TERMINAL NOTIONS IN SET THEORY

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Abstract. Certain set theoretical notions cannot be split into finer subnotions.

0. Introduction

In this paper I want to show that certain set theoretic notions are *terminal* in the sense that formulas commonly used with them cannot split the notions into finer subnotions, or restated, that all representatives of these notions share the same natural properties. This will be accomplished in two ways. First I will prove that in the presence of large cardinals formulas of limited complexity cannot distinguish different instances. Second, I will produce a model of ZFC+$\diamondsuit$ in which different instances are completely indiscernible, without any restriction on the complexity of the formulas used. The model is the same for all notions considered and below I call it the *homogeneous model*.

0.1. Ramsey ultrafilters. An ultrafilter F on natural numbers is called *Ramsey* if for every partition of pairs of natural numbers into two classes there is a set in F homogeneous for that partition. I will show that Ramsey ultrafilters share the same natural properties. To do this I first identify three syntactically defined classes of formulas.

A formula $\phi(\vec{x})$ will be called an *ultrafilter formula* if its free variables are reserved for ultrafilters and the validity of $\phi(\vec{x})$ depends only on the Rudin-Keisler equivalence classes of the ultrafilters on the sequence $\vec{x}$, where ultrafilters F, G are Rudin-Keisler equivalent if there is a permutation $\pi : \omega \to \omega$ such that $F = \{\pi'' X : X \in G\}$. A formula $\phi(\vec{x})$ is projective if its quantifiers range over reals only. Here the free variables are reserved for sets of reals, and ultrafilters are treated as sets of reals. And finally a formula $\phi(\vec{x})$ is Σ^2_1 if it is of the form $\exists Y \subset \mathbb{R}\, \psi(\vec{x}, Y)$ where ψ is projective. Most properties of ultrafilters relevant to set theory and topology are expressed by ultrafilter formulas. For example $\phi(\vec{x}) =$"$\vec{x}$ is a sequence of pairwise nonequivalent Ramsey ultrafilters" is a projective ultrafilter formula, $\phi(x) =$"x is a $P_{\aleph_1}$ ultrafilter" is a Σ^2_1 ultrafilter formula and $\phi(x) =$"there is a forcing extension in which x generates a Ramsey ultrafilter and all other Ramsey ultrafilters are equivalent to this one" is an ultrafilter formula which is neither projective nor Σ^2_1.

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0.1. Theorem.

(1) *Suppose large cardinals exist and $\phi(\vec{x})$ is a projective ultrafilter formula. Then ϕ holds for none or for all sequences of pairwise nonequivalent Ramsey ultrafilters of the appropriate length.*
(2) *Suppose large cardinals exist, the Continuum Hypothesis holds and $\phi(\vec{x})$ is a Σ^2_1 ultrafilter formula. Then ϕ holds for none or for all sequences of pairwise nonequivalent Ramsey ultrafilters of the appropriate length.*
(3) *Suppose $\phi(\vec{x})$ is an arbitrary ultrafilter formula. In the homogeneous model, ϕ holds for none or for all sequences of pairwise nonequivalent Ramsey ultrafilters of the appropriate length.*

In fact, I will prove much stronger statements than (1) and (2), showing that relevant properties of Ramsey ultrafilters are the same in all set generic extensions. If large cardinals exist and $\phi(\vec{x})$ is a projective ultrafilter formula that holds in some generic extension of some sequence of pairwise nonequivalent Ramsey ultrafilters of the appropriate length, then it holds in all generic extensions of all sequences of pairwise nonequivalent Ramsey ultrafilters of the appropriate length. And if large cardinals exist and $\phi(\vec{x})$ is a Σ^2_1 ultrafilter formula that holds in some generic extension of some sequence of pairwise nonequivalent Ramsey ultrafilters of the appropriate length, then it holds in all generic extensions satisfying CH, of all sequences of pairwise nonequivalent Ramsey ultrafilters of the appropriate length.

(1) and (2) above are variations on old results of Woodin [W1]. A strong version of (1) has been known to Todorcevic and Di Prisco [Fa]. A model in which all Ramsey ultrafilters satisfy the same projective ultrafilter formulas was produced from a Mahlo cardinal by [La].

It is consistent with large cardinals and not CH that there be two Ramsey ultrafilters, one of them is a $P_{\aleph_1}$ point and the other is not [Lo], so the assumption of CH in (2) is necessary. It is also consistent with CH and large cardinals that two Ramsey ultrafilters differ on a Δ^2_2 ultrafilter formula [AS] and so (2) cannot be much improved. However, it is quite possible that under a quotable hypothesis such as $\diamondsuit$ Ramsey ultrafilters cannot be distinguished by an ultrafilter formula over the model $\langle H_{\aleph_2}, \in\rangle$.

0.2. Souslin trees. Call an ω_1-tree T a *free tree* [AS, SZ] if for every finite sequence $\vec{t}$ of pairwise incomparable elements of T the product tree $\prod_{n<lth(\vec{t})} T \restriction \vec{t}(n)$ has no uncountable antichain. So this is a certain strengthening of the notion of a Souslin tree; while not all Souslin trees are free, it is not known whether an existence of a Souslin tree implies the existence of a free tree. Call trees T, S *equivalent* if the complete boolean algebras they determine are isomorphic and call $\phi(x)$ a forcing formula if its validity depends only on the isomorphism type of the boolean algebra determined by the poset x. The projective and Σ^2_1 formulas in this case gain meaning if we consider the trees coded as sets of reals. Thus for example $\phi(x) =$"x is not a free tree" is a Σ^2_1 forcing formula.

0.2. Theorem. *Theorem 0.1 holds with ultrafilter formulas replaced by forcing formulas of one free variable and sequences of ultrafilters replaced with single free trees.*

It is impossible in this case to pass to finite sequences of free trees. For example under $\Diamond$ one can easily produce pairs T_0, T_1 and S_0, S_1 of distinct free trees which differ on the Σ^2_1 formula $\phi(x, y) =$"$x \times y$ is not a Souslin tree". It seems difficult to concoct a natural Σ^2_1 forcing formula on which two free trees could disagree in the presence of large cardinals (and failure of the continuum hypothesis). The following formula is a mildly plausible and at least remotely natural candidate: $\phi(x) = x \Vdash$ there is a maximal almost disjoint family of size $\aleph_1$.

0.3. Lusin sets. A version of Theorem 0.1 for Lusin sets fails. There are simply several topologically distinguishable kinds of Lusin sets, so if we want a terminal notion, it is necessary to strengthen the notion of a Lusin set appropriately. Call a set $S \subset \mathbb{R}$ *strong Lusin* if for every number n and every meager set $M \subset \mathbb{R}^n$ there is a countable set $S' \subset S$ such that no n-tuple of distinct elements of the set $S \setminus S'$ belongs to M. Moreover require that S has uncountable intersection with every open set. This is a strengthening of the notion of a Lusin set considered previously [T, Chapter 6]. It is not known whether an existence of a Lusin set implies the existence of a strong Lusin set.

Strong Lusin sets are already hard to distinguish from each other, however I still need to rule out one kind of them. Call a set $S \subset \mathbb{R}$ *extendible strong Lusin* if it is strong Lusin and for no countable collection of meager finitary relations on $\mathbb{R}$ does S contain a maximal free set with respect to these relations. This may sound confusing at first, but in the presence of CH it is simply equivalent to saying that the set S can be extended to a larger strong Lusin set by adding an uncountable number of reals–hence the term. I do not know whether there can be a nonextendible strong Lusin set.

Call two extendible strong Lusin sets *equivalent* if their symmetric difference is countable and call a formula $\phi(x)$ a *Lusin formula* if it depends only on the equivalence class of the set x.

0.3. Theorem. *Theorem 0.1 holds with Lusin formulas of one free variable in place of ultrafilter formulas and single extendible strong Lusin sets in place of sequences of Ramsey ultrafilters.*

It is not hard to reformulate $\phi(x) =$"x is an extendible strong Lusin set" as a projective Lusin property of x. Again, one cannot strengthen Theorem 0.3 to sequences of extendible strong Lusin sets, since under CH one can easily produce pairs S_0, S_1 and T_0, T_1 of extendible strong Lusin sets such that $S_0 \cup S_1$ is strong Lusin while $T_0 \cup T_1$ is not.

0.4. Diamond sequences. Call a sequence $\langle r_\alpha : \alpha \in \omega_1 \rangle$ of reals *a good diamond sequence* if for every Borel relation $B \subset \mathbb{R}^{\aleph_0} \times \mathbb{R}$, every sequence $\langle s_\alpha : \alpha \in \omega_1 \rangle$ of reals and every closed unbounded set $C \subset \omega_1$ there is an ordinal $\alpha \in C$ such that either $\langle \{s_\beta : \beta \in \alpha\}, r_\alpha \rangle \in B$ or else for no real r $\langle \{s_\beta : \beta \in \alpha\}, r \rangle \in B$ holds. This is just an innocent reformulation of the good old diamond principle that makes it possible to formulate the absoluteness theorems in a succint and correct way. Call two diamond sequences equivalent if the are equal on a closed unbounded set and call $\phi(x)$ a *diamond formula* if its validity depends only on the equivalence class of the diamond sequence x.

