equivalent of $h(y)$. Thus, the model cannot contain any F-equivalent of y, since its h-image would be an F'-equivalent of $h(y)$. □

This completes the proof of the required properties of the poset Q and the proof of the theorem. □

Corollary 7.2.17. *Let J be the ideal of sets of asymptotic density zero on ω. The equivalence relation $=_J$ is F-ν-ergodic for every equivalence relation $F \in \mathfrak{E}$, where ν is the usual product Borel probability measure on 2^ω.*

Proof. The argument follows word by word the proof of Corollary 7.1.17, with a reference to Theorem 7.1.11 replaced by Theorem 7.2.10. □

Now it is time to produce some negative results. I will show how the failure of concentration of measure can lead to the existence of nonstabilizing homomorphisms. The central tool is the following easy computation:

Lemma 7.2.18. *For every $i \in \omega$ and every $\varepsilon > 0$ there is a number $n \in \omega$ and sets $a, b \subset 2^n$ of the same relative size $> \frac{1-\varepsilon}{2}$ each, such that for every $x \in a$ and $y \in b$ the set $\{m \in n : x(m) \neq y(m)\}$ contains at least i many elements.*

Proof. Fix i and ε. Elementary computation shows that there is $n \in \omega$ such that the size of the set $\{a \subset n : ||a| - \frac{n}{2}| < i + 1\}$ is less than $\varepsilon 2^n$. Let $a = \{x \in 2^n$: the set $\{m \in n : x(m) = 1\}$ contains at most $\frac{n}{2} - i$ many elements$\}$ and $b = \{x \in 2^n$: the set $\{m \in n : x(m) = 1\}$ contains at least $\frac{n}{2} + 1$ many elements$\}$. This works. □

As the first example of the failure of ergodicity in measure, consider the case of general summable ideals. Let $w : \omega \to \mathbb{R}^+$ be a function, and write J_w for the ideal $\{a \subset \omega : \sum_{n \in a} w(a) < \infty\}$ and $=_w$ for the equivalence $=^2_{J_w}$.

Example 7.2.19. There is a function $w : \omega \to \mathbb{R}^+$ tending to 0 and a Borel homomorphism $f : 2^\omega \to 2^\omega$ of $=_w$ to E_0 such that preimages of E_0-classes have zero ν-mass.

Proof. Choose consecutive intervals $\{I_i : i \in \omega\}$ such that $n_i = |I_i|$ works for i and $\varepsilon = 2^{-i}$ as in Lemma 7.2.18. Let w be the function defined by $w(j) = \frac{1}{i+1}$ if $j \in I_i$. I claim that the function w works as desired.

To construct the homomorphism, for every $i \in \omega$ let $a_i, b_i \subset 2^{I_i}$ be the sets witnessing the validity of the claim. Let $B = \{x \in 2^\omega : \forall^\infty i \in \omega\ x \restriction I_i \in a_i \cup b_i\}$; this is an F_σ-set of full ν-mass. Let $h : B \to 2^\omega$ be the continuous function defined by $h(x)(i) = 0$ if $x \restriction I_i \in a_i$. The choice of the sets a_i, b_i implies that h is a homomorphism of $=_w \restriction B$ to E_0 such that preimages of singletons (and therefore E_0-classes) are ν-null. Now just use Lemma 2.1.5 to extend h to a total Borel homomorphism of $=_w$ to E_0. □

Example 7.2.20. There is a Tsirelson ideal J on ω such that there is a Borel homomorphism from $=_J$ to E_0 such that preimages of E_0-classes are ν-null.

Here, the *Tsirelson ideals* are certain F_σ-ideals associated with *Tsirelson submeasures* on ω. I will define a certain special subclass of them. Let $\alpha > 0$ be a real number and $f : \omega \to \mathbb{R}^+$ be a function. In a typical case, the function f will converge to 0 and never increase. By induction on $n \in \omega$ define submeasures μ_n on ω by setting $\mu_0(a) = \sup_{i \in a} f(i)$, and $\mu_{n+1}(a) = \sup\{\mu_n(a), \alpha \sum_{b \in \vec{b}} \mu_n(b)\}$ where the variable $\vec{b}$ ranges over all sequences $\langle b_0, b_1, \dots b_j \rangle$ of finite subsets of a such that $j < \min(b_0) \leq \max(b_0) < \min(b_1) \leq \max(b_1) < \dots$. In the end, let the submeasure μ be the supremum of μ_n for $n \in \omega$. Some computations are necessary to verify that μ_n is really a lower semicontinuous submeasure on ω. The ideal $J = \{a \subset \omega : \lim_m \mu(a \setminus m) = 0\}$ turns out to be an F_σ P-ideal [4]. I will show that for positive real number $\alpha > 0$ there is a function f converging to 0 such that the derived ideal J is has the nonstabilizing homomorphism as in Example 7.2.20.

Proof. By induction on $i \in \omega$ choose intervals $I_i \subset \omega$ such that $\max(I_i) > \min(I_{i+1})$ and such that $\min(I_i) > i/\alpha$ there are sets $a_i, b_i \subset 2^{I_i}$ of the same relative size $\geq \frac{1-2^{-i}}{2}$ such that for any elements $x \in a_i, y \in b_i$ the set $\{m \in I_i : x(m) \neq y(m)\}$ has size at least i/α. This is easily possible by Lemma 7.2.18. Now, consider the function f defined by $f(m) = 1/i$ for $m \in (\max(I_{i-1}), \max(I_i)]$ and let μ be the derived submeasure and J the derived Tsirelson ideal. Observe that with this choice of the function f, for any $i \in \omega$ and elements $x \in a_i, y \in b_i$ the set $\{m \in I_i : x(m) \neq y(m)\}$ has μ-mass at least 1, since it has μ_1-mass at least 1.

The remainder of the proof is the same as in the previous example. Let $B = \{x \in 2^\omega : \forall^\infty i \in \omega\ x \restriction I_i \in a_i \cup b_i\}$; this is an F_σ-set of full ν-mass. Let $h : B \to 2^\omega$ be the continuous function defined by $h(x)(i) = 0$ if $x \restriction I_i \in a_i$. The choice of the sets a_i, b_i implies that h is a homomorphism of $=_J \restriction B$ to E_0 such that preimages of singletons (and therefore E_0-classes) are ν-null. Now just use Lemma 2.1.5 to extend h to a total Borel homomorphism of $=_J$ to E_0. □

Question 7.2.21. Is there a Tsirelson ideal J such that the equivalence relation $=_J$ is ν-F-generically ergodic for every proper-trim equivalence relation F?

As another example, recall the eventually different ideal. Let $C = \{\langle m, n \rangle \in \omega^2 : n \in m\}$ and let J be the ideal on C generated by those subsets of C whose vertical sections are bounded in size.

Example 7.2.22. Let J be the eventually different ideal. There is a Borel homomorphism from $=_J$ to E_0 such that preimages of E_0-classes are ν-null.

Proof. Pick real numbers $\varepsilon_i > 0$ for $i \in \omega$ such that $\prod_i (1 - \varepsilon_i) > 0$. Find pairwise distinct numbers $n_i \in \omega$ which exemplify Lemma 7.2.18 for ε_i and find sets $a_i, b_i \in 2^{n_i}$ as in that lemma. Choose an arbitrary bijection $\pi_i : C_{n_i} \to n_i$, extend it naturally to a bijection $\pi_i : 2^{C_{n_i}} \to 2^{n_i}$, and write $\bar{a}_i = \pi_i^{-1} a_i$ and $\bar{b}_i = \pi_i^{-1} b_i$ Let $B = \{x \in 2^C : \forall^\infty i \in \omega\ x \restriction C_{n_i} \in \bar{a}_i \cup \bar{b}_i\}$. This is an F_σ-set of full ν-mass. Let $h : B \to 2^\omega$ be the continuous function defined by $h(x)(i) = 0$ if $x \restriction C_{n_i} \in \bar{a}_i$. This is a Borel homomorphism of $=_J \restriction B$ to E_0 such that f-preimages of singletons and E_0-classes are ν-null. Now just use Lemma 2.1.5 to extend h to a total Borel homomorphism of $=_w$ to E_0. □

Chapter 8

Other reducibility invariants

8.1 Connections

Let E be an equivalence relation on a Polish space X, and let $V[G_0]$, $V[G_1]$ be generic extensions containing respective E-related points $x_0, x_1 \in X$. How far from the models $V[G_0], V[G_1]$ does one have to go to find a common representative of the class $[x_0]_E$? The answer to this question yields a good number of nonreducibility arguments. Their common feature is that they do not translate into ergodicity proofs. Therefore, they are useful for proving nonreducibility where many nontrivial homomorphisms are present. As one application, I provide an (modulo the forcing method) exceptionally simple and conceptual proof that E_1 is not Borel reducible to any orbit equivalence relation. As another application, I provide nonreducibility results complementary to the theorems of Chapter 7.

Definition 8.1.1. Let E be an analytic equivalence relation on a Polish space X. Let P be a Suslin forcing. Say that E has *P-σ-connections* if in every forcing extension, if $\{V[H_n] : n \in \omega\}$ are extensions of V containing respective points $x_n \in X$ which are pairwise E-related, and $V[H_0]$ is a P-extension of V, then in some further forcing extension there is a point $x \in X$ such that for every $n \in \omega$, $x \mathrel{E} x_n$ and x belongs to some P-extension of $V[H_n]$. Say that E has *P-connections* if the same conclusion holds in the case of merely two extensions $V[H_0]$ and $V[H_1]$

I will first verify that the notions defined above are really invariant under Borel reducibility.

Theorem 8.1.2. *If E, F are analytic equivalence relations on respective Polish spaces X, Y, $E \leq_{\mathrm{wB}} F$, and F has P-σ-connections, then E also has P-σ-connections. Similarly for P-connections.*

Proof. Suppose that $a \subset X$ is a countable set and $h : X \to Y$ is a Borel function which is a reduction of E to X on the set $X \setminus [a]_E$. Let $V[H_n]$ for $n \in \omega$ be models containing respective points $x_n \in X$ which are pairwise E-connected. If they are E-connected to some point $x \in a$, then the definition of P-σ-connections is automatically satisfied with that point x. If they are not E-connected to any point in a, then by the Shoenfield absoluteness their values $h(x_n) \in Y$ are F-connected. Since F satisfies P-σ-connections, there is a point $y \in Y$ which is F-connected to all of them and such that it is in some P-extension of each model $V[H_n]$.

By the Shoenfield absoluteness between $V[y]$ and $V[H_0][y]$, there must be a point $x \in V[y] \cap X$ such that x is not E-related to any point in a and $h(x)\ F\ y$– such a point, namely x_0, exists in the model $V[H_0][y]$. It is now obvious that the point x exemplifies the definition of P-σ-connections for E. $\square$

The connection invariant is mildly interesting already in the case of the trivial forcing P.

Theorem 8.1.3. *If P is the trivial forcing and E is an analytic equivalence relation with countable classes, then E has P-σ-connections. E_1 has P-connections.*

Proof. Suppose first that E has countable classes. If M is any transitive model of ZFC containing the code for E, it also satisfies that all classes of E are countable by the Mostowski absoluteness and the fact that the ideal of countable sets is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Thus, if $x \in M$ is a point in $\mathrm{dom}(E)$, then M contains a point $y \in \mathrm{dom}(E)^\omega$ which enumerates the equivalence class $[x]_E$ in M. By the Mostowski absoluteness for the model M again, y enumerates the equivalence class of $[x]_E$ even in V. Thus, the model M contains all E-equivalents of all of its points. Thus, if $V[H_n]$ for $n \in \omega$ are generic extensions containing respective E-related points x_n, then in fact $x_n \in V[H_m]$ for every $n, m \in \omega$. This proves the first sentence.

For the case of E_1, write $X = (2^\omega)^\omega = \mathrm{dom}(E_1)$. Any two models $V[G_0]$, $V[G_1]$ containing the respective representatives $x_0, x_1 \in X$ of the same E_1-class share a common tail of the sequences x_0, x_1 and so also a common representative of their E_1 class. $\square$

Question 8.1.4. Let P be the trivial forcing. Are the following equivalent for a Borel equivalence relation E?

1. E has P-σ-connections;
2. E is essentially countable.

Question 8.1.5. Let P be the trivial forcing. Are the following equivalent for a Borel equivalence relation E?

1. E has P-connections;
2. $E \leq_{\mathrm{B}} E_1 \times E_\infty$.

In the way of applications of this invariant, I will start with results that show key conceptual distinctions between the orbit equivalence relations and E_1.

Theorem 8.1.6. *Let E be an orbit equivalence relation of a continuous Polish group action. Then E has Cohen-σ-connections.*

Proof. Let $G \curvearrowright X$ be the action generating E. Let $V[H_n]$ for $n \in \omega$ be generic extensions of V containing respective points $x_n \in X$ which are E-related. Let $V[K]$ be a forcing extension containing all of them; in particular, for each n $V[G]$ contains an element $g_n \in G$ such that $g_n \cdot x_0 = x_n$. Let $g \in G$ be a P_G-generic point over $V[K]$ and let $x = g \cdot x_0$. It will be enough to show that x belongs to a P_G-extension of every model $V[H_n]$.

Indeed, fix a number $n \in \omega$ and consider the point $h_n = g \cdot g_n^{-1} \in G$. Since the meager ideal on the group G is invariant under multiplication, the point h_n is P_G-generic over $V[K]$. Thus, it is also P_G-generic over the smaller model $V[H_n]$. Also, $x \in V[H_n][h_n]$, since $x = h_n \cdot x_n$ by the definition of x and h_n. This concludes the proof. □

Theorem 8.1.7. *E_1 does not have P-σ-connections for any Suslin forcing.*

Proof. Write $X = (2^\omega)^\omega = \mathrm{dom}(E_1)$. Let $\langle y_n : n \in \omega \rangle \in X$ be a sequence added by the full support countable product of Sacks forcing over the ground model. Let $x_n \in X$ be the sequence defined by $x_n(m) = y_m$ if $m > n$, and $x_n(m) = 0$ if $m \leq n$. Thus, the models $V[x_n]$ for every $n \in \omega$ are obtained by the full support countable product of Sacks forcing as well, and the points x_n for $n \in \omega$ are pairwise E_1-related .