0.4. Theorem. *Theorem 0.1 holds with single good diamond sequences in place of sequences of Ramsey ultrafilters and diamond formulas in place of ultrafilter formulas.*

0.5. Concluding remarks. I have not tried to minimalize the large cardinal hypotheses needed for the proofs. For (1) of the theorems generally a class of Woodin cardinals is enough, and for (2) the assumption of a class of measurable Woodin cardinals is sufficient. The homogeneous model can be built from a variety of large cardinal or determinacy assumptions. For the results in Theorems 0.1-0.3(3) a weakly compact cardinal suffices. If one wants more homogeneity and simpler forcing, one needs to use stronger initial assumptions, which is the road we take. I do not know how to obtain Theorem 0.4(3) for diamond sequences without the use of a huge cardinal. It is not clear whether any of the possible constructions of the homogeneous model has a claim to canonicity.

This paper does not present all terminal notions that are known at this point. In fact it seems that most notions whose representatives can be coded as sets of reals have a strengthening that is in some interesting sense terminal. Let me just mention two somewhat amusing combinations. Suppose $\phi(\vec{F}, T, S)$ is a formula whose validity depends only on the equivalence class of ultrafilters on the sequence $\vec{F}$, equivalence class of the ω_1-tree T and the modulo nonstationary ideal equivalence class of the set $S \subset \omega_1$. Then in the homogeneous model we build in Section 4 the statement $\phi(\vec{F}, T, S)$ holds the same for all choices of a sequence of pairwise nonequivalent Ramsey ultrafilters, a free tree and a stationary costationary set. This loosely translates into the old dictum: ultrafilters, Souslin trees and stationary sets have nothing to do with each other. Another phenomenon is the following. Suppose ϕ is a sentence which quantifies only over reals, Ramsey ultrafilters and sets of equivalence classes of Ramsey ultrafilters. Then, granted large cardinals, every two set generic extensions with infinitely many equivalence classes of Ramsey ultrafilters agree on the sentence ϕ.

The notation in the paper follows the set theoretical standard as closely as possible. If $\vec{x}, \vec{y}$ are finite sequences and ϕ is a formula then the expression "$\phi(\vec{x}, \vec{y})$ coordinatewise" means "$\vec{x}$ and $\vec{y}$ have the same length, say n, and for every $i \in n$ $\phi(\vec{x}(i), \vec{y}(i))$ holds". If x is a set then $Coll(x)$ is the poset of finite functions from ω to x ordered by reverse inclusion. $Add(1, \aleph_1)$ is the poset of all countable functions from ω_1 to 2 ordered by reverse inclusion. The phrase "there is an external object such that ... " stands for "In some generic extension there is an object such that ... " or "for a large enough ordinal α, $Coll(\alpha) \Vdash$ there is an object such that ... " If P is a forcing τ a P-name and $G \subset P$ a filter then τ/G is the valuation of the name τ with respect to the filter G–see [J3]. Generally a poset and the complete Boolean algebra it determines are freely confused. If P is a poset and $g \subset P$ then $\bigwedge g$ is the greatest lower bound of the set g in the complete Boolean algebra of P. If Q is a regular subordering of P and $H \subset Q$ is a filter then $P/Q(H)$ is the residue poset $\{p \in P :$ the projection of p into the poset Q belongs to the filter $H\}$. If several models $V, V[G], M \dots$ of set theory are floating around, their respective sets of reals are denoted by $\mathbb{R} \cap V, \mathbb{R} \cap V[G], \mathbb{R} \cap M$ and their respective levels of the cumulative hierarchy by $V_\alpha \cap V, V_\alpha \cap V[G] \dots$ If $\langle I, < \rangle$ is a linearly ordered set, $F \subset \mathcal{P}(I)$ is a filter and $x \subset I$ is a set the phrase "x diagonalizes F" stands for

$\forall y \in F \exists i \in x\ \{j \in x : i \leq j\} \subset y$. ω_1-trees grow downwards and are treated as sets of functions from countable ordinals to ω ordered by reverse inclusion. For a tree T the expression T_α stands for the α-th level of T.

1. Symmetric Extensions

In this section I will fix some notation and terminology concerning choiceless extensions of models of set theory. Most of the results were no doubt known before, but I could not find a suitable reference.

Let P be a forcing, τ a P-name and X an external transitive set. I wish to characterize in terms of the theory of the model $V(X)$ exactly when there can be an external V-generic filter $G \subset P$ such that $\tau/G = X$. Work in $V(X)$.

Let $Y \in V$ be the set of all names mentioned in τ, that is $Y = \{\sigma : \exists p \in P\ \langle p, \sigma\rangle \in \tau\}$. The exact formula for Y may vary according to which manifestation of forcing names one is working with, the idea is that $P \Vdash \check{\tau}/\dot{G} \subset \{\sigma/\dot{G} : \sigma \in \check{Y}\}$. By transfinite induction on the ordinal α build partial orders $Q_\alpha \in V(X)$ as follows.

- (1.a) Q_0 is the set of all pairs $\langle p, g\rangle$ such that $p \in P$ and g is a function such that $\mathrm{dom}(g) \subset X$ is finite, $\mathrm{rng}(g)$ consists of finite subsets of Y and for every two sets $x_0, x_1 \in X$ and every two names $\sigma_0 \in g(x_0), \sigma_1 \in g(x_1)$ the following four conditions are satisfied: $x_0 = x_1 \leftrightarrow p \Vdash \sigma_0 = \sigma_1$, $\neg x_0 = x_1 \leftrightarrow p \Vdash \neg x_0 = x_1$, $x_0 \in x_1 \leftrightarrow p \Vdash \sigma_0 \in \sigma_1$ and $\neg x_0 \in x_1 \leftrightarrow p \Vdash \neg\sigma_0 \in \sigma_1$. The ordering is defined by $\langle p, g\rangle \geq \langle q, h\rangle$ if $p \geq q$ in the poset P, $\mathrm{dom}(g) \subset \mathrm{dom}(h)$ and $\forall x \in \mathrm{dom}(g)\ g(x) \subset h(x)$.
- (1.b) $Q_{\alpha+1}$ is the set of those conditions $\langle p, g\rangle \in Q_\alpha$ for which: for every open dense set $D \subset P$ in V, every finite set $a \subset X$ and every finite set $b \subset Y$ there is a condition $\langle q, h\rangle \leq \langle p, g\rangle$ in Q_α such that $q \in D, a \subset \mathrm{dom}(h)$ and for every name $\sigma \in b$ either $q \Vdash \neg\sigma \in \tau$ or else $\sigma \in \bigcup \mathrm{rng}(h)$. The ordering is inherited from Q_0.
- (1.c) $Q_\alpha = \bigcap_{\beta\in\alpha} Q_\beta$ for limit ordinals α.

Since the posets Q_α are inclusion decreasing, the construction has to stabilize at some point. Let Q denote this stable value. In general, the poset Q can be empty and very frequently it fails to be separative.

1.1. Definition. Suppose P, τ, X are a poset, a P-name and an external transitive set respectively. We say that $V(X)$ is a *τ-extension* of V if the stabilizing value Q as above is not empty. If X is not a transitive set we note $V(X) = V(\mathrm{trcl}(X))$ and say that $V(X)$ is a τ-extension of V if $V(\mathrm{trcl}(X))$ is a $\mathrm{trcl}(\tau)$-extension of V.

It is important to observe that $Q \neq 0$ is purely a fact of the theory of the model $V(X)$ with parameters P, τ, X and $\mathcal{P}(P) \cap V$. The following theorem shows that the terminology chosen is sound.

1.2. Theorem. *Suppose P, τ, X are a poset, a P-name and an external set respectively. $V(X)$ is a τ-extension of V if and only if there is an external V-generic filter $G \subset P$ such that $\tau/G = X$.*

Proof. A straightforward check. Without loss of generality the set X can be assumed transitive. For the left to right direction, suppose the construction of posets Q_α stabilized at some ordinal β so that $Q = Q_\beta = Q_{\beta+1} \neq 0$. Choose an external

$V(X)$-generic filter $H \subset Q$ and let $G = \{p \in P : \langle p, 0\rangle \in H\}$. The following can be routinely checked from the fact that $Q_\beta = Q_{\beta+1}$:

(1.d) $G \subset P$ is a V-generic filter
(1.e) $\tau/G = X$
(1.f) $H = \{\langle p, g\rangle \in Q : p \in G$ and $\forall x \in \mathrm{dom}(g) \forall\sigma \in g(x)\ x = \sigma/G\}$ so $G \in V(X)[H]$ and $X, H \in V[G]$ and $V[G] = V(X)[H]$.

For the right to left direction suppose $G \subset P$ is an external V-generic filter such that $\tau/G = X$. Then the set $H = \{\langle p, g\rangle \in Q_0 : p \in G$ and $\forall x \in \mathrm{dom}(g)\forall\sigma \in g(x)\ x = \sigma/G\}$ is included in all the sets Q_α as verified by induction on α. Thus the stabilizing value Q must be nonempty. In fact, one can invoke forcing theorem and the first paragraph of this proof to show that $H \subset Q$ is a $V(X)$-generic filter. □

For reasons obvious from the previous proof I will denote the stabilizing value Q by $P/\tau(X)$.

1.3. Definition. A P-name τ is called *homogeneous* if for all conditions $p_0, p_1 \in P$ there are strengthenings $q_0 \leq p_0$ and $q_1 \leq p_1$ in $RO(P)$ and an isomorphism $\pi : RO(P) \restriction q_0 \to RO(P) \restriction q_1$ such that $q_1 \Vdash \pi(\tau) = \tau$. See [J2] for the natural definition of extension of the isomorphism π to the space of all P-names.