Now suppose that $x \in X$ is a point in some further generic extension which is E_1 equivalent to all x_n. I will show that there is $n \in \omega$ such that x does not belong to any c.c.c. extension of $V[x_n]$. Just choose any number $n \in \omega$ such that $x(n) = y_n$. By the product forcing theorem applied in V to the product of Sacks forcing, the point y_n is generic over $V[x_n]$ for the Sacks forcing in the sense of V. This poset is nowhere c.c.c. in $V[x_n]$ since it is nowhere c.c.c. in V and $\aleph_1$ is preserved between V and $V[x_n]$. Thus, the point y_n as well as x cannot belong to any c.c.c. extension of $V[x_n]$. □

Theorem 8.1.8. *E_1^ω has Hechler connections, but not Cohen connections.*

Proof. Write $X = (2^\omega)^{\omega\times\omega} = \mathrm{dom}(E_1^\omega)$. For the first sentence, let $V[H]$ be a generic extension of V containing smaller extensions $V[G_0], V[G_1]$ which in turn contain E_1^ω-related points $x_0, x_1 \in X$. Let $g \in \omega^\omega$ be a function such that $\forall i \forall j > g(i)\ x_0(i,j) = x_1(i,j)$. Let $f \in \omega^\omega$ be a Hechler-generic function over $V[H]$ which pointwise dominates g. As Hechler forcing is Suslin c.c.c., f is Hechler generic over both models $V[G_0]$ and $V[G_1]$ by Fact 2.3.8. Let $x \in X$ be the point defined by $x(i,j) = x_0(i,j)$ if $j > f(i)$, and $x(i,j) = 0$ otherwise. Clearly, x is E_1^ω-related to both x_0, x_1 and it belongs to both models $V[G_0][f]$ and $V[G_1][f]$. This proves the Hechler connections of E_1^ω.

The second sentence is more difficult. Let $f \in \omega^\omega$ be a Hechler-generic real over the ground model. Let $y \in (2^\omega)^{\omega\times\omega\times 2}$ be a point generic over $V[f]$ for the random algebra with the usual product measure on $(2^\omega)^{\omega\times\omega\times 2}$. Let $x_0 \in X$ be defined by $x_0(i,j) = y(i,j,0)$ and let $x_1 \in X$ be defined by $x_1(i,j) = y(i,j,0)$ except when $f(i) = j$ when I set $x_1(i,j) = x_1(i,j,1)$. It is clear that $x_0\ E_1^\omega\ x_1$. I will show that in no forcing extension there is $x \in X$ which is E_1^ω-related to x_0, x_1 and $V[x_0][x]$, $V[x_1][x]$ are Cohen extensions of $V[x_0]$ and $V[x_1]$ respectively.

To this end, first use standard Fubini-type considerations in $V[f]$ to show that both x_0, x_1 are random-generic over $V[f]$ and for every $i,j \in \omega$, $y(i,j,1)$ is random-generic over $V[f][x_0]$. Since random forcing is c.c.c. and Suslin, x_0, x_1 are random-generic over V and for every $i,j \in \omega$, $y(i,j,1)$ is random-generic over $V[x_0]$ by Fact 2.3.8. Since the random forcing is bounding, it follows that $f \in \omega^\omega$ is still dominating over $V[x_0]$. Now, suppose that $x \in X$ is a point such that $x\ E_1\ x_0$ and $V[x_0][x]$ is a Cohen extension of $V[x_0]$. Let $g \in \omega^\omega$ in $V[x_0][x]$ be a function such that for every i, for every $j > g(i)$, $x(i,j) = x_0(i,j)$. Since Cohen forcing does not add a dominating real, there is $i \in \omega$ such that $f(i) > g(i)$ and so $x(i,f(i)) = x_0(i,f(i)) = y(i,f(i),0)$. Thus, $y(i,f(i),0) \in V[x] \subset V[x_1][x]$, it is a point random generic over $V[x_1]$, and therefore $V[x_1][x]$ is not a Cohen extension of $V[x_1]$.

□

Corollary 8.1.9. *E_1 is not Borel reducible to an orbit equivalence of a Polish group action.*

Proof. The previous theorems yield two different ways of proving this. The easier way will note that E_1 does not have Suslin σ-connections (Theorem 8.1.7) while every orbit equivalence relation has Cohen σ-connections (Theorem 8.1.6). Theorem 8.1.2 then completes the argument.

The more difficult way will note that if E_1 was Borel reducible to an orbit equivalence relation E, then E_1^ω would be reducible to E^ω. Now E^ω is still an orbit equivalence relation, generated by the product of the original action, and therefore has Cohen connections. On the other hand, E_1^ω does not have Cohen connections by Theorem 8.1.8. □

Theorem 8.1.10. *Let J be the branch ideal on $2^{<\omega}$ or one of the ideals J_α for $\alpha \in \omega_1$. Then $=_J$ does not have P-connections for any Suslin poset P.*

Proof. Let $X = 2^{2^{<\omega}}$. Let Q be the poset of finite partial functions from $2^{<\omega}$ to 2. Let $x_0 \in X$ be a Q-generic point over V. Let $G \subset \mathrm{Coll}(\omega, \mathfrak{c})$ be a filter generic over $V[x_0]$. Working in $V[x_0][H]$, let $z \in 2^\omega$ be any point coding a wellorder of ordertype ω_1^V, and let $a \subset \omega$ be an infinite set such that for every function $f \in \omega^\omega$ in $V[x_0]$ there is $n \in a$ such that $f(n)$ is smaller than the next element of a past n. Let $x_1 \in X$ be defined by $x_1(t) = 1 - x_0(t)$ if there is $n \in a$ such that $t = z \restriction n$ and $x_1(t) = x_0(t)$ otherwise. I claim that $x_0, x_1 \in X$ violate the definition of P-connections for any c.c.c. Suslin poset P.

First of all, $x_0 =_J x_1$ since the two points of X differ only on sequences along the branch $z \in 2^\omega$. Second, observe that $x_1 \in X$ is Q-generic over V. For every open dense set $D \subset Q$ in the ground model and every $n \in \omega$ there is $f(n) \in \omega$ such that the condition $x_0 \restriction 2^{<f(n)}$ belongs to D even when one disturbs it

arbitrarily on $2^{<n}$–this follows from the genericity of the point $x_0 \in X$. The function f belongs to $V[x_0]$ and so there is a number $n \in a$ such that $f(n) < m$ where m is the next element of a past n. Then $x_1 \restriction 2^{<f(n)} \in D$, proving the Q-genericity of x_1 over the ground model.

I claim that in no extension there is a point $x \in X$ which is $=_J$-related to both x_0, x_1 and at the same time c.c.c.-generic over both models $V[x_0], V[x_1]$. Since the branch ideal is a subset of the ideals J_α, it is enough to verify this for a fixed ordinal $\alpha \in \omega_1$ and the ideal $J = J_\alpha$. Let x be any point $=_J$-related to both x_0, x_1. Let $b_0 = \{t \in 2^{<\omega} : x(t) \neq x_0(t)\} \in V[x_0][x]$ and $b_1 = \{t \in 2^{<\omega} : x(t) \neq x_1(t)\} \in V[x_1][x]$. Since the sets b_0, b_1 are in the ideal J_α, their respective traces $\mathrm{tr}(b_0), \mathrm{tr}(b_1)$ are countable and by the Shoenfield absoluteness, all of their elements belong to the respective models $V[x_0][x]$ and $V[x_1][x]$. Now observe that $z \in \mathrm{tr}(b_0) \cup \mathrm{tr}(b_1)$. In the opposite case, there would be $n \in a$ such that $z \restriction n \notin b_0 \cup b_1$. Since $x_0(z \restriction n) \neq x_1(z \restriction n)$, this contradicts the choice of the sets b_0, b_1.

In conclusion, either $z \in V[x_0][x]$ or $z \in V[x_1][x]$. This means that one of these models must collapse $\aleph_1^V$ to $\aleph_0$, and impossibility for a c.c.c. extension of a Cohen extension. □

Theorem 8.1.11. *Let I be an analytic σ-ideal of compact sets on a compact metrizable space Y. Then $=_I^{2^\omega}$ has P-connections for some Suslin poset P.*

Proof. I will first identify the Suslin poset. Let $\mathcal{O}$ be a countable basis for Y closed under finite unions and intersections. Let P be the poset consisting of pairs $p = \langle t_p, a_p \rangle$ where $t_p \in \mathcal{O}$ and $a_p \in I$. The ordering is defined by $q \leq p$ if $t_p \subset t_q$, $a_p \subset a_q$, and $a_p \cap t_q \setminus t_p = 0$. The poset P adds an open set $\dot{O}_{gen} \subset Y$ which is the union of the first coordinates of the conditions in the generic filter.

Claim 8.1.12. *P is a σ-centered Suslin poset. It forces $Y \setminus \dot{O}_{gen} \in I$.*

Proof. The centeredness is immediate as two conditions with the same first coordinate are compatible in P. The complexity evaluation is clear as well. Note that conditions $p, q \in P$ are compatible if and only if the sets $a_q \cap t_p \setminus t_q$ and $a_p \cap t_q \setminus t_p$ are both empty. This is a Borel condition as the sets a_p, a_q are compact.

To see that $P \Vdash Y \setminus \dot{O}_{gen} \in I$, recall that by [14, Theorem 33.3], the σ-ideal I is in fact G_δ in $K(X)$. Thus, $I = \bigcap_n U_n$ where $U_n \subset K(X)$ is open. The open sets U_n can be chosen to be downward closed, and for each such a set there is a collection $\mathcal{O}_n$ of open subsets of X such that $U_n = \{K \in K(X) : \exists O \in \mathcal{O}_n\ K \subset O\}$. Let $p \in P$ be a condition and $n \in \omega$. It will be enough to find a condition $q \leq p$ and a set $O \in \mathcal{O}_n$ such that $q \Vdash \dot{O}_{gen} \cup O = X$. To do this, pick a set $O \in \mathcal{O}_n$ such that $a_p \subset O$; such a set must exist as $a \in I$. By a compactness argument, there is a set $O' \subset X$ in $\mathcal{O}_n$ such that $O' \cup O = X$ and $O' \cap a_p = 0$. The condition $q = \langle t_p, O_p \cup O' \rangle \leq p$ has the required properties. □

Let $C \subset Y$ be a countable dense set and let $X = (2^\omega)^C$ be the domain of $=_I$. Now suppose that $V[H]$ is some generic extension of V, containing smaller generic extensions $V[G_0]$, $V[G_1]$ which in turn contain representatives $x_0, x_1 \in X$ of the same $=_I$-class. Let $a \subset Y$ be the closure of the set $\{c \in C : x_0(c) \neq x_1(c)\}$; so $a \in I$. Let $O \subset Y$ be a set P-generic over $V[H]$ meeting the condition $\langle 0, a\rangle$. By Fact 2.3.8, the set O is P-generic over both models $V[G_0], V[G_1]$. Let $x \in X$ be defined by $x(c) = x_0(c)$ if $c \in O$ and $x(c) = 0$ otherwise. It is clear that x belongs to both models $V[G_0][O]$ and $V[G_1][O]$, and $x =_I x_0$. This concludes the proof. □

As the last remark in this section, I will show that the notion of P-connections is absolute throughout all forcing extensions.

Theorem 8.1.13. *Suppose that there are class many Woodin cardinals. Let P be a Suslin forcing, and let E be an analytic equivalence relation on a Polish space X. The truth value of the statement "E has P-connections" is the same in all forcing extensions.*

Proof. To begin, recall that c.c.c. of a given Suslin forcing is absolute among all forcing extensions by [2, Theorem 3.6.6], and therefore the poset given by the definition of P remains Suslin in all forcing extensions. Now, let Q be a poset. I will show that E has P-connections if and only if some condition of Q forces E to have P-connections; this will prove the theorem.

Suppose first that E has P-connections. If $G \subset Q$ is a filter generic over V, R_0, R_1, R_2 are posets in $V[G]$, and $H_0 \subset R_0$, $H_1 \subset R_1$, and $H_2 \subset R_2$ are filters generic over $V[G]$ such that $H_0, H_1 \in V[G][H_2]$, and $x_0, x_1 \in X$ are E-related points in the respective models $V[G][H_0]$ and $V[G][H_1]$, then by the P-connections of E, in some further generic extension of $V[G][H_2]$ there are filters $K_0 \subset P$ and $K_1 \subset P$ which are respectively generic over $V[G][H_0]$ and $V[G][H_1]$, and a point $x \in V[G][H_0][K_0] \cap V[G][H_1][K_1]$ which is E-related to x_0, x_1. This, however, verifies the P-connections of E in the model $V[G]$.

The converse is significantly more difficult. Suppose that some condition $q \in Q$ forces E to have P-connections. Suppose that R_0, R_1, R_2 are posets in V, $\dot H_0, \dot H_1$ are R_2-names and $\dot x_0, \dot x_1$ are respective R_0, R_1-names for elements of the space X such that $R_2 \Vdash \dot H_0 \subset R_0$ is generic over V, $\dot H_1 \subset \check R_1$ is generic over V, $\dot x_0/\dot H_0 \; E \; \dot x_1/\dot H_1$. Let $\kappa = |\mathcal{P}^5(R_2)|$. I will show that

(*) $R_2 \times \mathrm{Coll}(\omega, \kappa) \Vdash \exists K_0, K_1 \subset P$ respectively generic over the models $V[\dot H_0]$ and $V[\dot H_1]$ and a point $x \in V[\dot H_0][K_0] \cap V[\dot H_1][K_1]$ such that $x \; E \; \dot x_0/\dot H_0$.

Let δ be a Woodin cardinal larger than $|Q|, |R_2|$. Let L be a generic filter for the full stationary tower $\mathbb{P}_{<\delta}$, containing the condition consisting of all countable subsets of $\mathcal{P}(Q)$. Let $j : V \to M$ be the associated embedding into a transitive model M in $V[L]$. It is known [16, Theorem 2.5.8] that $j(\delta) = \delta$ and M is closed under $< \delta$-sequences in $V[L]$. I will show that (*) holds in M with all its parameters moved by j; the elementarity of j will conclude the argument.

Let $H_2 \subset j(R_2)$ be a filter generic over $V[L]$, and let $H_0 \subset j(R_0), H_1 \subset j(R_1)$ be the filters given by $H_0 = j(\dot H_0)/H_2$ and $H_1 = j(\dot H_1)/H_2$; let also $x_0 = j(\dot x_0)/H_0$ and $x_1 = j(\dot H_1)$. By the elementarity of the embedding j and the forcing theorem applied in the model M, H_0, H_1 are filters generic over M and $x_0\ E\ x_1$. Since M is closed under $< \delta$-sequences in $V[L]$, and the posets R_0, R_1 have size $< \delta$, the filters H_0, H_1 are generic over the model $V[L]$.