Obviously if ϕ is a formula, $x_0 \dots x_n$ are sets and τ is a homogeneous P-name then all conditions in P decide the value of $\phi(\check{x}_0 \dots \check{x}_n, \tau)$ in the same way and in fact this is the only fact about homogeneous names that I will use. If $P \Vdash V(\tau) \models \phi(\check{x}_0 \dots \check{x}_n, \tau)$ I will frequently write $\tau \Vdash \phi(\check{x}_0 \dots \check{x}_n, \tau)$ and by Theorem 1.2 this notation will have the expected meaning: in all τ-extensions $V(X)$ the formula $\phi(x_0 \dots x_n, X)$ holds.

1.4. Theorem.

(1) *Suppose τ is a homogeneous P-name and $V(X)$ is a τ-extension of V. Then for any condition $p \in P$ there is an external V-generic filter $G \subset P$ containing p such that $\tau/G = X$.*
(2) *Suppose P, Q are posets, τ, σ are P, Q names respectively, τ is homogeneous and $Q \Vdash V(\sigma)$ is a $\check{\tau}$-extension of V. Then for any conditions $p \in P, q \in Q$ we have: for every external V-generic filter $G \subset P$ containing p there is an external V-generic filter $H \subset Q$ containing q such that $\tau/G = \sigma/H$ and vice versa, for every external V-generic filter $H \subset Q$ containing q there is an external V-generic filter $G \subset P$ containing p such that $\tau/G = \sigma/H$.*

Proof. First show that if τ is a homogeneous P-name then $\tau \Vdash \forall p \in P\ \langle p, 0\rangle \in P/\check{\tau}(\tau)$. If this were not the case then by homogeneity there would be some fixed condition $p \in P$ such that $\tau \Vdash \neg\langle\check{p}, 0\rangle \in P/\tau$. Then choose an external V-generic filter $G \subset P$ containing p and note that writing $X = \tau/G$ we have $V(X) \models \langle p, 0\rangle \in P/\tau(X)$ as in the proof of Theorem 1.2. Contradiction.

Now (1) follows. Just force the filter $G \subset P$ below the condition $\langle p, 0\rangle \in P/\tau(X)$ in $V(X)$. (2) is very similar. Argue by homogeneity of the name τ that $\tau \Vdash \forall q \in Q\ \langle\check{q}, 0\rangle \in Q/\check{\sigma}(\tau)$ and then force the desired filters G, H as in (1). □

Let me now give a few examples of homogeneous τ-extensions.

1.5. Example. Suppose κ is an inaccessible cardinal, $P = Coll(\omega, <\kappa)$ and denote by κ_{sym} the P-name for all reals of the extension. Then κ_{sym} is a homogeneous name and whenever $V(\mathbb{R}^*)$ is an extension such that $\mathbb{R}^* = \mathbb{R} \cap V(\mathbb{R}^*)$ and $V(\mathbb{R}^*) \models \kappa = \aleph_1$ and every real is generic over V using a poset of size $< \kappa$ then $V(\mathbb{R}^*)$ is a κ_{sym}-extension of V. This is most easily proved by directly exhibiting the remainder forcing $P/\kappa_{sym}(\mathbb{R}^*)$. Work in $V(\mathbb{R}^*)$ and let $Q = \{g :$ for some ordinal $\alpha \in \kappa$ $g \subset Coll(\omega, <\alpha)$ is a V-generic filter$\}$ ordered by reverse inclusion. Suppose $H \subset Q$ is a $V(\mathbb{R}^*)$-generic filter. Elementary density arguments show that the filter $G = \bigcup H \subset P$ is a V-generic filter and $\mathbb{R}^* \subset V[G] \cap \mathbb{R}$. Moreover, $V[G] \cap \mathbb{R} \subset \mathbb{R}^*$ since by the κ-c.c. of the poset P in V every real $r \in V[G]$ belongs to the model $V[G \cap Coll(\omega, <\alpha)]$ for some ordinal $\alpha \in \kappa$; but $G \cap Coll(\omega, <\alpha) \in V(\mathbb{R}^*)$ and so $r \in \mathbb{R}^*$. It follows that $\mathbb{R}^* = V[G] \cap \mathbb{R}$ and by Theorem 1.2 $V(\mathbb{R}^*)$ is a κ_{sym}-extension of V. While it is not literally true that $P/\kappa_{sym}(\mathbb{R}^*) = Q$ it can be proved that the two posets' completions are isomorphic. It should be noted that if $V \models$AC then $V \models \kappa_{sym} \Vdash$DC. I will frequently use the following well known fact.

1.6. Claim. *If $V(\mathbb{R}^*)$ is a κ_{sym}-extension of V and $x \in V(\mathbb{R}^*)$ is a bounded subset of κ then $V[x](\mathbb{R}^*)$ is a κ_{sym}-extension of $V[x]$.*

1.7. Example. Suppose T is a free tree, P is the finite support product of ω many copies of T and let T_{sym} be the P-name for the set of the infinitely many branches of T added. Then T_{sym} is a homogeneous name and whenever b is an external set of cofinal branches of the tree T such that $\bigcup b = T$ then $V(b)$ is a T_{sym}-extension of V. For consider the forcing Q consisting of all finite injections from ω into the set b ordered by reverse inclusion. Then $Q \in V(b)$ and if $H \subset Q$ is a $V(b)$-generic filter then $\bigcup H : \omega \to b$ is a bijection which naturally induces a V-generic filter $G \subset P$ such that $T_{sym}/G = b$. To verify the genericity of the filter $G \subset P$ suppose $f \in Q$ and $D \subset P$ is an open dense subset in V. I will find conditions $g \in Q$ and $p \in D$ such that $f \subset g$, $\mathrm{dom}(p) \subset \mathrm{dom}(g)$ and $\forall n \in \mathrm{dom}(p)\ p(n) \in g(n)$, so $g \Vdash_Q \check{p} \in \dot{G} \cap \check{D}$. Note that conditions in P are finite functions from ω to the tree T. By a genericity argument applied in the model $V(b)$ to the forcing Q, the filter G must then be V-generic. To get the conditions g, p note that by the freeness of the tree T the branches in $\mathrm{rng}(f)$ are mutually V-generic, so there must be a condition $p \in D$ with $\forall n \in \mathrm{dom}(f) \cap \mathrm{dom}(p)\ p(n) \in f(n)$. Since $\bigcup b = T$ it is possible to find an injection $g \in Q$ with $f \subset g$, $\mathrm{dom}(p) \subset \mathrm{dom}(g)$ and $\forall n \in \mathrm{dom}(p)\ p(n) \in g(n)$, as desired.

It should be noted that the model $V(b)$ fails to satisfy the axiom of dependent choice–the set b is Dedekind finite in that model. Also, b contains all the cofinal branches of the tree T in the model $V(b)$ and $\mathbb{R} \cap V = \mathbb{R} \cap V(b)$.

1.8. Example. Let P be the product of infinitely many copies of the Cohen forcing $2^{<\omega}$ with finite support and let c_{sym} be the name for the infinitely many Cohen reals added. By roughly the same argument as in the previous example it follows that if X is a dense subset of reals such that finite subsets of X are mutually Cohen generic over V then $V(X)$ is a c_{sym}-extension of V.

2. The nonstationary tower

Basic familiarity with the nonstationary tower techniques is assumed throughout. A good source is [FM]. In this section I will prove the main facts about the nonstationary tower that will be used in later sections. The results are all due to W. Hugh Woodin. The first lemma is well known.

2.1. Lemma. [W2, FM] *Suppose $\kappa \in \lambda$ are Woodin cardinals.*

(1) *There is a condition $q_\kappa \in \mathbb{Q}_{<\lambda}$ such that q_κ is compatible with every element of $\mathbb{Q}_{<\kappa}$ and $q_\kappa \Vdash \dot{G} \cap \check{\mathbb{Q}}_{<\kappa}$ is a generic subset of $\check{\mathbb{Q}}_{<\kappa}$.*

Suppose moreover that κ is a weakly compact Woodin cardinal. Then

(2) *$\mathbb{Q}_{<\kappa} \Vdash V(\dot{\mathbb{R}})$ is a κ_{sym}-extension of V.*
(3) *Let $V(\mathbb{R}^*)$ be a κ_{sym}-extension of V. The residue forcing $\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)$ has a dense subset isomorphic to $D = \{g :$ for some Woodin cardinal $\alpha \in \kappa$ of V, $g \subset \mathbb{Q}_{<\alpha}$ is a V-generic filter$\}$ ordered by reverse inclusion, where the V-generic filter on $\mathbb{Q}_{<\kappa}$ is to be obtained as a union of the $V(\mathbb{R}^*)$-generic filter on D.*

2.2. Theorem. *Suppose $\kappa \in \lambda$ are a measurable Woodin and a Woodin cardinal respectively. Then there is a condition a in the full nonstationary tower $\mathbb{P}_{<\lambda}$ such that a forces the following.*

(1) $\check{\kappa} = \aleph_1$
(2) *$\dot{G} \cap \check{\mathbb{Q}}_{<\kappa}$ is a V-generic filter*
(3) *there is a closed unbounded set $C \subset \check{\kappa}$ such that every ordinal in C is a weakly compact Woodin cardinal in V and for every limit point α of C including $\alpha = \check{\kappa}$ the set $C \cap \alpha$ diagonalizes the weakly compact filter on α as evaluated in V.*

This theorem is the main tool for establishing Σ^2_1 absoluteness results. Before I prove it, let me show how it is used.

2.3. Lemma. *Assume the continuum hypothesis. Then $a \Vdash \mathbb{R} \cap V[\dot{G}] = \mathbb{R} \cap V[\dot{G} \cap \check{\mathbb{Q}}_{<\kappa}]$ and so $V(\dot{\mathbb{R}})$ is a κ_{sym}-extension of V.*

Proof. Fix an enumeration $e : \omega_1 \to \mathbb{R} \cap V$ of the reals in the ground model and choose a generic filter $G \subset \mathbb{P}_{<\lambda}$ containing the condition a. Writing $j : V \to M$ for the associated generic ultrapower we find that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\ j\ } & M \\ \Big\| & & \Big\uparrow{\scriptstyle k} \\ V & \xrightarrow{\ i\ } & N \end{array}$$

where $i : V \to N$ is the generic ultrapower associated with the V-generic filter $G \cap \mathbb{Q}_{<\kappa}$ and $k[f]_{G\cap\mathbb{Q}_{<\kappa}} = [f]_G$. Now the critical point of the elementary embedding k must be above κ since $\kappa = \aleph_1^N = \aleph_1^M$, and so $j(e) = k(i(e)) = i(e)$. However, by the elementarity of j and i the range of $i(e) = j(e)$ contains all the reals in the models M and N and so these models share the same reals. The basic facts about the nonstationary ultrapower imply that $\mathbb{R} \cap V[G \cap \mathbb{Q}_{<\kappa}] = \mathbb{R} \cap N = \mathbb{R} \cap M = \mathbb{R} \cap V[G]$. The proof of the lemma is concluded by a reference to Lemma 2.1(2). □

The set $C \subset \kappa$ of Theorem 2.2(3) is a poor man's Radin-generic club for a sequence of measures with a repeat point at κ. Such a club could be procured at the expense of increasing the large cardinal assumptions on κ and introducing another heap of technology into the proof. The basic application of the properties of the set C is

2.4. Lemma. *Assume the continuum hypothesis. Then $a \Vdash$ for every generic extension $V[g]$ of V using a poset of size $< \kappa$ the forcing $\mathbb{Q}^{V[g]}_{<\kappa}/\dot{\mathbb{R}}$ has a σ-closed dense subset.*

Proof. Let $G \subset \mathbb{P}_{<\lambda}$ be a generic filter containing the condition a, let $P \in V_\kappa$ be a poset and let $g \subset P$, $g \in V[G]$ be a V-generic filter. Then

(2.a) $V[g] \models \kappa$ is a measurable Woodin cardinal and so $\mathbb{Q}_{<\kappa} \Vdash V[g](\dot{\mathbb{R}})$ is a κ_{sym}-extension of $V[g]$, by Lemma 2.1(3), and

(2.b) $V[g](\mathbb{R} \cap V[G])$ is a κ_{sym}-extension of the model $V[g]$ by Lemma 2.1(2) and Claim 1.6.

So one can calculate the remainder poset $\mathbb{Q}^{V[g]}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R} \cap V[G])$ as in Lemma 2.1(3).

Now let $C \subset \kappa$ be a club in $V[G]$ from Theorem 2.2(3). Note that for every large enough ordinal $\alpha \in \kappa$ if α is a weakly compact Woodin cardinal in V then it is such a cardinal in $V[g]$ and the weakly compact filter on α as evaluated in $V[g]$ is generated by this filter as evaluated in V. Therefore passing to a tail of C if necessary we may assume that the set C has the properties of Theorem 2.2(3) with V replaced by $V[g]$.

I claim that the set $D = \{h \subset \mathbb{Q}^{V[g]}_{<\alpha} : \alpha \in C$ and h is $V[g]$-generic filter$\}$ is a σ-closed dense subset of the poset $\mathbb{Q}^{V[g]}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R} \cap V[G])$ as computed in Lemma 2.1(3). This will complete the proof. The density of D is clear. For the closedness assume that $h_n \subset \mathbb{Q}^{V[g]}_{<\alpha_n}$ is a descending chain of conditions in D. There is just one candidate for its lower bound, namely the filter $h = \bigcup_n h_n \subset \mathbb{Q}^{V[g]}_{<\alpha}$ where $\alpha = \bigcup_n \alpha_n$. Since the ordinals α_n for $n \in \omega$ were elements of the closed set C so must be their supremum α, and the only thing left to verify is the $V[g]$-genericity of the filter h. So assume $A \subset \mathbb{Q}^{V[g]}_{<\alpha}$ is a maximal antichain in $V[g]$. The set $\{\beta \in \alpha : A \cap \mathbb{Q}^{V[g]}_{<\beta}$ is a maximal antichain in the poset $\mathbb{Q}^{V[g]}_{<\beta}\}$ belongs to the weakly compact filter on α, so there must be an integer $n \in \omega$ such that the ordinal α_n belongs to this set. By the genericity of the filter h_n then, $h_n \cap A \cap \mathbb{Q}^{V[g]}_{<\alpha_n} \neq 0$, and since $h_n \subset h$ we have $h \cap A \neq 0$ as desired. $\square$

Finally, to prove Theorem 2.2, choose an inaccessible cardinal θ between κ and λ. By standard nonstationary tower arguments it is enough to show that the set a is stationary, where a consists of those elementary submodels M of H_θ for which $\kappa \in M, o.t.M \cap \kappa = \omega_1$ and there is a relatively closed unbounded set $C_M \subset M \cap \kappa$ such that for every $\alpha \in C_M$, α is a weakly compact Woodin cardinal, the model M is selfgeneric at α, and if α is a relatively limit point of C_M including the case $\alpha = \kappa$ then the set $C_M \cap \alpha$ diagonalizes the weakly compact filter at α intersected with M.

So suppose $f : H_\theta^{<\omega} \to H_\theta$ is a function; a closure point $M \in a$ must be found. To this end, choose a normal measure U on κ and a countable submodel

$M_0 \prec \langle H_\theta, \in, f\rangle$ containing κ and U as elements. By induction on $\alpha \in \omega_1$ build countable models M_α and ordinals $\gamma_\alpha \in \kappa$ so that

(2.c) if $\alpha \in \beta$ then $M_\alpha \subset M_\beta$ and $M_\beta \cap V_\kappa$ endextends $M_\alpha \cap V_\kappa$, that is to say $\forall x \in M_\alpha \cap V_\kappa\ x \cap M_\alpha = x \cap M_\beta$
(2.d) if $\alpha \neq 0$ then M_α is selfgeneric at κ–see [FM] for the definition
(2.e) $\gamma_\alpha = min(\kappa \cap M_{\alpha+1} \setminus M_\alpha)$, $\gamma_\alpha \in \bigcap(M_\alpha \cap U)$, $M_{\alpha+1} \cap V_{\gamma_\alpha+1} = \{A \cap V_{\gamma_\alpha+1} : A \in M_\alpha \cap V_{\kappa+1}\}$ and the weakly compact filter at γ_α intersected with $M_{\alpha+1}$ is included in the set $\{A \cap \gamma_\alpha : A \in M_\alpha \cap U\}$
(2.f) $M_\beta = \bigcup_{\alpha\in\beta} M_\alpha$ for limit ordinals $\beta \in \omega_1$.

This is not hard to do. To obtain the model $M_{\alpha+1}$ and the ordinal γ_α from the model M_α, first get a model $M_{\alpha.5}$ and an ordinal γ_α satisfying (2.c,e). Look at the ultrapower $i : V \to N$ associated with the measure U and compare the models iM_α and $M_\alpha \cap N \cap H_\theta$ in the model N. The model $M_\alpha \cap N \cap H_\theta \prec \langle iH_\theta, \in, if\rangle$ and the ordinal κ satisfy (2.c,e) with respect to iM_α in the model N, so by the elementarity of the embedding i, there must be a model $M_{\alpha.5}$ and an ordinal γ_α satisfying (2.c,e) with respect to M_α. Once the model $M_{\alpha.5}$ is obtained, one can get a model $M_{\alpha+1}$ selfgeneric at κ such that $M_{\alpha.5} \subset M_{\alpha+1}$ and $M_{\alpha+1} \cap V_{\gamma_\alpha+1} = M_{\alpha.5} \cap V_{\gamma_\alpha+1}$ using basic facts on the nonstationary tower. Such a model clearly satisfies all of (2.c,d,e). Finally the selfgenericity of the models M_α survives at the limit stages due to the requirement (2.c).

Now the model $M = \bigcup_{\alpha\in\omega_1} M_\alpha$ is closed under the function f and it is an element of the set a as witnessed by the set $C_M = \{\gamma_\alpha : \alpha \in \omega_1\}$. For given an ordinal $\alpha \in \omega_1$ the ordinal γ_α is weakly compact Woodin cardinal by the second requirement in (2.e), the model M is selfgeneric at γ_α by (2.d) and the third requirement in (2.e) applied at α. Furthermore, if α is a limit ordinal then by the second requirement of (2.e) applied at ordinals $\beta \in \alpha$ the set $\{\gamma_\beta : \beta \in \alpha\}$ diagonalizes the filter $U \cap M_\alpha$ and by the last requirement of (2.e) it must diagonalize the weakly compact filter at γ_α intersected with M.

3. The absoluteness results

In this section the first two clauses of theorems mentioned in the introduction are proved.