The model M (and therefore also the model $V[L]$) contains a filter $G \subset Q$ generic over V: the choice of the initial condition in the nonstationary tower forcing guarantees that $j''\mathcal{P}(Q)$ is countable in M and therefore the generic filter G can be obtained in M. Thus, by the P-connections of E in the model $V[G]$, in some further forcing extension $V[L][H_2][K]$ there are filters $K_0, K_1 \subset P$ such that

(**) $K_0, K_1 \subset P$ are respectively generic over $V[L][H_0]$ and $V[L][H_1]$ and there is a point $x \in V[L][H_0][K_0] \cap V[L][H_1][K_1]$ which is E-related to both x_0, x_1.

Let $K' \subset \mathrm{Coll}(\omega, j(\kappa))$ be a filter generic over $V[L][H_2][K]$. By the Mostowski absoluteness between the models $V[L][H_2][K]$ and $V[L][H_2][K']$, the filters K_0, K_1 satisfying (**) exist also in $V[L][H_2][K']$ (as $\mathcal{P}(P) \cap V[H_2]$ is countable there). Now, $V_\delta \cap V[L][H_2][K'] = V_\delta \cap M[H_2][K']$, since the model M is closed under $< \delta$-sequences in $V[L]$ and the poset $j(R_2) \times \mathrm{Coll}(\omega, j(\kappa))$ has size $< \delta$ in the model $V[L]$. Thus, the filters K_0, K_1 satisfying (**) exist in the model $M[H_2][K']$. This confirms (*) in M and concludes the proof of the theorem. $\square$

Question 8.1.14. Does Theorem 8.1.13 hold without the large cardinal assumptions?

Question 8.1.15. Let P be a Suslin forcing, and let E be an analytic equivalence relation on a Polish space X. What is the complexity of the collection $I = \{A \subset X : A \text{ is analytic and } E \restriction A \text{ has } P\text{-connections}\}$? Is it $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$?

It is rather immediate that I is a σ-ideal of analytic sets.

8.2 The uniformity cardinal

In the spirit of the study of cardinal invariants of the continuum, one can consider the following natural definition:

Definition 8.2.1. Let E be an analytic equivalence relation on a Polish space X. The uniformity number $\mathtt{non}(E)$ is the smallest possible cardinality of a set $A \subset X$ such that there is no Borel set $B \subset X$ such that $A \subset B$ and $E \restriction B$ is smooth. If E is smooth then let $\mathtt{non}(E) = \infty$.

The collection of all Borel sets $B \subset X$ such that $E \restriction B$ is smooth is a σ-ideal by [12, Corollary 7.3.2] at least in the case of a Borel equivalence relation E. Thus, the invariant $\mathtt{non}(E)$ is nothing else but the uniformity number of this σ-ideal as defined in [2, Definition 1.3.]. Its value may depend on the structure of real line and statements such as Martin's Axiom. The most important point here is that uniformity is a Borel reducibility invariant, even though in a sense opposite to the pinned cardinal $\kappa(E)$:

Theorem 8.2.2. *Suppose that E, F are analytic equivalence rleations on respective Polish spaces X, Y. If $E \leq_{\mathrm{B}} F$ then $\mathtt{non}(E) \geq \mathtt{non}(F)$.*

Proof. Let $h : X \to Y$ be the Borel reduction of E to F. Suppose that $A \subset X$ is a set such that there is no Borel set $B \subset X$ such that $A \subset B$ and $E \restriction B$ is smooth. Let $A' = h''A \subset Y$. Then there is no Borel set $B' \subset Y$ such that $A' \subset B'$ and $F \restriction B'$ is smooth–if $B' \subset Y$ were such a set, then its preimage $B = h^{-1}B'$ would contradict the assumed properties of the set $A \subset X$. The inequality claimed in the theorem immediately follows. □

Thus, high values of $\mathtt{non}(E)$ should indicate a simple equivalence relation E. The first observation is that in ZFC, $\mathtt{non}(E_0) \leq \mathtt{non}(I)$ holds where I is either the meager ideal or the null ideal on 2^ω. This follows immediately from the well-known fact that if a Borel set $B \subset 2^\omega$ is non-meager or non-null, then $E_0 \restriction B$ is not smooth. Thus, in models where $\mathtt{non}(I) = \aleph_1$, the uniformity invariant trivializes: the smooth equivalence relations have uniformity equal to ∞, while the nonsmooth Borel equivalence relations have uniformity equal to $\aleph_1$ by the Glimm–Effros dichotomy. If one wants to find finer distinctions in the uniformity invariant, it is necessary to move to other models of set theory. The following result places E_{K_σ} among the equivalence relations with large uniformity.

Theorem 8.2.3. *Assume that Martin's Axiom for κ holds.Then $\mathtt{non}(E_{K_\sigma}) > \kappa$.*

Proof. I will phrase the proof in a way that will be useful later.

Lemma 8.2.4. *Suppose that Martin's Axiom for κ holds. Let $X = 2^\omega$ or $X = \omega^\omega$. Let $G \subset [X]^2$ be a nonempty G_δ graph invariant under E_0. Suppose that $A \subset X$ is a set of size κ such that $[A]^2 \subset G$. Then there is a perfect set $P \subset X$ such that $[P]^2 \subset G$ and $[A]_{E_0} \subset [P]_{E_0}$.*

Proof. For definiteness, consider the case $X = \omega^\omega$. Write $G = \bigcap_n O_n$ for some open sets $O_n \subset [X]^2$. Consider the poset Q consisting of pairs $q = \langle f_q, g_q \rangle$ such that for some $n_q \in \omega$, $f_q : 2^{n_q} \to \omega^{<\omega}$ is a map and $g_q : A \to 2^{n_q}$ is a partial injection, and for distinct strings $t \neq u \in 2^{n_q}$, $[f_q(t)] \times [f_q(u)] \subset \bigcap_{n \in n_q} O_n$. The ordering is defined by $r \leq q$ if $n_q \leq n_r$, $\forall t \in 2^{n_r}\ f_q(t \restriction n_q) \subset f_r(t)$, $\mathrm{dom}(g_q) \subset \mathrm{dom}(g_r)$, and $\forall x \in \mathrm{dom}(g_q)\ g_q(x) \subset g_r(x)$ and $f_r(g_r(x)) \setminus f_q(g_q(x)) \subset x$. It is not difficult to see that Q is indeed a partial ordering.

Claim 8.2.5. *The following sets are dense in Q:*

1. *$\{q \in Q : n_q > n\}$ for every natural number n;*
2. *$\{q \in Q : x \in \mathrm{dom}(g_q)\}$ for every point $x \in A$.*

Proof. First argue that for every number $n \in \omega$, every point $x \in \mathrm{dom}(G)$ and every sequence $s \in \omega^{<\omega}$ there is an extension $s \subset t \in \omega^{<\omega}$ such that $[x \restriction \mathrm{dom}(t)] \times [t] \subset \bigcap_{m \in n} O_m$. To see this, use the invariance of the graph G under E_0 to find a point $y \in [s]$ such that $\{x, y\} \in G$ and then let t be a sufficiently long initial segment of y. By the same reasoning, for every $n \in \omega$

and $s_0, s_1 \in \omega^{<\omega}$ there are extensions $s_0 \subset t_0$ and $s_1 \subset t_1$ in $\omega^{<\omega}$ such that $[t_0] \times [t_1] \subset \bigcap_{m \in n} O_m$.

To prove (1), let $q \in Q$ be arbitrary. It will be enough to show that there is $r \leq q$ such that $n_r = n_q + 1$ and $\mathrm{dom}(g_q) = \mathrm{dom}(g_r)$. To find r, by a repeated use of the preceding paragraph find a function $f_r : 2^{n_r} \to \omega^{<\omega}$ such that for all $t \in 2^{n_q}$ $f_q(t) \subset f_r(t^\frown 0)$ and $f_q(t) \subset f_r(t^\frown 1)$, and if $x \in \mathrm{dom}(g_q)$ is such that $t = g_q(x)$ then $f_r(t^\frown 0) \setminus f_q(t) \subset x$, and moreover, for distinct $s_0, s_1 \in 2^{n_r}$, $[s_0] \times [s_1] \subset \bigcap_{n \in n_r} O_n$. Define the function g_r by $\mathrm{dom}(g_r) = \mathrm{dom}(g_q)$ and $g_r(x) = g_q(x)^\frown 0$. Observe that the condition $r = \langle f_r, g_r \rangle$ is as required.

To prove (2), given condition $q \in Q$ and $x \in A$, just use the preceding paragraph to find a condition $r \leq q$ such that $n_r = n_q + 1$ and $\mathrm{dom}(g_q) = \mathrm{dom}(g_r)$. Then $\mathrm{rng}(g) \neq 2^{n_r}$. If $x \in \mathrm{dom}(g_r)$, the condition $r \leq q$ is in the requested set. If $x \notin \mathrm{dom}(g_r)$, find a string $t \in 2^{n_r}$ such that $t \notin \mathrm{rng}(g_r)$, let $g_s = g_r \cup \langle x, t \rangle$, and observe that the condition $s = \langle f_r, g_s \rangle \leq r \leq q$ is in the requested set. This completes the proof of (2). □

Claim 8.2.6. *The poset Q is c.c.c.*

Proof. By the usual Δ-system and counting arguments, it is enough to show the following. Let $q, r \in Q$ be conditions such that $n_q = n_r$, $f_q = f_r$ and, writing $a = \mathrm{dom}(g_q) \cap \mathrm{dom}(g_r)$, $g_q \restriction u = g_r \restriction u$. Then q, r are compatible in Q. For the simplicity of the notation, assume that $a = 0$.

Let $n_s = n_q + 1$. Define g_s to be a function with $\mathrm{dom}(g_s) = \mathrm{dom}(g_q) \cup \mathrm{dom}(g_r)$ and $\forall x \in \mathrm{dom}(g_q)\ g_s(x) = g_q(x)^\frown 0$ and $\forall x \in \mathrm{dom}(g_r)\ g_s(x) = g_r(x)^\frown 1$. For every $x \in \mathrm{dom}(g_s)$, write $x' = x$ rew $f_q(g_s(x) \restriction n_q)$. Note that the invariance of the graph G under E_0 changes means that $[\{x' : x \in \mathrm{dom}(g_s)\}]^2 \subset G$. Now it is not difficult to find a function $f_s : 2^{n_s} \to \omega^{<\omega}$ so that for distinct sequences $t \neq u \in 2^{n_s}$, $[f_s(t)] \times [f_s(u)] \subset \bigcap_{n \in n_s} O_n$ and for all $x \in \mathrm{dom}(g_s)$, $f_s(g_s(x)) \subset x'$. It is immediate that the condition $s = \langle f_s, g_s \rangle$ is the desired lower bound of the conditions q, r. □

Now, use Martin's Axiom to find a filter $H \subset Q$ that meets all the open dense subsets of Q indicated in the previous claim. Define a continuous function $f : 2^\omega \to X$ by setting $f(y) = \bigcup \{f_q(y \restriction n_q) : q \in H\}$. I claim that the set $P = \mathrm{rng}(f)$ works as required. The set P is a homeomorphic copy of 2^ω and therefore perfect. $[P]^2 \subset G$ follows from the definition of the poset Q. To see that every element of the set A has an E_0-equivalent in the set P, for every $x \in A$ consider the point $y = \bigcup \{g_q(x) : q \in Q \text{ and } x \in \mathrm{dom}(g_q)\} \in 2^\omega$ and use the definition of the poset Q to show that $f(y)\ E_0\ x$ holds as desired. □

Now, suppose that κ is a cardinal, Martin's Axiom for κ holds, and $A \subset \omega^\omega$ is a set of size κ. I must produce a Borel set $B \subset \omega^\omega$ such that $A \subset B$ and $E_{K_\sigma} \restriction B$ is smooth. Just use Lemma 8.2.4 to the complement of E_{K_σ} to produce a perfect set $P \subset \omega^\omega$ such that $[A]_{E_0} \subset [P]_{E_0}$ such that P consists of pairwise E_{K_σ}-unrelated elements. Let $B = [P]_{E_0}$. This is a Borel set containing A. Moreover, $E \restriction B$ is smooth since the Borel map $h : B \to P$ defined by

$h(x)$ =the unique element of P which is E_0-equivalent to x, reduces $E \restriction B$ to the identity on the set P. $\square$

Other equivalence relations with high values of the uniformity invariant are difficult to find. One tool that can reach beyond E_{K_σ} is the following theorem:

Theorem 8.2.7. *Suppose that J is an analytic ideal on ω. Suppose that E_n are equivalence relations for each $n \in \omega$. Then $\mathtt{non}(\prod_J E_n) \geq \min\{\mathtt{non}(=_J^{2^\omega}), \mathtt{non}(E_n) : n \in \omega\}$.*

Proof. Let X_n be the respective Polish domains of the equivalence relations E_n. For simplicity assume that these Polish spaces are pairwise disjoint. Let $X = \prod_n X_n$ and let $E = \prod_J E_n$. Let $\kappa = \min\{\mathtt{non}(=_J^{2^\omega}), \mathtt{non}(E_n) : n \in \omega\}$. Suppose that $A \subset X$ is a set of size $< \kappa$; I must produce a Borel set $B \subset X$ containing A such that $E \restriction B$ is smooth. Let $A_n = \{y \in X_n : \exists x \in A\ y = x(n)\}$; thus $|A_n| < \kappa$ for every $n \in \omega$. Thus, there is a Borel set $B_n \subset X_n$ containing A_n such that $E_n \restriction B_n$ is smooth. The relation $\bigcup_n E_n \restriction \bigcup_n B_n$ is still smooth, let $h : \bigcup_n B_n \to 2^\omega$ be its Borel reduction to the identity. For every $x \in \prod_n B_n$ define $k(x) = \langle h(x(n)) : n \in \omega\rangle \in (2^\omega)^\omega = \mathrm{dom}(=_J^{2^\omega})$. Since $|k''A| \leq |A| < \mathtt{non}(=_J^{2^\omega})$, there is a Borel set $C \subset (2^\omega)^\omega$ containing $k''A$ on which $=_J^\omega$ is smooth, as witnessed by a Borel reduction $l : C \to 2^\omega$. It is not difficult to verify that $k^{-1}C$ is a Borel set containing A on which E is smooth as witnessed by the reduction $l \circ k$. $\square$

Low values of the invariant $\mathtt{non}(E)$ should indicate a complicated equivalence relation E. Indeed, it turns out that the unpinned equivalence relations have a provably low uniformity.