3.1. Ramsey ultrafilters.

The main property of Ramsey ultrafilters used throughout this paper is described in the following well-known definition and lemma:

3.1.1. Definition. Suppose F is an ultrafilter on natural numbers. Define a poset P_F for the diagonalization of F by $P_F = \{\langle a, A\rangle : a \subset \omega \text{ is finite and } A \in F\}$ ordered by $\langle a, A\rangle \geq \langle b, B\rangle$ if b endextends a and $B \subset A$. If $\vec{F}$ is a finite sequence of ultrafilters then let $P_{\vec{F}} = \prod_{n\in lth(\vec{F})} P_{\vec{F}(n)}$. A generic filter for $P_{\vec{F}}$ is naturally identified with the finite sequence of subsets of ω it determines, which will be denoted by $\vec{x}_{gen}$.

3.1.2. Lemma. [SS] *Suppose $\vec{F}$ is a finite sequence of pairwise nonequivalent Ramsey ultrafilters of length n and $\vec{x}$ is an external sequence of subsets of ω diagonalizing $\vec{F}$ coordinatewise. Then $\vec{x}$ is a generic sequence for the poset $P_{\vec{F}}$.*

Recall also that there is a natural poset for adding a finite sequence of pairwise nonequivalent Ramsey ultrafilters. Let $\prod \mathcal{P}(\omega)/fin$ be a product of finitely many copies of the poset $\mathcal{P}(\omega)$ ordered by modulo finite inclusion and let $\vec{F}_{gen}$ denote the sequence of the generic filters.

3.1.3. Lemma.

(1) *$\prod \mathcal{P}(\omega)/fin$ is a σ-closed poset*
(2) *$\prod \mathcal{P}(\omega)/fin \Vdash \vec{F}_{gen}$ is a sequence of pairwise nonisomorphic Ramsey ultrafilters*
(3) *whenever ϕ is an ultrafilter formula then all conditions in $\prod \mathcal{P}(\omega)/fin$ decide the validity of $\phi(\vec{F}_{gen})$ in the same way.*

Proof. All of this is well-known. For (3) argue by contradiction. If $\vec{a}, \vec{b} \in \prod \mathcal{P}(\omega)/fin$ are conditions–finite sequences of infinite subsets of ω such that $\vec{a} \Vdash \phi(\vec{F}_{gen})$ and $\vec{b} \Vdash \neg\phi(\vec{F}_{gen})$ then strengthening the conditions if necessary one can find a sequence $\vec{\pi}$ of permutations of ω such that $\vec{\pi}''\vec{a} = \vec{b}$ coordinatewise. Now suppose $G \subset \prod \mathcal{P}(\omega)/fin$ is a generic filter containing the condition $\vec{a}$. Let $\vec{E} \in V[G]$ be the finite sequence of ultrafilters given by $\vec{E} = \{\vec{\pi}''x : x \in \vec{F}_{gen}/G\}$ coordinatewise and let $H \subset \prod \mathcal{P}(\omega)/fin$ be the filter given by the equation $\vec{F}_{gen} = \vec{E}$. Then $H \subset \prod \mathcal{P}(\omega)/fin$ is a V-generic filter containing the condition $\vec{b}$, $V[G] = V[H]$ and by the forcing theorem, $V[G] = V[H] \models \phi(\vec{F}_{gen}/G) \wedge \neg\phi(\vec{E})$ while $\vec{F}_{gen}/G$ is equivalent to $\vec{E}$ coordinatewise, contradicting the assumption that ϕ is an ultrafilter formula. □

The basic connection between Ramsey ultrafilters and the nonstationary tower is established in the following.

3.1.4. Lemma. *Suppose $\vec{F}$ is a finite sequence of pairwise nonequivalent Ramsey ultrafilters and κ is a weakly compact Woodin cardinal. Then*

$$\mathbb{Q}_{<\kappa} \Vdash j(\vec{F}) \text{ is } V(\dot{\mathbb{R}})\text{-generic subset of } \prod \mathcal{P}(\omega)/fin.$$

Here j denotes the canonical $\mathbb{Q}_{<\kappa}$ generic ultrapower.

3.1.5. Corollary.

(1) *Suppose there is a weakly compact Woodin cardinal. Then every finite sequence of pairwise nonequivalent Ramsey ultrafilters is a $L(\mathbb{R})$-generic subset of $\prod \mathcal{P}(\omega)/fin$.*
(2) *Suppose there is a weakly compact Woodin cardinal and $\phi(\vec{x})$ is a projective ultrafilter formula. Then for every finite sequence $\vec{F}$ of pairwise nonequivalent Ramsey ultrafilters of the appropriate length,*

$$\phi(\vec{F}) \leftrightarrow L(\mathbb{R}) \models \prod \mathcal{P}(\omega)/fin \Vdash \phi(\vec{F}_{gen}).$$

The corollary was known to Todorcevic and Di Prisco. It immediately implies Theorem 0.1 (1)–the right hand side of depends only on the theory of $L(\mathbb{R})$ which is invariant under set forcing. The assumptions in the Corollary could be reduced to the existence of infinitely many Woodin cardinals. Consistency-wise the conclusion of Corollary 3.1.5(1) is only as strong as the existence of a Mahlo cardinal.

Proof of Corollary. Suppose κ is a weakly compact Woodin cardinal, $G \subset \mathbb{Q}_{<\kappa}$ a generic filter and $j : V \to M$ the derived generic ultrapower. By the Lemma the sequence $j(\vec{F})$ is $V(\mathbb{R} \cap V[G])$-generic and so $L(\mathbb{R} \cap V[G]) = L(\mathbb{R} \cap M)$-generic. But this is a fact of the theory of the model M and so it can be pulled back to the ground model using the elementarity of the embedding j. (1) follows.

For (2) first note that any projective property of $\vec{F}$ is absolute between V and the model $L(\mathbb{R})[\vec{F}]$. Then (2) follows immediately from (1) and Lemma 3.1.3(3) applied in the model $L(\mathbb{R})$. □

Proof of the Lemma. Let $V(\mathbb{R}^*)$ be a κ_{sym}-extension of V and work in $V(\mathbb{R}^*)$. Choose a condition $g \in \mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)$. Thus for some Woodin cardinal $\alpha \in \kappa$ of V, $g \subset \mathbb{Q}_{<\alpha}$ is a V-generic filter and one can form the generic ultrapower $j_g : V \to M_g$. I will show

(3.1.a) for every sequence $\vec{x}$ diagonalizing the sequence $j_g(\vec{F})$ coordinatewise there is a condition $h \in \mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)$ such that $g \subset h$ and $h \Vdash \vec{x} \in j(\vec{F})$ coordinatewise, where j denotes the generic $\mathbb{Q}_{<\kappa}$ ultrapower.

This will prove the lemma since it immediately follows that

(3.1.b) $\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*) \Vdash j(\vec{F})$ is a $V(\mathbb{R}^*)$-generic subset of $\prod \mathcal{P}(\omega)/fin$, and

(3.1.c) $\bigwedge j_g(\vec{F})$ is the projection of the condition g into the poset $\prod \mathcal{P}(\omega)/fin$ associated with the name $j(\vec{F})$.

Let me first deal with the following general situation. Suppose that $\beta \in \kappa$ is any Woodin cardinal of V between α and κ and that $h \subset \mathbb{Q}_{<\beta}$ is an arbitrary V-generic filter extending g, with $j_h : V \to M_h$ denoting the generic ultrapower. Then

(3.1.d) $V[g] \models j_g(\vec{F})$ is a sequence of pairwise nonequivalent Ramsey ultrafilters and $g \Vdash_{\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)} j_g(\vec{F}) \subset j(\vec{F})$ coordinatewise

(3.1.e) $V[h] \models j_h(\vec{F})$ is a sequence of pairwise nonequivalent Ramsey ultrafilters, $j_g(\vec{F}) \subset j_h(\vec{F})$ coordinatewise and $h \Vdash_{\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)} j_h(\vec{F}) \subset j(\vec{F})$ coordinatewise

(3.1.f) $V[h] \models$ there is a finite sequence $\vec{x}$ of subsets of ω coordinatewise diagonalizing the filter sequence $j_g(\vec{F})$ such that $\vec{x} \in j_h(\vec{F})$ coordinatewise.

The proofs of (3.1.d) and (3.1.e) are the same, let me argue for (3.1.d). By the elementarity of the embedding j_g, $M_g \models j_g(\vec{F})$ is a sequence of pairwise nonequivalent Ramsey ultrafilters; but the model M_g has the same reals as $V[g]$, so $V[g]$ satisfies the same sentence. For the second part note that $g \in \mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)$ forces that the $\mathbb{Q}_{<\kappa}$ embedding j can be factored as $k \circ j_g$ for some elementary embedding k and so $j_g(\vec{F}) = k'' j_g(\vec{F}) \subset k j_g(\vec{F}) = j(\vec{F})$ coordinatewise as desired. (3.1.f) follows from the fact that the filters on the sequence $j_h(\vec{F})$ are P-points in the model $V[h]$ and the filters on the sequence $j_g(\vec{F})$ are their countable subsets.

Now the sequence $\vec{x}$ from (3.1.f) represents a $V[g]$-generic filter on $P_{j_g(\vec{F})}$ by Lemma 3.1.2. It follows that in $V[g]$ there is an embedding of $P_{j_g(\vec{F})}$ into the poset $Q = \mathbb{Q}_{<\beta} \restriction q_\alpha/\mathbb{Q}_{<\alpha}(g)$ such that $Q \Vdash \vec{x}_{gen} \in i(\vec{F})$ coordinatewise where $\vec{x}_{gen}$ is the name for the $P_{j_g(\vec{F})}$ generic sequence and i is the name for the $\mathbb{Q}_{<\beta}$ generic ultrapower.