Theorem 8.2.8. *Suppose that E is unpinned equivalence relation on a Polish space X. Then $\mathtt{non}(E) = \aleph_1$.*

Proof. Theorem 8.3.10 below shows that there is a sequence $\langle x_\alpha : \alpha \in \alpha_1\rangle$ of pairwise E-unrelated points of X such that for every analytic E-invariant set $C \subset X$, either $\{\alpha \in \omega_1 : x_\alpha \in C\}$ or $\{\alpha \in \omega_1 : x_\alpha \notin C\}$ contains a club. I claim that the set $A = \{x_\alpha : \alpha \in \omega_1\}$ witnesses the fact that $\mathtt{non}(E) = \aleph_1$.

Indeed, suppose for contradiction that $B \subset X$ is a Borel set containing A such that $E \restriction B$ is smooth, as witnessed by a Borel reduction $h : B \to 2^\omega$ of $E \restriction B$ to id. By the countable completeness of the nonstationary ideal, there must be a clopen set $O \subset 2^\omega$ such that the set $h^{-1}O$ contains points x_α for stationary-costationary set of α. However, the analytic E-invariant set $C = \{x \in X : \exists y \in B\ x\ E\ y \text{ and } h(y) \in O\}$ then contradicts the assumed properties of the original sequence $\langle x_\alpha : \alpha \in \alpha_1\rangle$. $\square$

The uniformity invariant is connected with separation cardinal invariants that I proceed to define now.They relate in a rather obvious way to the classical Martin–Solovay c.c.c. coding procedure [10, Theorem 16.20]. I will first recall an easy restatement of the Martin–Solovay result.

Lemma 8.2.9. *Assume that Martin's Axiom for κ holds. Let X be a Polish space and $A_0, A_1 \subset X$ be disjoint sets of size κ. Then there is a Borel set $B \subset X$ such that $A_0 \subset B$ and $A_1 \cap B = 0$.*

Proof. Let $X' \subset \mathcal{P}(\omega)$ be a perfect collection of pairwise almost disjoint infinite subsets of ω. Let $h : X \to X'$ be a Borel bijection, and put $A'_0 = h''A$, $A'_1 = h''A_1$. Let P be the poset of all pairs $p = \langle t_p, a_p \rangle$ where $t_p \subset \omega$ is finite and $a_p \subset A'_0$ is finite, ordered by $q \leq p$ if $t_p \subset t_q$, $a_p \subset a_q$, and $(t_q \setminus t_p) \cap \bigcup a_p = 0$. This is clearly a σ-centered partial order as conditions with the same first coordinate are compatible. For every $x \in A'_0$ let $D_x \subset P$ be the open desne subset of P of all conditions $p \in P$ with $x \in a_p$. For every $x \in A'_1$ and every $n \in \omega$ let $D_{x,n}$ be the open dense subset of P consisting of all conditions p with $t_p \cap x \setminus n \neq 0$. Use the Martin's Axiom to find a filter $G \subset P$ which meets all the open dense sets named. It is not difficult to see that writing $y = \bigcup_{p \in G} t_p$, the set y has infinite intersection with all sets $x \in A'_1$, and finite intersection with all sets $x \in A'_0$. Write $B' = \{x \in Y : x \cap y \text{ is finite}\}$; so $B' \subset Y$ is Borel and $A'_0 \subset B'$ and $B' \cap A'_1 = 0$. The set $B = h^{-1}B'$ works as desired in the lemma. $\square$

A curious landscape appears when one attempts to generalize this result to quotient spaces X/E for various analytic equivalence relations E. A natural definition suggests itself:

Definition 8.2.10. Let E be an analytic equivalence relation on a Polish space X. Let κ, λ be cardinals. Say that E has *κ, λ-separation property* if for every pair $A_0, A_1 \subset X$ of sets such that $|A_0| < \kappa$, $|A_1| < \lambda$ and $[A_0]_E \cap A_1 = 0$ there is an E-invariant analytic set $B \subset X$ such that $A_0 \subset B$ and $B \cap A_1 = 0$.

As it was the case with the pinned cardinal, the status of separation property of a given relation depends on the specific forcing extension while the nonreducibility consequences of it are absolute among all forcing extensions. The nontrivial cases occur when $\aleph_1 < \kappa \leq \mathfrak{c}$ and $1 < \lambda \leq \mathfrak{c}$ and Martin's Axiom or a similar principle holds. I will first verify that the separation property is indeed a reducibility invariant.

Theorem 8.2.11. *If E, F are analytic equivalence relations on Polish spaces X, Y and κ, λ be cardinals. If $E \leq_{\mathrm{B}} F$ and F has the κ, λ-separation property, then E has the κ, λ-separation property as well.*

Proof. Let $h : X \to Y$ be a Borel reduction of E to F. Let $A_0, A_1 \subset X$ be sets such that $|A_0| < \kappa$, $|A_1| < \lambda$, and $[A_0]_E \cap A_1 = 0$. Let $A'_0 = h''A_0$ and $A'_1 = h''A_1$ be subsets of Y. Since h is a reduction, $[A'_0]_F \cap A_1 = 0$. By the κ, λ-separation property of F, there is an F-invariant analytic set $B' \subset Y$ such that $A'_0 \subset B'$ and $B' \cap A'_1 = 0$. The analytic E-invariant set $B = h^{-1}B'$ then separates A_0 from A_1. $\square$

The main tool for securing separation properties for larger cardinals is the Martin–Solovay coding together with forcings increasing the uniformity invariant.

Theorem 8.2.12. *Suppose that κ is a cardinal and Martin's Axiom for κ holds. If E is an analytic equivalence relation on a Polish space X and $\mathtt{non}(E) > \kappa$, then E has the κ^+, κ^+-separation property.*

Proof. Let $A_0, A_1 \subset X$ be sets of size κ such that $[A_0]_E \cap A_1 = 0$. Find a Borel set $C \subset X$ containing $A_0 \cup A_1$ such that $E \restriction C$ is smooth, as witnessed by some Borel reduction $h : C \to 2^\omega$ of $E \restriction C$ to the identity. Use Lemma 8.2.9 and the Martin's Axiom assumption to find a Borel set $D \subset 2^\omega$ such that $h''A_0 \subset D$ and $h''A_1 \cap D = 0$. The E-invariant analytic set $B \subset X$ defined by $B = \{x \in X : \exists y \in C\ x\ E\ y \wedge h(y) \in D\}$ separates the sets A_0 and A_1 as desired. $\square$

The main tool for refuting separation properties is the unpinned property of equivalence relations:

Theorem 8.2.13. *If E is an analytic equivalence relation an a Polish space X and E is unpinned, then E does not have the $\aleph_2, \aleph_2$-separation property.*

Proof. Theorem 8.3.10 below yields a sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$ of pairwise E-unrelated points such that for every analytic E-invariant set $A \subset X$, either the set $\{\alpha \in \omega_1 : x_\alpha \in A\}$ or its complement contains a closed unbounded set. Let $S \subset \omega_1$ be a stationary co-stationary set, and let $A_0 = \{x_\alpha : \alpha \in S\}$ and $A_1 = \{x_\alpha : \alpha \in \omega_1 \setminus S\}$. The sets $A_0, A_1 \subset X$ witness the failure of the $\aleph_2, \aleph_2$-separation property of E. $\square$

The results in this section leave many questions open.

Question 8.2.14. Let J be the eventual density zero ideal on ω. Evaluate $\mathtt{non}(=_J)$ and the separation properties of $=_J$ in the context of Martin's Axiom.

Question 8.2.15. Under Martin's Axiom and $\mathfrak{c} > \aleph_1$, are the following equivalent for every analytic equivalence relation E?

1. E is pinned;
2. $\mathtt{non}(E) = \mathfrak{c}$ (or $\mathtt{non}(E) = \infty$ to cover the case of smooth E);
3. E has the $\mathfrak{c}, \mathfrak{c}$-separation property.

8.3 Ideal sequences

The failure of the separation properties is often exemplified in a particularly spectacular fashion captured in the following definition:

Definition 8.3.1. Let κ be a cardinal and I an ideal on κ. Let E be an analytic equivalence relation on a Polish space X. An *I-sequence* for E is a sequence $\langle x_\alpha : \alpha \in \kappa \rangle$ of pairwise E-unrelated elements of the space X such that for every E-invariant analytic set $A \subset X$, either $\{\alpha \in \kappa : x_\alpha \in A\} \in I$ or $\{\alpha \in \kappa : x_\alpha \notin A\} \in I$.

The interesting cases include the nonstationary ideal on κ as well as its restrictions to various stationary sets. The existence of ideal sequences is intimately tied with the pinned property, as Theorem 8.3.10 below shows. Once again, the nonexistence of I-sequences is a Borel reducibility invariant:

Theorem 8.3.2. *If E, F are analytic equivalence relations on Polish spaces X, Y, $E \leq_{\mathrm{B}} F$, I is an ideal on a cardinal κ, and F has no I-sequence, then E has no I-sequence either.*

Proof. If $h : X \to Y$ is a Borel reduction of E to F and $\langle x_\alpha : \alpha \in \kappa\rangle$ is a I-sequence for E, then $\langle h(x_\alpha) : \alpha \in \kappa\rangle$ is a I-sequence for F. $\square$

I will start this section with showing that a number of equivalence relations does not have interesting ideal sequences.

Theorem 8.3.3. *Let κ be a cardinal and I be a nonprincipal σ-ideal on κ. Let E be an analytic equivalence relation on a Polish space X such that $\mathtt{non}(E) > \kappa$. Then E does not have an I-sequence.*

Proof. Let $\langle x_\alpha : \alpha \in \kappa\rangle$ be a sequence of pairwise E-unrelated elements of the space X. Let $B \subset X$ be a Borel set containing all points x_α for $\alpha \in \kappa$ such that $E \restriction B$ is smooth as witnessed by a Borel reduction $h : B \to 2^\omega$ of $E \restriction B$ to the identity. Use the σ-additivity of the ideal I to find a clopen set $O \subset 2^\omega$ such that neither the set $\{\alpha \in \kappa : h(x_\alpha) \in O\}$ nor its complement are in I. Let $A = \{x \in X : \exists y \in B\ x\ E\ y \text{ and } h(y) \in O\}$. The analytic E-invariant set A shows that $\langle x_\alpha : \alpha \in \kappa\rangle$ is not a I-sequence. $\square$

Together with Theorem 8.2.3, I get the first conclusion about nonexistence of ideal sequences:

Corollary 8.3.4. *Suppose that κ is a cardinal and Martin's Axiom for κ holds. If I is a nonprincipal σ-complete ideal on κ then E_{K_σ} has no I-sequence.*

Theorem 8.3.5. *Suppose that κ is a cardinal and Martin's Axiom for κ holds. If I is a nonprincipal σ-complete normal ideal on κ and J an analytic P-ideal on ω, then $=_J$ does not have a I-sequence.*

Proof. Use a theorem of Solecki [23] to find a lower semicontinuous submeasure ϕ on ω such that $J = \{a \subset \omega : \lim_n \phi(a \setminus n) = 0\}$. For points $x, y \in 2^\omega$ define $d(x, y) = \limsup_n \phi(\{m > n : x(m) \neq y(m)\})$. This is a quasimetric on 2^ω and $x =_J y$ if and only if $d(x, y) = 0$.

Suppose that $\langle x_\alpha : \alpha \in \kappa\rangle$ is a sequence of pairwise $=_J$-inequivalent elements of 2^ω. I must find an analytic $=_J$-invariant set $A \subset 2^\omega$ such that neither the set $\{\alpha \in \kappa : x_\alpha \in A\}$ nor its complement belong to I. The treatment divides into two cases.

Case 1. There is $y \in 2^\omega$ and a real $\varepsilon > 0$ such that neither the set $\{\alpha \in \kappa : d(y, x_\alpha) < \varepsilon\}$ nor its complement belong to I. In this case, just let $A = \{x \in 2^\omega : d(x, y) < \varepsilon\}$ and observe that this is an analytic $=_J$-invariant set with the requested properties.

Case 2. If Case 1 fails, I will first find a set $C \subset \kappa$ whose complement belongs to I and a real number $\varepsilon > 0$ such that the points x_α for $\alpha \in C$ have pairwise d-distance $> \varepsilon$. Towards its construction, observe the following.

Claim 8.3.6. *There is a rational number $\varepsilon > 0$ such that for every $y \in 2^\omega$, the set $D_{y,\varepsilon} = \{\alpha \in \kappa : d(y, x_\alpha) \leq \varepsilon\}$ belongs to I.*

Proof. If this failed for every $\varepsilon > 0$ as witnessed by $y_\varepsilon \in 2^\omega$, then by the failure of Case 1 the complements of the sets $D_{y_\varepsilon,\varepsilon}$ would belong to I, and by the σ-completeness of I this would be also the case for their intersection D. Let $\alpha \neq \beta$ be two distinct ordinals in D, and let $\varepsilon > 0$ be a rational such that $2\varepsilon < d(x_\alpha, x_\beta)$. Then $d(y_\varepsilon, x_\alpha), d(y_\varepsilon, x_\beta) \leq \varepsilon$ by the definition of the set $D_{y_\varepsilon,\varepsilon}$ while $2\varepsilon < d(x_\alpha, x_\beta)$, contradicting the triangle inequality for the quasimetric d. □

Fix the rational $\varepsilon > 0$ that works as in the claim. Use the normality of the ideal I to conclude that the diagonal union D of sets $D_{x_\alpha,\varepsilon}$ is in I. Let $C = \kappa \setminus D$ and observe that for distinct ordinals $\alpha \neq \beta \in C$, $d(x_\alpha, x_\beta) > \varepsilon$ as desired.

Now, consider the G_δ graph $G \subset [2^\omega]^2$ connecting x, y if $d(x, y) > \varepsilon$. Use Lemma 8.2.4 and Martin's Axiom to find a perfect set $P \subset 2^\omega$ such that $[P]^2 \subset G$ and for every ordinal $\alpha \in C$, P contains some E_0-equivalent of x_α. By the σ-additivity of the ideal I, there must be a relatively open set $O \subset P$ such that neither the set $\{\alpha \in \kappa : [x_\alpha]_{E_0} \cap O\}$ nor its complement belong to the ideal I. Let $A = \{x \in 2^\omega : \exists y \in O\ x =_J y\}$. This is an analytic $=_J$-invariant set, and it confirms that $\langle x_\alpha : \alpha \in \kappa\rangle$ is not a I-sequence. □

For the following theorem, recall the definition of trace and the ideals connected with the Cantor–Bendixson ranks of countable sets of Theorem 6.6.18. If $a \subset 2^{<\omega}$ is a set then write $\mathrm{tr}(a) = \{x \in 2^\omega : \forall n\ \exists t \in a\ x \restriction n \subset t\}$.

Theorem 8.3.7. *Suppose that κ is a cardinal and Martin's Axiom for κ holds. Suppose that I is a nonprincipal normal σ-complete normal ideal on κ. Let J be the ideal of sets $a \subset 2^{<\omega}$ such that $\mathrm{tr}(a)$ is a closed set of Cantor-Bendixson rank 1. Then the equivalence relation $=_J$ has no I-sequence.*

Proof. Let $\langle x_\alpha : \alpha \in \kappa\rangle$ be a sequence of pairwise $=_J$-unrelated points of $X = 2^{2^{<\omega}}$. I must find an analytic $=_J$-invariant set $A \subset X$ such that neither the set $\{\alpha \in \kappa : x_\alpha \in A\}$ nor its complement are in I. The treatment divides into two cases.

Case 1. There is a binary string $t \in 2^{<\omega}$ and a function $y \in 2^{[t]}$ such that neither the set $\{\alpha \in \kappa : x_\alpha \restriction [t] =_J y\}$ nor its complement belong to the ideal I. In this case, fix such t, y and let $A = \{x \in 2^\omega : x \restriction [t] =_J y\}$; this is an analytic $=_J$-invariant set that shows that the sequence $\langle x_\alpha : \alpha \in \kappa\rangle$ is not an I-sequence.

Case 2. If Case 1 fails, consider the set $T = \{t \in 2^\omega : \forall y \in 2^{[t]}\ \{\alpha \in \kappa : x_\alpha \restriction t =_J y\} \in I\}$. It is immediate from the definition of J that this is a downward closed set of binary strings without endpoints. The treatment splits into two subcases.

Case 2a. The tree T has only finitely many branches. In this case, for the simplicity of notation assume that T has only one branch and this branch has zero entries only. For every $n \in \omega$ write t_n for the unique binary string consisting of first n many zero and then a single 1. Let $y_n \in 2^{[t_n]}$ be a sequence such that $D_n = \{\alpha \in \kappa : x_\alpha \restriction [t_n] = y_n\} \notin I$. By the failure of Case 1, the complement of the set D_n is in I and by the σ-completeness of the ideal I the same is true of the set $D = \bigcap_n D_n$.

Consider the space $Z = \prod_n 2^{[t_n]}$ and the equivalence relation F on Y which is the modulo finite product of E_0 equivalence relations on the coordinates of the space Z. Consider the function $h : X \to Z$ defined by $h(x) = \langle x \restriction [t_n] : n \in \omega \rangle$. It is not difficult to see that g is a continuous homomorphism of $=_J$ to F. I will first show that the points $h(x_\alpha)$ are pairwise F-unrelated for $\alpha \in D$. Indeed, if $h(x_\alpha)\ F\ h(x_\beta)$ was true for some $\alpha \neq \beta$, then it would be the case that for all $n \in \omega$ $x_\alpha \restriction [t_n] =_J x_\beta \restriction [t_n]$ (since $\alpha, \beta \in D$) and for all but finitely many $n \in \omega$, $x_\alpha \restriction [t_n]\ E_0\ x_\beta \restriction [t_n]$ (since $h(x_\alpha)\ F\ h(x_\beta)$). This together implies that $x_\alpha =_J x_\beta$, which contradicts the assumptions on the sequence $\langle x_\alpha : \alpha \in \kappa \rangle$.

Now, the equivalence relation F does not have I-sequences by the conjunction of Theorems 8.3.3 and 8.2.7. Thus, there is a $=_J$-invariant analytic set $B \subset Z$ such that neither the set $\{\alpha \in D : h(x_\alpha) \in B\}$ nor its complement belong to the ideal I. Let $A = \{x \in X : h(x) \in B\}$ and observe that the set A has the requested properties.

Case 2b. The tree T has infinitely many branches. In this case, for every $\alpha \in D$ consider the set $D_\alpha = \{\beta \in \kappa : \exists t \in T\ x_\alpha \restriction [t] =_J x_\beta \restriction [t]\} \subset \kappa$. By the definition of the tree T and the σ-completeness assumption on the ideal I, the set D_α belongs to I. Use the normality of the ideal I to find a set $C \subset \kappa$ such that for every $\alpha \in \beta \in C$, $\beta \notin D_\alpha$ and $\kappa \setminus C \in I$. Now, consider the G_δ graph G on X defined by $x\ G\ y$ if for every $t \in T$ the set $\{s \supset t : x(s) \neq y(s)\}$ is infinite. The graph G is disjoint from the equivalence relation $=_J$, since whenever $x\ G\ y$ are two G-connected points in X, the set $\mathrm{tr}(\{s \in 2^{<\omega} : x(s) \neq y(s)\})$ contains all branches of T, and the case assumption then implies $x \neq_J y$. The construction of the set C implies that if $\alpha \neq \beta \in C$ are distinct elements, then $x_\alpha\ G\ x_\beta$. Lemma 8.2.4 now shows that there is a perfect set $P \subset X$ consisting of pairwise G-unrelated points such that $\{x_\alpha : \alpha \in C\}_{E_0} \subset [P]_{E_0}$. Use the σ-completeness of the ideal I to find a relative basic open set $O \subset P$ such that neither the set $\{\alpha \in C : \text{the unique } E_0\text{-equivalent of } x_\alpha \text{ in } C \text{ belongs to } O\}$ nor its complement belong to the ideal I. Let $A = [O]_{=_J}$ and observe that the analytic $=_J$-invariant set $A \subset X$ has the requested properties. □

Theorem 8.3.8. *Suppose that κ is a cardinal and Martin's Axiom for κ holds. If I is a nonprincipal σ-complete normal ideal on κ and K is an analytic σ-ideal of closed sets on a Polish space X, then the equivalence relation $=_K$ has no I-sequence.*

Proof. Let $B \subset X$ be a countable dense set and let $=_K$ be realized as the equivalence relation on 2^B by setting $x =_K y$ if the closure of the set $\{b \in B : x(b) \neq y(b)\}$ is in the ideal K. Let $\langle x_\alpha : \alpha \in \kappa \rangle$ be a sequence of pairwise

$=_K$-unrelated points in the set 2^ω. I must find an analytic $=_J$-invariant set $A \subset 2^D$ such that neither the set $\{\alpha \in \kappa : x_\alpha \in A\}$ nor its complement belong to I.

Let $\mathcal{O}$ be a countable basis of the space X. For every set $O \in \mathcal{O}$ and every point $z \in 2^{B \cap O}$, let $D_{O,z} = \{\alpha \in \kappa : x_\alpha \restriction O =_K z\}$. The treatment separates into two cases.

Case 1. There is a set $O \in \mathcal{O}$ and a point $z \in 2^{D \cap O}$ such that neither the set $D_{O,z} \subset \kappa$ nor its complement belongs to I. In this case, let $A = \{x \in 2^B : x \restriction O =_K z\}$ and note that $A \subset 2^B$ is a $=_K$-invariant analytic set that works as required.

Case 2. If Case 1 fails, I will first produce a set $C \subset \kappa$ whose complement belongs to I and a set $U \subset \mathcal{O}$ such that $X \setminus \bigcup U \notin K$ and $\forall \alpha \neq \beta \in C\ \forall O \notin U\ x_\alpha \restriction O \neq_K x_\beta \restriction O$. Let $U = \{O \in \mathcal{O} : \exists z \in 2^{B \cap O}$: the complement of the set $D_{O,z}$ belongs to $I\}$.

Claim 8.3.9. *The set $X \setminus \bigcup U$ is not in the ideal K.*

Proof. Suppose for contradiction that this fails. Enumerate $U = \{O_n : n \in \omega\}$ and for every $O \in U$ pick a function $z_n \in 2^{D \setminus O_n}$ such that the complement of the set D_{O_n,z_n} belongs to I. Let $y \in 2^D$ be any point such that for every $n \in \omega$, y agrees with z_n on the set $O_n \setminus \bigcup_{m<n} O_m$. The complement of the set $D = \bigcap_n D_{O_n,z_n}$ belongs to I by the σ-additivity of the ideal I. I will show that for every $\alpha \in D$, $x_\alpha =_K y$ holds. As the ideal I is nonprincipal, the set D must contain more than one point, and so this contradicts the assumption that the points x_α for $\alpha \in \kappa$ are pairwise $=_K$-unrelated.

Thus, let $\alpha \in D$ and consider the set $A = \{b \in B : y(b) \neq x_\alpha(b)\}$, and the sets $A_n = \{b \in B \cap O_n : y(b) \neq z_n(b)\}$. Note that for every $n \in \omega$, the closure $\bar{A}_n$ belongs to K as $\alpha \in D$. It will be enough to show that $\bar{A} \subset (X \setminus \bigcup U) \cup \bigcup \bar{A}_n$, since the sets on the right hand side are all in K and K is a σ-ideal of closed sets. To confirm the inclusion, suppose that $v \in \bar{A}$ is a point which belongs to $\bigcup U$. Then there is $n \in \omega$ such that $v \in O_n$, and so v must be in the closure of the set $\bigcup_{m \leq n} A_m$ and so $\bar{A}_m$ for some $m \leq n$. This completes the proof. $\square$

For every $\alpha \in \kappa$, let $D_\alpha = \bigcup_{O \notin U} D_{O,x_\alpha}$. By the failure of Case 1 and the definition of the set U, the sets entering this union belong to I. By the σ-additivity of the ideal I, even the union D_α belongs to I. By the normality of the ideal I, the diagonal union D of D_α for $\alpha \in \kappa$ belongs to I as well. Let $C = \kappa \setminus D$. This set has the required properties.

Now, consider the G_δ graph $G \subset [2^D]^2$ connecting points x, y if for every open set $O \in \mathcal{O} \setminus U$, the set $\{d \in B \cap O : x(d) \neq y(d)\}$ is infinite. By Claim 8.3.9, the graph G is disjoint from $=_K$: if $x\ G\ y$ then the closure of the set $\{b \in B : x(b) \neq y(b)\}$ must contain the set $X \setminus \bigcup U$ and therefore cannot be in the ideal K. By Lemma 8.2.4, there is a perfect set $P \subset 2^B$ such that $[P]^2 \subset G$ and for every ordinal $\alpha \in C$, the set P contains some E_0-equivalent of the point x_α.

As the final step, by the σ-additivity of the ideal I, there must be a relatively open set $O \subset P$ such that neither the set $\{\alpha \in \kappa : [x_\alpha]_{E_0} \cap O\}$ nor its complement belong to the ideal I. Let $A = \{x \in 2^B : \exists y \in O\ x =_J y\}$. This is an analytic $=_J$-invariant set, and it confirms that $\langle x_\alpha : \alpha \in \kappa \rangle$ is not a I-sequence. $\square$

The most natural case of I-sequences comes with I equal to the nonstationary ideal on ω_1, where it is intimately connected with the pinned property of the equivalence relation in question, as showed by the following theorem.

Theorem 8.3.10. *Let E be an analytic equivalence relation on a Polish space X and let I be the nonstationary ideal on ω_1. If E is not pinned, then E has a I-sequence.*

Proof. Suppose that E is not pinned. Then, by Theorem 3.3.2(1), there is a poset P of size $\aleph_1$ and a nontrivial E-pinned P-name τ. Let $\langle M_\alpha : \alpha \in \omega_1 \rangle$ be a continuous $\in$-tower of countable elementary submodels of a large enough structure containing E, X, P, τ. For every $\alpha \in \omega_1$ find a filter $g_\alpha \subset P$ in the model $M_{\alpha+1}$ which is generic over M_α, and let $x_\alpha = \tau/g_\alpha$. It will be enough to show that $\langle x_\alpha : \alpha \in \omega_1 \rangle$ is a I-sequence for E.

First of all, the points x_α for $\alpha \in \omega_1$ are pairwise E-unrelated. This follows from the fact that τ is a nontrivial P-name. Thus, for countable ordinals $\beta \in \alpha$ the forcing theorem applied in M_α to the poset P implies that $M_\alpha[g_\alpha] \models \neg x_\alpha\ E\ x_\beta$, and the Mostowski absoluteness for the model $M_\alpha[g_\alpha]$ implies that $x_\alpha\ E\ x_\beta$ indeed fails in V.

Now, suppose that B is an analytic E-invariant set. It certainly remains E-invariant in the P-extension by the Shoenfield absoluteness. As τ is a pinned name, it must be the case that the largest condition in P decides the statement $\tau \in \dot{B}$. I will show that if $P \Vdash \tau \in \dot{B}$, then the set $\{\alpha \in \omega_1 : x_\alpha \in B\}$ contains a closed unbounded set. The case when $P \Vdash \tau \notin \dot{B}$ is symmetrical.

Let N be a countable elementary submodel of a large enough structure containing all the objects mentioned above, in particular the $\in$-tower of models, and write $\alpha = N \cap \omega_1$. It will be enough to show that $x_\alpha \in B$. Note that $P \cap N = P \cap M_\alpha$ by the continuity of the $\in$-tower. Indeed, let $h \subset P \cap N$ be a filter generic over both the countable models N and $M_\alpha[g_\alpha]$. Then, by the forcing theorem applied in the model N to P, $N[h] \models \tau/h \in \dot{B}/h$, and by the Mostowski absoluteness for $N[h]$, $\tau/g \in B$ holds in V. By the product forcing theorem applied in the model M_α to the poset $P \times P$, the filter $g_\alpha \times h$ is $P \times P$-generic over M_α; by the forcing theorem $M_\alpha[g_\alpha, h] \models x_\alpha = \tau/g_\alpha\ E\ \tau/h$, and by the Mostowski absoluteness for this model, $x_\alpha\ E\ \tau/g$ holds even in V. As the set B is E-invariant, it follows that $x_\alpha \in B$. $\square$

A natural question appears about the equivalence of ideal sequences and the pinned property:

Question 8.3.11. Assume that Martin's Axiom for $\aleph_1$ holds. Are the following equivalent for every analytic equivalence relation E?

1. E is pinned;
2. E has no I-sequence, where I is the nonstationary ideal on ω_1.

I can only resolve this question in the fairly restrictive case of equivalence relations Borel reducible to F_2.

Theorem 8.3.12. *Suppose that Martin's Axiom for $\aleph_1$ holds, let I be the nonstationary ideal on ω_1 and let E be an analytic equivalence relation on a Polish space X such that $E \leq_{\mathrm{wB}} F_2$. E is not pinned if and only if E has a I-sequence.*

Proof. By virtue of Theorem 8.3.10, I need to prove only the right-to-left implication. Let $B_0 \subset X$ be a set consisting of countably many equivalence classes and $h : X \to (2^\omega)^\omega$ be a Borel function which is a reduction of E to F_2 on the set $X \setminus B_0$. Suppose that E is pinned, and $\langle x_\alpha : \alpha \in \omega_1 \rangle$ is a sequence of pairwise E-unrelated elements of x. I must find an analytic E-invariant set $B \subset \omega_1$ such that the set $\{\alpha \in \omega_1 : x_\alpha \in B\}$ is stationary and costationary. Removing countably many points from the sequence if necessary, I may assume that none of the points on the sequence belongs to B_0. Write $y_\alpha = \mathrm{rng}(h(x_\alpha)) \subset 2^\omega$. I proceed in several cases.