Finally we can return to the proof of (3.1.a). Whenever $\vec{x}$ is a sequence of subsets of ω diagonalizing the filter sequence $j_g(\vec{F})$ coordinatewise, it is $V[g]$-generic for the poset $P_{j_g(\vec{F})}$ by Lemma 3.1.2. By the previous paragraph then one can find a V-generic filter $h \subset \mathbb{Q}_{<\beta}$ such that $g \subset h$ and $\vec{x} \in j_h(\vec{F})$ coordinatewise. By (3.1.e) above, $V(\mathbb{R}^*) \models h \Vdash_{\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)} \vec{x} \in j(\vec{F})$ coordinatewise as desired. $\square$

Now for the Σ^2_1 absoluteness. Assume that there is a class of measurable Woodin cardinals. Assume that ϕ is a Σ^2_1 ultrafilter formula, P_0, P_1 are posets, $\vec{F}_0, \vec{F}_1$ are P_0, P_1 names respectively and

(3.1.g) $P_0 \Vdash \vec{F}_0$ is a sequence of the appropriate length of pairwise nonequivalent Ramsey ultrafilters and the Continuum Hypothesis holds

(3.1.h) $P_1 \Vdash \vec{F}_1$ is a sequence of the appropriate length of pairwise nonequivalent Ramsey ultrafilters and $\phi(\vec{F}_1)$ holds.

For Theorem 0.1(2) it is enough to deduce from these assumptions that $P_0 \Vdash \phi(\vec{F}_0)$. To do that, choose cardinals $\kappa \in \lambda$, a measurable Woodin and a Woodin cardinal respectively such that $P_0, P_1 \in V_\kappa$, and choose a V-generic filter $g_0 \subset P_0$ and a $V[g_0]$-generic filter $G_0 \subset \mathbb{P}^{V[g_0]}_{<\lambda}$ containing a condition a as in Theorem 2.2. It must be proved that $V[g_0] \models \phi(\vec{F}_0/g_0)$. Writing $j : V[g_0] \to M$ for the G_0-generic ultrapower, by the elementarity of j this is the same as to show that $M \models \phi(j(\vec{F}_0/g_0))$.

To this end note that $V[g_0][G_0] \models \kappa = \aleph_1$. So in this model one can find a V-generic filter $g_1 \subset P_1$ and a sequence $\vec{E}$ of Ramsey ultrafilters such that $j(\vec{F}_0/g_0)$ is equivalent to $\vec{E}$ and $\vec{F}_1/g_1 \subset \vec{E}$ coordinatewise. Write $\mathbb{R}^* = \mathbb{R} \cap V[g_0][G_0]$. By Lemmas 2.3 and 3.4 applied in the model $V[g_0][G_0]$,

(3.1.i) the sequence $j(\vec{F}_0/g_0)$–and so the sequence $\vec{E}$ as well–is a $V(\mathbb{R}^*)$-generic subset of $\prod \mathcal{P}(\omega)/fin$ and the model $V(\mathbb{R}^*)$ is a κ_{sym}-extension of V

and by Lemmas 2.1(2) and 2.4 applied in the model $V[g_1]$,

(3.1.j) $V[g_1] \models \mathbb{Q}_{<\kappa} \Vdash V(\dot{\mathbb{R}})$ is a κ_{sym}-extension of V and $i(\vec{F}_1/g_1)$ is a $V(\dot{\mathbb{R}})$-generic subset of the poset $\prod \mathcal{P}(\omega)/fin$, where $i : V[g_1] \to M$ is the $\mathbb{Q}_{<\kappa}$ generic ultrapower.

So by Theorem 1.4(2) applied in the model $V(\mathbb{R}^*)[\vec{E}]$ the remainder poset $\mathbb{Q}^{V[g_1]}_{<\kappa}/\dot{\mathbb{R}}, i(\vec{F}_1/g_1)(\mathbb{R}^*, \vec{E})$ is nonempty and a direct calculation based on (3.1.c) shows that it has a dense subset isomorphic to

(3.1.k) $\mathbb{S} = \{h \subset \mathbb{Q}^{V[g_1]}_{<\alpha} : \alpha \in \kappa$ is a Woodin cardinal of $V[g_1]$, h is a $V[g_1]$-generic filter and $j_h(\vec{F}_1/g_1) = M_h \cap \vec{E}$ coordinatewise, where $j_h : V[g_1] \to M_h$ is the generic ultrapower associated with the filter $h\}$ ordered by reverse inclusion.

An argument almost identical to the one in Lemma 2.4 shows that for some club $C \subset \kappa$ in the model $V[g_0][G_0]$ the set $D \in V[g_0][G_0]$, $D = \{h \in \mathbb{S}$: for some $\alpha \in C$ $h \subset \mathbb{Q}^{V[g_1]}_{<\alpha}$ is a $V[g_1]$-generic filter$\}$ is a σ-closed dense subset of the poset $\mathbb{S}$. Since $V[g_0][G_0] \models |\mathcal{P}(\mathbb{S}) \cap V(\mathbb{R}^*)[\vec{E}]| = \aleph_1$ one can find in $V[g_0][G_0]$ a $V(\mathbb{R}^*)[\vec{E}]$-generic filter on $\mathbb{S}$ and a posteriori a $V[g_1]$-generic filter $G_1 \subset \mathbb{Q}^{V[g_1]}_{<\kappa}$ such that writing

$i : V[g_1] \to N$ for the associated generic ultrapower, $\mathbb{R} \cap V[g_1][G_1] = \mathbb{R} \cap N = \mathbb{R}^*$ and $i(\vec{F}_1/g_1) = \vec{E}$.

Finally it is possible to chase the formula ϕ from the model $V[g_1]$ to $V[g_0]$ as desired. Namely: $V[g_1] \models \phi(\vec{F}_1/g_1)$ by the forcing theorem and (3.1.h). $N \models \phi(i(\vec{F}_1/g_1))$ by the elementarity of the embedding i. Since ϕ is Σ^2_1, $N \subset V[g_0][G_0]$ and $\mathbb{R} \cap N = \mathbb{R} \cap V[g_0][G_0]$, this means that $V[g_0][G_0] \models \phi(i(\vec{F}_1/g_1))$. Since ϕ is an ultrafilter formula and $i(\vec{F}_1/g_1) = \vec{E}$ is coordinatewise equivalent to the sequence $j(\vec{F}_0/g_0)$, necessarily $V[g_0][G_0] \models \phi(j(\vec{F}_0/g_0))$. Since the models $V[g_0][G_0]$ and M agree on reals and sets of reals, it follows that $M \models \phi(j(\vec{F}_0/g_0))$. And lastly, by the elementarity of the embedding j it follows that $V[g_0] \models \phi(\vec{F}_0/g_0)$, which is what we set out to prove.

3.2. Free trees.

The proof in this case is a little different since equivalence of trees is a Σ^2_1 statement while equivalence of ultrafilters is projective. The argument is based on the following

3.2.1. Lemma. *Suppose T is a free tree and $\alpha \in \beta$ are a measurable Woodin and a Woodin cardinal respectively. Let $g \subset \mathbb{Q}_{<\alpha}$ be a generic filter and $j_g : V \to M_g$ be the associated ultrapower, and let b be an external collection of cofinal branches of the tree $j_g(T)$ such that $\bigcup b = j_g(T)$. Then*

1. *$V[g] \models j_g(T)$ is a free tree*
2. *there is an external V-generic filter $h \subset \mathbb{Q}_{<\beta}$ such that $g \subset h$ and b is exactly the set of branches of $j_g(T)$ that have lower bound in the tree $j_h(T)$– in other words $(j_h(T))_\alpha = \{\bigcup x : x \in b\}$–where $j_h : V \to M_h$ is the generic ultrapower associated with the filter h.*

The hypotheses of the Lemma are not really optimal. For (1) is a Π^1_1 statement about V_α and so it reflects down at a measurable cardinal, and (2) really uses (1) only. However I do not know how to prove (1) from Woodinness of α only.

Proof. For (1) note that by Theorem 2.2 there is an external V-generic filter $G \subset \mathbb{P}_{<\beta}$ such that $g \subset G$ and $V[G] \models \alpha = \aleph_1$. Writing $i : V \to N$ for the G-generic ultrapower we have $j_g(T) = i(T)$ and $N \cap \mathcal{P}(\alpha) = V[G] \cap \mathcal{P}(\alpha)$. So $N \models$"$i(T) = j_g(T)$ is a free tree" by the elementarity of the embedding i, $V[G] \models$"$j_g(T)$ is a free tree" by the agreement between the models N and $V[G]$, and finally $V[g] \models$"$j_g(T)$ is a free tree" since $V[g] \subset V[G]$.