Case 1. There is a stationary set $S \subset \omega_1$ such that for every $\alpha \in S$, $y_\alpha \subset \bigcup_{\beta \in \alpha} y_\beta$.

Case 1a. There is a point $z \in 2^\omega$ such that the set $S_z = \{\alpha \in \omega_1 : z \in y_\alpha\}$ is stationary costationary. Then, let $A = \{w \in (2^\omega)^\omega : z \in \mathrm{rng}(w)\}$. This is a Borel F_2-invariant set, and so $B = B_0 \cup h^{-1}A \subset X$ is an E-invariant analytic set. The set $B \subset X$ works as desired by the case assumption.

Case 1b. Assume now that Case 1a fails and the set $y = \{z \in 2^\omega : S_z$ contains a club$\}$ is countable. In this case, let M be a countable elementary model of a large structure such that $\alpha = M \cap \omega_1 \in S$ and argue that $y_\alpha = y$. Indeed, since $\alpha \in S$, it follows by the Case 1 assumption that $y_\alpha \subset M$, and then $y_\alpha = y$ by the definition of the set y and the elementarity of M. Now, if M_0 and M_1 are two such countable submodels with $\alpha_0 = M_0 \cap \omega_1 \in S$ and $\alpha_1 = M_1 \cap \omega_1 \in S$, it follows that $y_{\alpha_0} = y_{\alpha_1} = y$ and therefore $x_{\alpha_0} \mathrel{E} x_{\alpha_1}$. This contradicts the initial assumptions on the sequence $\langle x_\alpha : \alpha \in \omega_1 \rangle$, and so this case is impossible.

Case 1c. If both Cases 1a and 1b fail, consider the poset P of all stationary subsets of S ordered by inclusion. In the P-extension, form the usual generic ultrapower $j : V \to N$ into a model N. While the model N may not be wellfounded, by the normality of the nonstationary ideal its ordinals contain an initial segment isomorphic to $\omega_1^V + 1$. Write $\alpha = \omega_1^V$. By elementarity of the embedding j, $\alpha \in j(S)$ and so $y_\alpha \subset \bigcup_{\beta \in \alpha} y_\beta \subset V$. By the case assumption, $y = y_\alpha$. Let $\dot{x}$ be a P-name for some element of X such that $N \models \mathrm{rng}(h(\dot{x})) = \check{y}$. Since h remains a reduction in the P-extension by the Shoenfield absoluteness, N is correct about h by the Borel absoluteness between N and the P-extension, $\dot{x}$ is an E-pinned P-name. By the Case 1c assumption, the set $y \subset 2^\omega$ is uncountable in V, and therefore $\dot{x}$ is a nontrivial E-pinned name. This contradicts the assumption that E is pinned, and therefore this case is again impossible.

Case 2. If Case 1 fails, then for all but nonstationarily many $\alpha \in \omega_1$ there is a point $z_\alpha \in 2^\omega$ such that $z_\alpha \in y_\alpha \setminus \bigcup_{\beta \in \alpha} y_\beta$. To simplify the notation, assume that the nonstationary set of exceptions is actually empty.

Case 2a. There is $\alpha \in \omega_1$ such that the set $S_\alpha = \{\gamma \in \omega_1 : z_\alpha \in y_\gamma\}$ is stationary costationary. Then let $A = \{w \in (2^\omega)^\omega : z_\alpha \in \mathrm{rng}(w)\}$. This is a Borel F_2-invariant set, and so $B = B_0 \cup h^{-1}A$ is an analytic E-invariant set. The set B works as required by the case assumption.
Case 2b. Suppose that Case 2a fails and the set $T = \{\alpha : S_\alpha \text{ is nonstationary}\}$ is stationary. Let $\bar{T} = \{\alpha \in \omega_1 : \alpha \in T$ and for all $\beta \in \alpha$ such that $\beta \in T$, $\alpha \notin S_\beta\}$; thus, $\bar{T} \subset \omega_1$ is again stationary. Note that for every $\alpha \in \bar{T}$, α is the unique ordinal $\gamma \in \bar{T}$ such that $z_\alpha \in y_\alpha$. Let $T = T_0 \cup T_1$ be a partition into disjoint stationary subsets. A standard application of Solovay coding yields a Borel set $C \subset 2^\omega$ such that $C \cap \bigcup_{\alpha\in\omega_1} y_\alpha = \{z_\alpha : \alpha \in T_0\}$. Let $A = \{w \in (2^\omega)^\omega : C \cap \mathrm{rng}(w) \neq 0\}$. This is clearly an analytic F_2-invariant set. The set $B = B_0 \cup h^{-1}A \subset X$ is analytic and E-invariant. The definitions show that $\{\alpha \in \omega_1 : x_\alpha \in B\} \supset T_0$ and $\{\alpha \in \omega_1 : x_\alpha \notin B\} \supset T_1$ and so the set B works as required.
Case 2c. Suppose that the set $T = \{\alpha : S_\alpha \text{ contains a closed unbounded set}\}$ is stationary. Let $\bar{T} = \{\alpha \in \omega_1 : \alpha \in T$ and for all $\beta \in \alpha$ such that $\beta \in T$, $\alpha \in S_\beta\}$; thus, $\bar{T} \subset \omega_1$ is again stationary. For every ordinal $\alpha \in \bar{T}$ write α^+ for the smallest ordinal in $\bar{T}$ greater than α. Observe that for every $\alpha \in \bar{T}$, α is the only ordinal $\gamma \in \bar{T}$ such that $z_\alpha \in y_\alpha$ and $z_{\alpha^+} \notin y_\alpha$.

Let $\bar{T} = T_0 \cup T_1$ be a partition into disjoint stationary subsets, and consider the set $f = \{\langle z_\alpha, z_{\alpha^+}\rangle : \alpha \in \bar{T}\}$. A routine application of Solovay almost disjoint c.c.c. coding 8.2.9 yields a Borel function $g : 2^\omega \to 2^\omega$ such that $f \subset g$ and a Borel set $C \subset 2^\omega$ such that $C \cap \bigcup_{\alpha\in\omega_1} y_\alpha = \{z_\alpha : \alpha \in T_0\}$. Let $A = \{w \in (2^\omega)^\omega : \exists z \in C\ z \in \mathrm{rng}(w) \wedge g(z) \notin \mathrm{rng}(w)\}$. This is clearly an analytic F_2-invariant set. The set $B = B_0 \cup h^{-1}A \subset X$ is analytic and E-invariant. The definitions show that $\{\alpha \in \omega_1 : x_\alpha \in B\} \supset T_0$ and $\{\alpha \in \omega_1 : x_\alpha \notin B\} \supset T_1$ and so the set B works as required. □

The reader may wonder whether I-sequences can appear in other ideals besides the nonstationary ideal on ω_1. This may occur if perhaps less frequently. I conclude this section with two theorems addressing this situation.

Theorem 8.3.13. *Suppose that κ is a cardinal and Martin's Axiom for κ holds. If I is a normal $< \aleph_2$-complete ideal on κ and E is an equivalence relation classifiable by countable structures, then E has no I-sequence.*

Proof. I will first show that it is enough to consider the case of Borel equivalence relations classifiable by countable structures. Let X be the space of all graphs on ω and E the relation of isomorphism on X; thus E is universal among all equivalence relations classifiable by countable structures. Let $\langle x_\alpha : \alpha \in \kappa\rangle$ be a sequence of pairwise E-unrelated elements of the space X. By ??? there are Borel E-invariant sets $B_\delta \subset X$ for $\delta \in \omega_1$ such that $E \restriction B_\delta$ is Borel, and the sets B_δ form an increasing sequence exhausting the whole space X. For every $\delta \in \omega_1$, let $D_\delta = \{\alpha \in C : x_\alpha \in B_\delta\}$. If for some ordinal δ neither the set D_δ nor its complement beongs to I, then the sequence $\langle x_\alpha : \alpha \in C\rangle$ is not a I-sequence. Otherwise, since the sets D_δ form an increasing union which exhausts all of κ, the $< \aleph_2$-completeness of the ideal I shows that the complement of one of them must belong to I. On that set D_δ, the equivalence relation E is Borel. Thus, it is enough to consider only the case of Borel equivalence relations classifiable by countable structures.

Every Borel equivalence relation classifiable by countable structures is Borel reducible to equality of transitive countable sets of some fixed rank. Let $\delta \in \omega_1$ be any ordinal, let B be the Borel set of extensional relations on ω that are well-founded of rank $\leq \beta$, and let E be the equivalence relation of isomorphism on B. Let $\langle x_\alpha : \alpha \in \kappa \rangle$ be a sequence of pairwise E-unrelated elements of the set B; I must find an analytic E-invariant set $A \subset B$ such that neither the set $\{\alpha \in \kappa : x_\alpha \in A\}$ nor its complement belongs to I.

For every $\alpha \in \kappa$ write $\bar{x}_\alpha$ for the transitive isomorph of x_α. For every hereditarily countable set y, let $D_y = \{\alpha \in \kappa : y \in \bar{x}_\alpha\}$. The treatment divides into two cases:

Case 1. There is a set y such that neither the set $D_y \subset \kappa$ nor its complement belongs to I. In such a case, the Borel set $A = \{x \in B :$ the transitive isomorph of x contains $y\}$ is analytic and E-invariant and works as required.

Case 2. If Case 1 fails, consider the set $u = \{y$: the complement of D_y belongs to $I\}$. It follows directly from the definitions that the set u is transitive.

Claim 8.3.14. *The set u is countable.*

Proof. Suppose for contradiction that the set u is uncountable, containing distinct elements y_δ for $\delta \in \omega_1$. Let $\alpha \in \kappa$ be an ordinal in the intersection of all the sets D_{y_δ} for $\delta \in \omega_1$. Such an ordinal exists by the $< \aleph_2$-completeness of I. Then, for every $\delta \in \omega_1$, it has to be the case that $y_\delta \in \bar{x}_\alpha$, contradicting the fact that the set $\bar{x}_\alpha$ is countable. $\square$

Let $D = \bigcap_{y \in u} D_y$; thus $\kappa \setminus D \in I$. For every $\alpha \in D$ with perhaps one exception, the set $\bar{x}_\alpha$ is not equal to u, and so it contains some extra subsets of u. For every $\alpha \in D$, let $C_\alpha = \bigcup\{D_z : z \subset u, z \in \bar{x}_\alpha \setminus u\}$. The sets forming the union are in I by the definition of u and failure of Case 1, and the union is in I as well by the completeness of I. By the normality of I, the diagonal union C of C_α for $\alpha \in D$ is in I as well. For distinct elements $\alpha \neq \beta \in D \setminus C$, the sets $(\bar{x}_\alpha \setminus u) \cap \mathcal{P}(u)$ and $(\bar{x}_\beta \setminus u) \cap \mathcal{P}(u)$ are nonempty and pairwise disjoint.

Use the σ-completeness of I to find a set $S \subset D \setminus C$ such that neither it nor its complement are in I. By the Martin–Solovay coding theorem 8.2.9 and Martin's Axiom, there is a Borel set $B \subset \mathcal{P}(u)$ such that for every $\alpha \in S$ and every $v \in \bar{x}_\alpha \cap \mathcal{P}(u)$, $v \in B$, while for every $\alpha \in D \setminus (C \cup S)$, for every $v \in \bar{x}_\alpha \cap \mathcal{P}(u)$, $v \notin B$. Consider the set $A = \{x \in X$: for some $v \in B$, v belongs to the transitive isomorph of $x\}$. The set A is Borel, E-invariant, and neither the set $\{\alpha \in C : x_\alpha \in A\}$ nor its complement are in I. $\square$

Theorem 8.3.15. *Suppose that κ is a regular uncountable cardinal and Martin's Axiom for κ holds. Let I be the nonstationary ideal on the set of ordinals in κ of countable cofinality. There is a I-sequence for the mutual domination equivalence.*

Proof. Let $C \subset \omega$ denote the set of all ordinals of countable cofinality in κ. Write E for the mutual domination equivalence relation on the space $X = (\omega^\omega)^\omega$. Use Martin's Axiom to find a modulo finite increasing sequence $\langle y_\gamma : \gamma \in \kappa \rangle$ of elements of ω^ω. For every ordinal $\alpha \in C$ choose a point $x_\alpha \in X$ so that if α is a limit of some countable set $C_\alpha \subset \kappa$ then x_α enumerates the set $\{y_\gamma : \gamma \in C_\alpha\}$. I claim that $\langle x_\alpha : \alpha \in C \rangle$ is a I-sequence for E.

To verify this, first note that if $\alpha \neq \beta \in C$ are distinct limit points of the set C then $x_\alpha \; E \; x_\beta$ fails since the sequence $\langle y_\gamma : \gamma \in \omega_2 \rangle$ is modulo finite increasing. Now, suppose that $A \subset X$ is an analytic E-invariant set; I must show that the set $D = \{\alpha \in C : x_\alpha \in A\}$ or its complement contains a relative club in C. Let P be any poset forcing $\mathtt{cof}(\omega_2^V) = \omega$, let $\langle \dot\gamma_n : n \in \omega \rangle$ be a P-name for a sequence of ordinals cofinal in ω_2^V. Let $\dot x$ be a P-name for an element of X given by $\dot x(n) = y_{\dot\gamma_n}$ for every $n \in \omega$. Let $p \in P$ be a condition deciding the statement $\dot x \in \dot A$. (In fact, p can be taken to be the largest condition in P.) I will show that if the decision is affirmative then the set D contains a relative club in C, and if the decision is negative then the complement of the set D contains a relative club in C. This will complete the proof of (1).

Suppose for definiteness that $p \Vdash \dot x \notin \dot A$. Let M be any elementary submodel of a large enough structure containing the condition p as well as the sequence $\langle y_\gamma : \gamma \in \kappa \rangle$, such that $M \cap \kappa \in C$. It will be enough to show that $x_\alpha \notin A$. For this, move to a generic extension $V[G]$ in which there is a filter $H \subset P \cap M$ which is generic over the model M and contains the condition p. Note that $\dot x/H \; E \; x_\alpha$ as the sequence $\langle y_\gamma : \gamma \in M \cap \kappa \rangle$ is modulo finite increasing. Now, $M[H] \models \dot x/H \notin A$ by the forcing theorem. $V[G] \models \dot x/H \notin A$ holds by the Mostowski absoluteness between the wellfounded models $M[H]$ and $V[G]$. $V[G] \models x_\alpha \notin A$ holds as the set A is E-invariant in the model $V[G]$ by the Shoenfield absoluteness between the models V and $V[G]$. Finally, $V \models x_\alpha \notin A$ by the Mostowski absoluteness between the wellfounded models V and $V[G]$. This completes the proof. □

Question 8.3.16. Let κ be a regular cardinal $> \omega_1$, and let I be the nonstationary ideal on the set of ordinals in κ of cofinality ω_1. Is there a I-sequence for the mutual domination equivalence?

Question 8.3.17. In the theorems of this section, can the assumption of Martin's Axiom for κ be replaced by $\kappa < \mathfrak{c}$?

8.4 Linear orderability

Consider the following definition in the context of choiceless set theory.

Definition 8.4.1. An analytic equivalence relation E on a Polish space X is *linearly orderable* if there is a linear ordering on the set of all E-equivalence classes.

Clearly, the axiom of choice implies that every set can be linearly ordered and so every analytic equivalence relation is linearly orderable. I will study the linear orderability in models where the axiom of choice fails. It turns out that this study boils down to observations about names for elements of the underlying Polish spaces close to the considerations of Chapter 6.

The main motivation resides in the following easy observation:

Theorem 8.4.2. *(ZF) If $E \leq_{\mathrm{wB}} F$ are analytic equivalence relations and F is linearly orderable, then so is E.*

Proof. Let $X = \mathrm{dom}(E)$ and $Y = \mathrm{dom}(F)$ and let $h : X \to Y$ be a Borel function which is a reduction of E to F except on a countable set a of E-equivalence classes. Fix an enumeration $a = \{b_n : n \in \omega\}$. Let $\leq_F$ be a linear ordering of the quotient space Y/F. Define a relation $\leq_E$ on the quotient space X/E by setting $[x_0]_E \leq_E [x_1]_E$ if $[h(x_0)]_F \leq_F [h(x_1)]_F$ if $x_0, x_1 \notin \bigcup a$. Since h is a reduction of E to F outside of the set $\bigcup a$, the relation $\leq_E$ is well defined and it is a linear ordering on the set $X/E \setminus a$. Extend $\leq_E$ to a linear ordering of the whole quotient space X/E by appending the classes of a to the end of it, ordered by the chosen enumeration. This completes the proof of the theorem. $\square$

Thus, one may attempt to detect nonreducibility between equivalence relations by checking which of them are linearly orderable in which models of ZF+DC set theory. In this section, I will consider two such models. Let κ be an inaccessible cardinal and $G \subset \mathrm{Coll}(\omega, <\kappa)$ be a generic filter over V; the first model considered is $V(\mathbb{R}) = V(\mathbb{R} \cap V[G])$. This is the usual choiceless Solovay model. The second model differs from the Solovay model by containing a nonprincipal ultrafilter on ω. The ultrafilter will be of the following common special kind:

Definition 8.4.3. A nonprincipal ultrafilter U on ω is *Ramsey* if for every partition $\pi : [\omega]^2 \to 2$ there is a homogeneous set $a \in U$, i.e. a set such that $\pi \restriction [a]^2$ is constant.

Suppose that $U \subset \mathcal{P}(\omega)$ modulo finite is a filter generic over $V[G]$. In the model $V[G][U]$, U is a Ramsey ultrafilter on ω. The second model is the model $V(\mathbb{R})[U]$–the choiceless Solovay model with Ramsey ultrafilter adjoined. In fact, it is not particularly relevant which forcing is used to adjoin the Ramsey ultrafilter over the choiceless Solovay model as all the resulting models have the same theory. The model $V(\mathbb{R})[U]$ satisfies the axiom of dependent choices as it is closed under ω-sequences in the AC model $V[G][U]$.

Which nonreducibilities are detected by linear orderability in these two models? In the case of the choiceless Solovay model, the answer is quite simple and known for some time:

Fact 8.4.4. [11] *In the choiceless Solovay model $V(\mathbb{R})$, the following are equivalent for any analytic equivalence relation E:*

1. *E is linearly orderable;*
2. *$E_0 \leq_{\mathrm{B}} E$ fails.*

In the model $V(\mathbb{R})[U]$, the answer is considerably more complicated. The ultrafilter can be used to linearly order a good number of equivalence relations.

Theorem 8.4.5. *(ZF+DC) The class of linearly orderable equivalence relations is closed under the following operations:*

1. *countable product;*
2. *if there is a nonprincipal ultrafilter on natural numbers, then also the modulo finite product.*

Proof. Let $\{F_n : n \in \omega\}$ be analytic equivalence relations on respective spaces X_n. Let $\leq_n$ be linear orderings of the respective quotient spaces X_n/F_n for every $n \in \omega$. Let $X = \prod_n X_n$, and write $E = \prod_n F_n$ and $F = \prod_n F_n$ modulo finite.

To prove (1), just let $\leq$ be the relation on X defined by $x \leq y$ if for the least number $n \in \omega$ such that $[x(n)]_{F_n} \neq [y(n)]_{F_n}$, it is the case that $[x(n)]_{F_n} \leq_n [y(n)]_{F_n}$. It is not difficult to see that the relation $\leq$ respects the E-equivalence classes and it induces a linear ordering on X/E. To prove (2), fix a nonprincipal ultrafilter U on ω and let $\leq$ be the relation on X defined by $x \leq y$ if the set $a_{x,y} \subset \omega$ is in U, where $m \in a_{x,y}$ if for the least number $n > m$ such that $[x(n)]_{F_n} \neq [y(n)]_{F_n}$, it is the case that $[x(n)]_{F_n} \leq_n [y(n)]_{F_n}$. It is not difficult to see that $\leq$ respects the F-classes and it induces a linear ordering on the quotient space X/F.

□

Corollary 8.4.6. *In the model $V(\mathbb{R})[U]$, the equivalence relations E_0, E_1, E_3 are linearly orderable.*

Proof. These equivalence relations are obtained from the linearly orderable id by the operations of modulo finite product, infinite product, and reducibility. □

Proving that an equivalence relation is not linearly orderable in $V(\mathbb{R})[U]$ is quite challenging. One way of doing it is again considering forcing names for elements of the underlying Polish space, as in the following definition.

Definition 8.4.7. Let E be an analytic equivalence relations on a Polish space X. Let P be a poset and τ, σ be P-names for elements of X. Call the names σ, τ *E-interchangeable* if for every condition $p \in P$, in some generic extensions there are filters $G_0, G_1 \subset P$ separately generic over the ground model, both containing p, such that $\tau/G_0 \; E \; \sigma/G_1$ and $\sigma/G_0 \; E \; \tau/G_1$. The pair σ, τ is *nontrivial* if $P \Vdash \neg\sigma \; E \; \tau$.

Thus, every name is interchangeable with itself and all names that are forced to be E-related to it. Of course, the nontrivial case is where all the interest lies. Interchangeability is clearly symmetric, but it does not appear to be an equivalence relation. A similar definition will be useful also for names for E-pinned names:

Definition 8.4.8. Let E be an analytic equivalence relations on a Polish space X. Let P be a poset and τ, σ be P-names for E-pinned names of elements of X on some posets. Call the names σ, τ *$\bar{E}$-interchangeable* if for every condition

$p \in P$, in some generic extensions there are filters $G_0, G_1 \subset P$ separately generic over the ground model, both containing p, such that $\tau/G_0\ \bar{E}\ \sigma/G_1$ and $\sigma/G_0\ \bar{E}\ \tau/G_1$. The pair σ, τ is *nontrivial* if $P \Vdash \neg\sigma\ \bar{E}\ \tau$.

Theorem 8.4.9. *Let κ be a strongly inaccessible cardinal. Let E be an analytic equivalence relation on a Polish space X. Suppose that V_κ satisfies*

(*) in every forcing extension, there is a Ramsey ultrafilter preserving poset with a nontrivial pair of E-interchangeable names or a nontrivial pair of $\bar{E}$-interchangeable names.

Then, in the model $V(\mathbb{R})[U]$, E is not linearly orderable.

Here, a poset P *preserves Ramsey ultrafilters* if for every Ramsey ultarfilter u, the upwards closure of u generates a Ramsey ultrafilter in the P-extension.

Proof. I will start with a couple of abstract facts. Let u be a Ramsey ultrafilter on ω. Let Q_u be the usual c.c.c. partial order associated with the ultrafilter u. That is, Q_u is the poset of all conditions $q = \langle t_q, a_q\rangle$ where $t_q \subset \omega$ is a finite set, $a_q \in u$, and $r \leq q$ if $t_q \subset t_r$, $a_r \subset a_q$, and $t_r \setminus t_q \subset a_q$. The poset Q_u adds a generic set $\dot{a}_{gen} \subset \omega$, the union of the first coordinates of conditions in the generic filter. The set $\dot{a}_{gen}$ *diagonalizes* u in the sense that it is modulo finite included in every set in u. The following is the main forcing property of Ramsey ultrafilters:

Fact 8.4.10. [22] *In every forcing extension, if $a \subset \omega$ is an infinite set diagonalizing u then a is Q_u-generic over V.*

This fact enables an abstract trick typically used to determine the theory of the model $V(\mathbb{R})[U]$. Let $V[g]$ be a generic extension of V via a poset of size $< \kappa$. Suppose that $V[g] \models u$ is a Ramsey ultrafilter. The following holds in $V[g]$, where $\dot{R}$ is the $Q \times \mathrm{Coll}(\omega, <\kappa)$ name for the poset $\mathcal{P}(\omega)$ modulo finite, and $\dot{U}$ is the $\dot{R}$-name for the generic ultrafilter on ω.

Claim 8.4.11. *Suppose that ϕ is a formula with parameters in $V[g]$ and one free variable. Then, in the three step iteration $Q_u * \mathrm{Coll}(\omega, <\kappa) * \dot{R}$, the condition $\langle 1, 1, \dot{a}_{gen}\rangle$ decides the formula $\phi(\dot{U})$.*

Proof. Suppose for contradiction that this fails and let $p = \langle p_0, p_1, p_2\rangle$ and $p' = \langle p'_0, p'_1, p'_2\rangle$ be conditions below $\langle 1, 1, \dot{a}_{gen}\rangle$ that decide the formula $\phi(\dot{U})$ differently. Let $h_0 \subset Q$ and $h_1 \subset \mathrm{Coll}(\omega, \kappa)$ be mutually generic filters over $V[g]$ such that $p_0 \in h_0$ and $p_1 \in h_1$. In the model $V[g][h_0][h_1]$, evaluate the set $p_2 \subset \omega$. Since $p \leq \langle 1, 1, \dot{a}_{gen}\rangle$, the set p_2 diagonalizes the Ramsey filter $u \in V[g]$. Thus, p_2 is a set Q_u-generic over $V[u]$ by Fact 8.4.10. Adjusting the set p_2 on finitely many positions if necessary, I may assume that the filter $h'_0 \subset Q_u$ generic over $V[u]$ defined by p_2 meets the condition p'_0. By 2.2.5, there is a filter $h'_1 \subset \mathrm{Coll}(\omega, <\kappa)$ generic over $V[g][h'_0]$ such that $V[g][h'_0][h'_1] = V[g][h_0][h_1]$; h'_1 may be chosen so as to contain the condition p'_1. Consider the set $p'_2 \subset \omega$. Since

$p' \leq \langle 1, 1, \dot{a}_{gen} \rangle$, $p'_2 \subset p_2$. By the forcing theorem applied to the filters $h_0 \times h_1$ and $h'_0 \times h'_1$. in the model $V[g][h_0][h_1]$, the conditions p'_2, p_2 are compatible elements of $\mathcal{P}(\omega)$ modulo finite which decide the statement $\phi(\dot{U})$ differently. This is a contradiction. $\square$

Now I am ready to address the specific issues raised by the theorem. Suppose for contradiction that E is linearly orderable in $V(\mathbb{R})[U]$. In the model $V(\mathbb{R})[U]$, every element is definable from a ground model parameter, a real, and the ultrafilter U, since $V(\mathbb{R})[U]$ is a generic extension of the model $V(\mathbb{R})$ in which every element is definable from some ground model parameter and a real number. Go back into the ground model. Suppose that some condition in the iteration $\mathrm{Coll}(\omega, < \kappa) * \mathcal{P}(\omega)$ modulo finite forces that ϕ is a formula which defines the linear ordering of the quotient space X/E. More specifically, ϕ is a formula with a ground model parameter, some real parameters, parameter U, and two free variables such that the relation $\{\langle x, y \rangle \in X \times X : \phi(x, y)\} \subset X \times X$ respects the equivalence relation E and induces a linear ordering on the quotient space. For the simplicity of notation assume that the condition identifying ϕ is the largest condition in the iteration. For the simplicity of notation assume that the ground model parameter in ϕ is 0. Passing to a generic extension by a poset of size $< \kappa$ if necessary, I may assume that the real parameter of ϕ is also in the ground model. For the simplicity of notation assume that the real parameter is 0 as well. Passing to a further generic extension if necessary, I may assume that the ground model contains a Ramsey ultrafilter u.

Use (*) above to find a u-preserving poset P and a nontrivial E-interchangeable pair of P-names σ, τ in the ground model (the case of $\bar{E}$-interchangeable names is nearly identical). Since P preserves the Ramseyness of the filter u, Claim 8.4.11 applied in the P-extension shows that there is some condition $p \in P$ such that in the four step iteration $P * Q_u * \mathrm{Coll}(\omega, < \kappa) * \dot{R}$, the condition $\langle p, 1, 1, \dot{a}_{gen} \rangle$ decides the formula $\phi(\sigma, \tau, \dot{U})$. Assume for definiteness that the decision is in the affirmative. In some generic extension $V[h]$ by a poset of size $< \kappa$, find filters $g_0, g_1 \subset P$ separately generic over V, both containing the condition p, such that $\sigma/g_0 \; E \; \tau/g_1$ and $\tau/g_0 \; E \; \sigma/g_1$. Let $k \subset \mathrm{Coll}(\omega, < \kappa)$ is a filter generic over $V[h]$, and in the model $V[h][k]$ find a set $a \subset \omega$ which diagonalizes the filter u. The following two formulas hold in the model $V[h][k]$, where R is the poset $\mathcal{P}(\omega)$ modulo finite:

- $a \Vdash_R \phi(\sigma/g_0, \tau/g_0, \dot{U})$. To see this, note that the set a is Q_u-generic over the model $V[g_0]$ since u still generates a Ramsey ultrafilter in that model and Fact 8.4.10 applies. Moreover, $V[h][k]$ is a $\mathrm{Coll}(\omega, < \kappa)$-extension of $V[g_0][a]$ by Fact 2.2.5. Now, $a \Vdash_R \phi(\sigma/g_0, \tau/g_0, \dot{U})$ follows from the choice of the condition p and the forcing theorem applied in the ground model.
- $a \Vdash_R \phi(\sigma/g_1, \tau/g_1, \dot{U})$. This is proved in the same way as the previous item, with g_1 replacing g_0.