For (2) work in $V[g]$ and note that $\mathbb{Q}_{<\beta} \restriction q_\alpha/\mathbb{Q}_{<\alpha}(g) \Vdash V[g](\dot{c})$ is a $(j_g(\check{T}))_{sym}$-extension of the model $V[g]$, where $\dot{c}$ is the set of all cofinal branches of the tree $j_g(T)$ which have a lower bound in the tree $i(T)$ where i is the $\mathbb{Q}_{<\beta}$ ultrapower. This follows from Example 1.7, which by the same token implies that $V[g](b)$ is a $(j_g(T))_{sym}$-extension of the model $V[g]$. The existence of the desired filter $h \subset \mathbb{Q}_{<\beta}$ then follows from Theorem 1.4. □

Towards the proof of Σ^2_1 absoluteness for free trees, assume that there is a class of measurable Woodin cardinals. Assume that ϕ is a Σ^2_1 forcing formula, P_0, P_1 are posets, $\dot{T}_0, \dot{T}_1$ are P_0, P_1 names respectively and

(3.2.a) $P_0 \Vdash \dot{T}_0$ is a free tree and the Continuum Hypothesis holds

(3.2.b) $P_1 \Vdash \dot{T}_1$ is a free tree and $\phi(\dot{T}_1)$ holds.

For Theorem 0.2(2) it is enough to deduce from these assumptions that $P_0 \Vdash \phi(\dot{T}_0)$. To do that, choose cardinals $\kappa \in \lambda$, a measurable Woodin and a Woodin cardinal respectively such that $P_0, P_1 \in V_\kappa$, and choose a V-generic filter $g_0 \subset P_0$ and a $V[g_0]$-generic filter $G_0 \subset \mathbb{P}^{V[g_0]}_{<\lambda}$ containing a condition a as in Theorem 2.2. It must be proved that $V[g_0] \models \phi(\dot{T}_0/g_0)$. Writing $j : V[g_0] \to M$ for the G_0-generic ultrapower, by the elementarity of j this is the same as to show that $M \models \phi(j(\dot{T}_0/g_0))$.

Work in the model $V[g_0][G_0]$. Remember that $\kappa = \aleph_1$ by Theorem 2.2 and that the Continuum Hypothesis holds, since it holds in the model M by the elementarity of the embedding j, and the models M and $V[g_0][G_0]$ share the same reals and sets of reals. Fix a V-generic filter $g_1 \subset P_1$ and a club $C \subset \kappa$ as in Theorem 2.2(3). By passing to a tail if necessary, we may assume that C consists of weakly compact Woodin cardinals of $V[g_1]$ which satisfy Lemma 3.2.1(1), and that for every limit point α of C including $\alpha = \kappa$ the weakly compact filter on α as evaluated in $V[g_1]$ is diagonalized by the set $C \cap \alpha$. Finally, let $\{r_\alpha : \alpha \in \kappa\}$ be an enumeration of all the reals in $V[g_0][G_0]$.

By induction on $\alpha \in \kappa$ build ordinals $\gamma_\alpha \in C$, filters h_α and functions f_α so that

(3.2.c) γ_α's form a continuous strictly increasing sequence

(3.2.d) $h_\alpha \subset \mathbb{Q}^{V[g_1]}_{<\gamma_\alpha}$ are continuously inclusion increasing $V[g_1]$-generic filters and $r_\alpha \in V[g_1][h_{\alpha+2}]$

(3.2.e) f_α's are continuously inclusion increasing isomorphisms between the trees $\{t \in j(\dot{T}_0/g_0) : \exists \beta \in \alpha\ t \in (j(\dot{T}_0/g_0))_{\gamma_\beta}\}$ and $\{t \in i_\alpha(\dot{T}_1/g_1) : \exists \beta \in \alpha\ t \in (j_\alpha(\dot{T}_1/g_1))_{\gamma_\beta}\}$, where i_α is the h_α-generic ultrapower of the model $V[g_1]$.

To do this, at limit stages just take unions of the objects constructed so far. The genericity of the filters survives this operation by an argument taken directly from the proof of Lemma 2.4. At the successor stage $\alpha = \alpha' + 1$ the cases of α' successor and limit are treated differently. First, if $\alpha' = \alpha'' + 1$ is a successor, just find an ordinal $\gamma_\alpha \in C$ such that the real $r_{\alpha''}$ is $V[g_1][h_{\alpha'}]$-generic for a poset of size $< \gamma_\alpha$ and then choose a $V[g_1]$-generic filter $h_\alpha \subset \mathbb{Q}^{V[g_1]}_{<\gamma_\alpha}$ such that $h_{\alpha'} \subset h_\alpha$ and $r_{\alpha''} \in V[g_1][h_\alpha]$. It is a matter of a trivial surgery on the map $f_{\alpha'}$ to extend it into a map f_α satisfying the condition (3.2.e). Second, if α' is limit define for every node $t \in (j(\dot{T}_0/g_0))_{\gamma_{\alpha'}}$ a cofinal branch x_t of the tree $i_{\alpha'}(\dot{T}_1/g_1)$ to be the upwards closure of the set $\{f_{\alpha'}(s) : s \subset t\}$ in the tree $i_{\alpha'}(\dot{T}_1/g_1)$. Let $b = \{x_t : t \in (j(\dot{T}_0/g_0))_{\gamma_{\alpha'}}\}$. Lemma 3.2.1 shows that setting γ_α =the next element of the set C above $\gamma_{\alpha'}$ there must be a $V[g_1]$-generic filter $h_\alpha \subset \mathbb{Q}^{V[g_1]}_{<\gamma_\alpha}$ such that $h_{\alpha'}$ and $(i_\alpha(\dot{T}_1/g_1))_{\gamma_{\alpha'}} = \{\bigcup x : x \in B\}$, where i_α is the h_α-ultrapower. The function f_α is then obtained as a union of $f_{\alpha'}$ and the function $\{t \mapsto \bigcup x_t : t \in (j(\dot{T}_0/g_0))_{\gamma_{\alpha'}}\}$.

Once this is done, let $G_1 = \bigcup_{\alpha\in\kappa} h_\alpha$ and let $F = \bigcup_{\alpha\in\kappa} f_\alpha$. The filter $G_1 \subset \mathbb{Q}^{V[g_1]}_{<\kappa}$ is $V[g_1]$-generic and the function F is an isomorphism of the trees $j(\dot{T}_0/g_0)$ and $i(\dot{T}_1/g_1)$ on the club $\{\gamma_\alpha : \alpha \in \kappa\} \subset \kappa$, where $i : V[g_1] \to N$ is the G_1-ultrapower. Now chase the formula ϕ from the model $V[g_1]$ into $V[g_0]$ as in the end of the previous subsection, using the fact that the trees $j(\dot{T}_0/g_0), i(\dot{T}_1/g_1)$ are isomorphic on a club, therefore they have to have isomorphic complete Boolean algebras and so must agree on the formula ϕ.

This completes the proof of Σ^2_1 absoluteness for free trees. The projective absoluteness is a consequence of this noting that projective statements are absolute under the σ-closed forcing making the Continuum Hypothesis true.

3.3. Lusin sets.

The main property of extendible strong Lusin sets used below is stated in the following.

3.3.1. Lemma. *Let $X \subset \mathbb{R}$ be an extendible strong Lusin set, θ a large regular cardinal and M a countable elementary submodel of H_θ containing the set X, with the transitive isomorph denoted by $\bar{M}$. Then*

(1) *$\bar{M}(X \setminus M)$ is a c_{sym}-extension of the model $\bar{M}$*
(2) *for every poset $P \in \bar{M}$ there is an $\bar{M}$-generic filter $g \subset P$ such that $\bar{M}[g](X \setminus M)$ is a c_{sym}-extension of the model $\bar{M}[g]$.*

Of course, (1) is well known. One does not need the extendibility of the set X for it, however (2) is exactly the property that distinguishes the extendible sets from the hypothetical nonextendible ones.

Proof. Fix the model M. For (1) it is enough to show that every finite sequence x of pairwise distinct reals in the set $X \subset M$ consists of reals mutually Cohen over the model M; (1) then follows from Example 1.8. Suppose that $|x| = n$ and $O \subset \mathbb{R}^n$ is an open dense set in M. We must show that $x \in O$. Since the set X is strongly Lusin there is a countable set $a \subset X$ such that all sequences y of length n consisting of pairwise distinct reals in $X \setminus a$ belong to the set O. By elementarity such a set a can be found in the model M, so $a \subset M$. Therefore x consists of reals in the set $X \setminus a$ and $x \in O$ as desired.

For (2) fix a poset $P \in \bar{M}$ and choose a countable model $N \prec H_\theta$ such that $\bar{M} \in N$. By elementarity $N \models |P| = \aleph_0$ and there is an isomorphism $\pi \in N$ between the Cohen poset and a dense subset of P. Now extendibility of the set X and (1) applied to the model N can be used to find a real number $r \notin X$ such that the set $(X \setminus N) \cup \{r\}$ consists of reals mutually Cohen over the model N. Let $h \subset$the Cohen poset be the N-generic filter associated with the real r and let $g \subset P$ be the upwards closure of the set $\pi'' h \subset P$ in P. I claim that the filter h is as desired. First of all it is $\bar{M}$-generic since it is even N-generic and $\bar{M} \subset N$. Second, we must prove that any finite set x of reals in $X \setminus M$ consists of reals mutually Cohen generic over the model $\bar{M}[g]$. Let $x_0 = x \cap N$ and $x_1 = x \setminus x_0$. Then

(3.3.a) the reals in x_0 are mutually generic over the model $\bar{M}[g]$. To see this note that the filter g is N-generic, so $\bar{M}[x_0]$-generic and apply a mutual geericity argument.
(3.3.b) the reals in x_1 are mutually Cohen generic over the model $\bar{M}[h][x_0]$ since they are generic even over the model $N[h]$.