However, the two items contradict each other in view of the fact that ϕ is forced to define a linear ordering on the quotient space X/E and $\sigma/g_0 \; E \; \tau/g_1$ and

$\tau/g_0 \; E \; \sigma/g_1$. ☐

Theorem 8.4.12. *In the model $V(\mathbb{R})[U]$, F_2 is not linearly orderable.*

Proof. In ZFC, I will produce a proper, Ramsey ultrafilter preserving poset P carrying a nontrivialpair of $\bar{F}_2$-interchangeable names. This will conclude the proof of the theorem in view of Theorem 8.4.9.

Let Q be the countable support product of ω_1 many Sacks posets. The poset Q adds an uncountable set $\{x_\alpha : \alpha \in \omega_1\} \subset 2^\omega$ of Sacks generic reals. I will identify this set with an F_2-pinned $\mathrm{Coll}(\omega, \omega_1)$-name for an element of X which enumerates all elements of this set. Let $P = Q_0 \times Q_1$ be the product of two copies of Q with their respective names σ_0, σ_1. Thus, a P-generic filter is given by a double sequence $\langle x^0_\alpha, x^1_\alpha : \alpha \in \omega_1\rangle$ of points in 2^ω.

Claim 8.4.13. *The names σ_0, σ_1 are $\bar{E}$-interchangeable.*

Proof. Let $p \in P$ be an arbitrary condition. Thus, $p = \langle q_0, q_1\rangle$ where $q_0, q_1 \in Q$, and strengthening p if necessary I may assume that $\mathrm{supp}(q_0) = \mathrm{supp}(q_1) = \alpha$ for some ordinal $\alpha \in \omega_1$. Consider the conditions $\bar{q}_0 \leq q_0, \bar{q}_1 \leq q_1 \in Q$ obtained in the following way. $\mathrm{supp}(\bar{q}_0) = \alpha + \alpha$, and for every $\beta \in \alpha$, $\bar{q}_0(\beta) = q_0(\beta)$ and $\bar{q}_0(\alpha + \beta) = q_1(\beta)$. Similarly, $\mathrm{supp}(\bar{q}_1) = \alpha + \alpha$, and for every $\beta \in \alpha$, $\bar{q}_1(\beta) = q_1(\beta)$ and $\bar{q}_1(\alpha + \beta) = q_0(\beta)$.

Now, let a double sequence $\vec{x} = \langle x^0_\alpha, x^1_\alpha : \alpha \in \omega_1\rangle$ of points in 2^ω be P-generic over V meeting the condition $\bar{p} = \langle \bar{q}_0, \bar{q}_1\rangle$. Consider the double sequence $\vec{y} = \langle y^0_\alpha, y^1_\alpha : \alpha \in \omega_1\rangle$ given by $y^0_\beta = x^1_{\alpha+\beta}$, $y^0_{\alpha+\beta} = x^1_\beta$ for all $\beta \in \alpha$ and $y^0_\gamma = x^1_\gamma$ for all $\gamma > \alpha + \alpha$, and similarly $y^1_\beta = x^0_{\alpha+\beta}$, $y^1_{\alpha+\beta} = x^0_\beta$ for all $\beta \in \alpha$ and $y^1_\gamma = x^0_\gamma$ for all countable ordinals $\gamma > \alpha + \alpha$. The double sequence $\vec{y}$ is obtained by a ground model permutation of $\vec{x}$ and so is P-generic over V. At the same time, the double sequence $\vec{y}$ also meets the condition p, and $\{x^0_\alpha : \alpha \in \omega_1\} = \{y^1_\alpha : \alpha \in \omega_1\}$ and $\{x^1_\alpha : \alpha \in \omega_1\} = \{y^0_\alpha : \alpha \in \omega_1\}$. This confirms the statement of the claim. ☐

Claim 8.4.14. *The poset P preserves Ramsey ultrafilters.*

Proof. This is fairly well-known on a folklore level. The first, weaker result in this direction was provided by Laver [17], who showed that under Martin's Axiom there is a Ramsey ultrafilter preserved by P. This would be strong enough for the purpose of the present proof. I will provide the argument for the full statement with references to all major steps.

The poset P is a countable support product of $\aleph_1$ many copies of the Sacks forcing. Since every subset of ω in the P-extension already belongs to an extension given by countably many Sacks reals, it is enough to show that the countable support product of countably many Sacks reals preserves Ramsey ultrafilters.

The σ-ideal associated with the countable support product of countably many Sacks reals is computed via [26, Theorem 5.2.6]; the only important part of the conclusion is that the σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The product is proper, bounding, and does not add independent reals by infinite-dimensional Halpern–Läuchli theorem as proved in [17]. Posets with these properties preserve Ramsey ultrafilters by [26, Theorem 3.4.1]. □

□

Theorem 8.4.15. *In the model $V(\mathbb{R})[U]$, E_2 is not linearly orderable.*

Proof. In ZFC, I will produce a proper poset P preserving Ramsey ultrafilters and a nontrivial pair of E_2-interchangeable P-names σ, τ. This will conclude the proof of the theorem in view of Theorem 8.4.9.

Let $\omega = \bigcup_n I_n$ be a partition of ω into successive intervals. Write $X_n = 2^{I_n}$ for every $n \in \omega$ and let $X = \prod_n X_n$; the space X is naturally identified with 2^ω via the bijection $\pi : x \mapsto \bigcup x$ from X to 2^ω. Let d_n be the metric on X_n given by $d_n(u, v) = \{\frac{1}{m+1} : u(m) \neq v(m)\}$. Let μ_n be the normalized counting measure on X_n multiplied by $n+1$. The concentration of measure computations as in [21, Theorem 4.3.19] show that the sequence $\langle I_n : n \in \omega\rangle$ can be chosen in such a way that for every $n > 0$ and every $a, b \subset X_n$ of μ_n-mass at least 1 there are binary strings $u \in a$ and $v \in b$ such that $d_n(u, v) \leq 2^{-n}$.

Let T_{ini} be the tree of all finite sequences t such that for all $n \in \text{dom}(t)$, $t(n) \in X_n$. Finally, let P be the poset all all trees $T \subset T_{\text{ini}}$ such that the numbers $\{\mu_{|s|}(\{u \in X_{|s|} : s^\frown u \in T\}) : s \in T\}$ converge to ∞. The ordering is that of inclusion.

The forcing P is of the fat tree kind studied for example in [2, Section 7.3.B] or [26, Section 4.4.3]. It adds a generic point $\dot{x}_{gen} \in X$ which is the unique element of X which is a branch through all trees in the generic filter. Let $\sigma = \pi(\dot{x}_{gen}) \in 2^\omega$ and $\tau = 1 - \sigma \in 2^\omega$. The following two claims complete the proof.

Claim 8.4.16. *The names σ, τ for a nontrivial E_2-interchangeable pair.*

Proof. As for every number n, $\sigma(n) = 1 - \tau(n)$, the names are certainly forced to be E_2-inequivalent. The interchangeability uses the concentration of measure assumptions.

Let $T \in P$ be an arbitrary condition. Let $V[H]$ be a forcing extension in which $\mathcal{P}(\mathcal{P}(\omega))^V$ is a countable set. The usual fusion arguments for the forcing P as in [2, Section 7.3.B] show that in $V[H]$, there is a tree $S \subset T$ in $P^{V[H]}$ such that all its branches yield P-generic filters over the ground model. Let $s_0 \in S$ be a node such that all nodes of S extending s_0 have the set of immediate successors in S of submeasure at least 1. For simplicity of notation assme that $s_0 = 0$. By induction on $n \in \omega$ build nodes $s_n, t_n \in S$ so that

- $t_0 = s_0 = 0$, t_{n+1} is an immediate successor of t_n and s_{n+1} is an immediate successor of s_n;
- writing $u_n, v_n \in X_n$ for the binary strings such that $s_n^\frown u_n = s_{n+1}$ and $t_n^\frown v_n = t_{n+1}$, it is the case that $d_n(u_n, 1 - v_n) \leq 2^{-n}$.

Once this is done, let $x = \bigcup_n s_n$ and $y = \bigcup_n t_n$. These are branches through the tree S, therefore P-generic over the ground model. The second item immediately implies that $\sigma/x\ E_2\ \tau/y$ and $\tau/x =E_2\ \sigma/y$ as desired.

The induction step of the construction above is obtained as follows. Suppose that $t_n, s_n \in S$ have been found. Let $a = \{u \in X_n : s_n^u \in S\}$ and $b = \{v \in X_n : t_n^\frown(1-v) \in S\}$. Then, $\mu_n(a)$, $\mu(b)$ are both numbers greater than 1, and therefore there are $u \in a$ and $v \in b$ such that $d_n(u,v) \leq 2^{-n}$. Setting $s_{n+1} = s_n^\frown u$ and $t_{n+1}^\frown(1-v)$ completes the induction step.

□

Claim 8.4.17. *The poset P preserves Ramsey ultrafilters.*

Proof. The forcing properties of posets similar to P are investigated in [26, Section 4.4.3]. [26, Theorem 4.4.8] shows that P is proper, bounding, and does not add independent reals. The associated σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by [26, Theorem 3.8.9]. Posets with these properties preserve Ramsey ultrafilters by [26, Theorem 3.4.1]. □

□

For most of the equivalence relations considered in this book, I do not know if they are linearly orderable in the model $V(\mathbb{R})[U]$ or not. The following is just a sample of my current ignorance:

Question 8.4.18. Is $=_J$ for the branch ideal J on $2^{<\omega}$ linearly orderable in $V(\mathbb{R})[U]$?

Question 8.4.19. Let $\alpha \in \omega_1$ be a nonzero ordinal and let F_α be the Borel equivalence relation with $\kappa(F_\alpha) = \aleph_\alpha$ produced in Corollary 4.4.8. Is F_α linearly orderable in $V(\mathbb{R})[U]$?

Bibliography

[1] Bohuslav Balcar, Thomas Jech, and Jindřich Zapletal. Semi-cohen boolean algebras. *Annals of Pure and Applied Logic*, 87:187–208, 1997.

[2] Tomek Bartoszynski and Haim Judah. *Set Theory. On the structure of the real line.* A K Peters, Wellesley, MA, 1995.

[3] Howard Becker and Alexander Kechris. *The descriptive set theory of Polish group actions.* London Mathematical Society Lecture Notes Series 232. Cambridge University Press, London, 1996.

[4] Ilijas Farah. Ideals induced by Tsirelson submeasures. *Fundamenta Mathematicae*, 159:243–258, 1999.

[5] Matthew Foreman and Menachem Magidor. Large cardinals and definable counterexamples to the continuum hypothesis. *Ann. Pure Appl. Logic*, 76:4797, 1995.

[6] Su Gao. *Invariant Descriptive Set Theory.* CRC Press, Boca Raton, 2009.

[7] Su Gao and Alexander S. Kechris. On the classification of Polish metric spaces up to isometry. *Mem. Amer. Math. Soc*, 161:78, 2000.

[8] András Hajnal and A. Máté. Set mappings, partitions, and chromatic numbers. In *Logic Colloquium 1973*, pages 347–379. North Holland, Amsterdam, 1975.

[9] Jaime Ihoda (Haim Judah) and Saharon Shelah. Souslin forcing. *The Journal of Symbolic Logic*, 53:1188–1207, 1988.

[10] Thomas Jech. *Set Theory.* Springer Verlag, New York, 2002.

[11] Vladimir Kanovei. Ulm classification of analytic equivalence relations in generic universes. *Mathematical Logic Quarterly*, 44:287–303, 1998.

[12] Vladimir Kanovei. *Borel Equivalence Relations.* University Lecture Series 44. American Mathematical Society, Providence, RI, 2008.

[13] Alexander Kechris. *Actions of Polish groups and classification problems*, pages 115–187. London Mathematical Society Lecture Note Series 262. Cambridge University Press, Cambridge, 2003.

[14] Alexander S. Kechris. *Classical Descriptive Set Theory.* Springer Verlag, New York, 1994.

[15] Peter Komjath and Saharon Shelah. Coloring finite subsets of uncountable sets. *Proceedings of the American Mathematical Society*, 124:3501–3505, 1996. math.LO/9505216.

[16] Paul B. Larson. *The stationary tower forcing.* University Lecture Series 32. American Mathematical Society, Providence, RI, 2004. Notes from Woodin's lectures.

[17] Richard Laver. Products of infinitely many perfect trees. *Journal of London Mathematical Society*, 29:385–396, 1984.

[18] Dominique Lecomte. Potential Wadge classes. *Mem. Amer. Math. Soc.*, 221, 2013.

[19] Menachem Magidor. On the singular cardinals problem. I. *Israel Journal of Mathematics*, 28:1–31, 1977.

[20] Krzysztof Mazur. F_σ-ideals and $\omega_1\omega_1*$ gaps in the Boolean algebra $P(\omega)$/I. *Fundamenta Mathematicae*, 138:103–111, 1991.

[21] Vladimir Pestov. *Dynamics of Infinite-Dimensional Groups.* University Lecture Series 40. Amer. Math. Society, Providence, 2006.

[22] Saharon Shelah and Otmar Spinas. The distributivity numbers of $p(\omega)/\mathrm{fin}$ and its square. *Transactions of the American Mathematical Society*, 352:2023–2047, 2000. math.LO/9606227.

[23] Slawomir Solecki. Analytic ideals and their applications. *Annals of Pure and Applied Logic*, 99:51–72, 1999.

[24] Jacques Stern. On Lusin's restricted continuum problem. *Annals of Mathematics*, 120:7–37, 1984.

[25] Stevo Todorcevic. Remarks on Martin's Axiom and the Continuum Hypothesis. *Canadian Journal of Mathematics*, 43:832–851, 1991.

[26] Jindřich Zapletal. *Forcing Idealized.* Cambridge Tracts in Mathematics 174. Cambridge University Press, Cambridge, 2008.

[27] Jindřich Zapletal. Pinned equivalence relations. *Mathematical Research Letters*, 18:1149–1156, 2011.

Index