(2) follows. □

Note also that there is a natural way of forcing a strongly Lusin set of reals by countable approximations. Let P_L be a poset consisting of pairs $\langle a, \mathcal{O} \rangle$ where a is a countable set of reals and $\mathcal{O}$ is a countable collection of open dense sets in the spaces $\mathbb{R}^n$ for various $n \in \omega$, ordered by $\langle a, \mathcal{O}\rangle \geq \langle b, \mathcal{P}\rangle$ if $a \subset b$, $\mathcal{O} \subset \mathcal{P}$ and for every sequence $\vec{x} \subset b \setminus a$ and every open dense set $D \subset \mathbb{R}^n, D \in \mathcal{O}$,

(3.3.c) if the length of $\vec{x}$ is exactly n then $\vec{x} \in D$

(3.3.d) if the length of $\vec{x}$ is $m \in n$ then the set $E \subset \mathbb{R}^{n-m}, E = \{\vec{y}$: any sequence combining the reals on $\vec{y}$ and $\vec{x}$ is an element of the set $D\}$ is open dense and is in $\mathcal{P}$.

The following is easy to verify.

3.3.2. Lemma.

(1) *P_L is a σ-closed notion of forcing and if $G \subset P_L$ is a generic filter then the set $X_{gen} = \bigcup\{a : \langle a, 0\rangle \in G\}$ is an extendible strong Lusin set in $V[G]$ from which the filter $G \subset P_L$ can be reconstructed*

(2) *if ϕ is a Lusin formula then all conditions in P_L decide the validity of $\phi(\dot{X}_{gen})$ in the same way.*

The connection between extendible strong Lusin sets and the nonstationary tower is described in the following

3.3.3. Lemma. *Suppose κ is a weakly compact Woodin cardinal and $X \subset \mathbb{R}$ is an extendible strong Lusin set. Then $\mathbb{Q}_{<\kappa} \Vdash j\check{X}$ is a $V(\dot{\mathbb{R}})$-generic subset for the poset $\dot{P}_L$.*

Granted this lemma, Theorem 0.3(1,2) is proved almost exactly like Theorem 0.1(1,2) in Subsection 3.1.

Towards the proof of the Lemma, fix a κ_{sym}-extension $V(\mathbb{R}^*)$ of V and work in $V(\mathbb{R}^*)$. Let $\alpha \in \kappa$ be a Woodin cardinal of V, let $g \in \mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)$, $g \subset \mathbb{Q}_{<\alpha}$ be a V-generic filter, let $j_g : V \to M_g$ be the associated ultrapower and consider the condition $p_g \in P_L$ given by $p_g = \langle j_g(X), \{D$: for some $n \in \omega$, $D \subset \mathbb{R}^n$ is an open dense set in the model $V[g]\}\rangle$. Similarly as in Section 3.1 I will prove

(3.3.e) for every condition $q \in P_L$ with $q \leq p_g$ there is a condition $k \in \mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)$ with $k \leq g$ such that $k \Vdash$"the filter on $\check{P}_L$ associated with the set $j(X)$ contains the condition $\check{q}$" where j is the name for the $\mathbb{Q}_{<\kappa}$ ultrapower.

This shows that $\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*) \Vdash$"$j(X)$ is a $V(\mathbb{R}^*)$-generic set for $\check{P}_L$" and p_g is the associated projection of the condition g into P_L, completing the proof of the lemma.

First let me consider the following general situation. Suppose $\beta \in \kappa$ is a Woodin cardinal of V above α and $h \subset \mathbb{Q}_{<\beta}$ is a V-generic filter such that $g \subset h$, with $j_h : V \to M_h$ denoting the associated ultrapower. Then

(3.3.f) $V[g](j_h(X) \setminus j_g(X))$ is a c_{sym}-extension of the model $V[g]$

(3.3.g) $g \Vdash_{\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)} p_g$ is in the filter on P_L determined by the set $j(X)$

(3.3.h) Suppose $P \in V[g]$ is a poset of size $< \beta$. Then there is in $V[h]$ a $V[g]$-generic filter $k \subset P$ such that $V[g][k](j_h(X) \setminus j_g(X))$ is a c_{sym}-extension of the model $V[g][k]$.

I will prove (3.3.f). Note that it is enough to show that $N_g(j_h(X) \setminus j_g(X))$ is a c_{sym}-extension of the model N_g, since N_g and $V[g]$ share the same reals. Write $k : N_g \to N_h$ for the usual factor embedding such that $j_h = k \circ j_g$ and choose an inaccessible cardinal θ of $V[g]$ between α and β. Then (3.3.f) follows from Lemma 3.3.1(1) applied in the model N_h to the cardinal $k(\theta)$, the countable model $k''(H_\theta)^{N_g}$

and the strong Lusin set $j_h(X)$. (3.3.g) follows from (3.3.f) and for (3.3.h) apply the proof of Lemma 3.3.1(2) in the model $V[h]$.

Now back to the proof of (3.3.e). Let $q = \langle a_q, b_q\rangle \in P_L$ be any condition below p_g. By strengthening the condition q if necessary, we may assume that $V[g](a_q \setminus j_g(X))$ is a c_{sym}-extension of the model $V[g]$–this happens whenever the set $a_q \setminus j_g(X) \subset \mathbb{R}$ is dense by the definition of the condition p_g and Example 1.8. Choose a Woodin cardinal $\beta \in \kappa$ of V above α. It follows from (3.3.f) and Theorem 1.4 that there is a V-generic filter $h \subset \mathbb{Q}_{<\beta}$ such that $g \subset h$ and $j_h(X) = a_q$. Now choose a partial order $P \in V[h]$ so that the set b_q is $V[h]$-generic for the poset P. Note that b_q is essentially a real and so–as $V(\mathbb{R}^*)$ is a κ_{sym}-extension of $V[h]$–it is possible to find a Woodin cardinal γ of V between β and κ such that $|P| < \gamma$ in $V[h]$. It follows from (3.3.h) that there is a V-generic filter $k \subset \mathbb{Q}_{<\gamma}$ such that $h \subset k$ and $b_q \in V[k]$ and $V[h][b_q](j_k(X) \setminus j_h(X))$ is a c_{sym}-extension of the model $V[h][b_q]$. Consider the condition $p_k \in P_L$. By inspection $p_k \leq q \leq p_g$; moreover $k \Vdash_{\mathbb{Q}_{<\kappa}/\dot{\mathbb{R}}(\mathbb{R}^*)} \check{p}_k$ is in the filter on $\check{P}_L$ determined by the set $j(X)$ by (3.3.g) applied to the filter k, completing the proof of (3.3.e) and the Lemma.

3.4. Diamond sequences.

Since our definition of good diamond sequences is somewhat nonstandard, it is perhaps worth it to see that it has the expected content.

3.4.1. Lemma. *There is a good diamond sequence if and only if $\diamondsuit$ holds.*

Proof. Suppose first that diamond holds and $\langle x_\alpha : \alpha \in \omega_1\rangle$ is the usual diamond sequence, i.e. $x_\alpha \subset \alpha$ and for every set $A \subset \omega_1$ and every club $C \subset \omega_1$ there is an ordinal $\alpha \in C$ such that $A \cap \alpha = x_\alpha$. A good diamond sequence must be produced. Choose a bijection e between $\mathcal{P}(\omega)$ and the set of all Borel relations on $\mathbb{R}^{\aleph_0} \times \mathbb{R}$, for each ordinal $\alpha \in \omega_1$ let $a_\alpha = \{s \subset \omega :$ there is a nonzero limit ordinal $\beta \in \alpha$ with $n \in s \leftrightarrow \beta + n \in x_\alpha\}$ and let r_α be any real such that $\langle a_\alpha, r_\alpha\rangle \in e(x_\alpha \cap \omega)$ if such exists, otherwise let $r_\alpha = 0$. It is trivial to verify that $\langle r_\alpha : \alpha \in \omega_1\rangle$ is a good diamond sequence.

Now assume that a good diamond sequence $\langle r_\alpha : \alpha \in \omega_1\rangle$ is given and we want to construct a diamond sequence of the usual kind. Fix any Borel bijection e between $\mathbb{R}$ and the set of all models with universe ω, and for every ordinal $\alpha \in \omega_1$, let $x_\alpha \subset \alpha$ be a set such that the model $e(r_\alpha)$ is isomorphic to $\langle \alpha, \in, x_\alpha\rangle$ if such set exists, and $x_\alpha = 0$ otherwise. Again, the easy verification that $\langle x_\alpha : \alpha \in \omega_1\rangle$ is a diamond sequence is left to the reader. □

The crucial property of good diamond sequences is described in the following Lemma. The lemma granted, the proof of the absoluteness theorems is completed in the same way as in Subsection 3.2.

3.4.2. Lemma. *Suppose $\kappa \in \lambda$ are a measurable Woodin and a Woodin cardinal respectively and d is a good diamond sequence. Suppose $g \subset \mathbb{Q}_{<\kappa}$ is a V-generic filter and s is an external real number such that s is $V[g]$-generic via a poset of size $< \lambda$. Then there is an external V-generic filter $h \subset \mathbb{Q}_{<\lambda}$ such that $g \subset h$ and $j_h(d)(\kappa) = s$, where j_h is the h-generic ultrapower of V.*

Proof. Work in V. Choose a $\mathbb{Q}_{<\kappa}$-name $\dot{P}$ for a poset in V_λ and let θ be an inaccessible cardinal between κ and λ such that $\dot{P} \in H_\theta$. Consider the following set

(3.4.a) $b = \{M \prec H_\theta : M$ is a countable model containing κ and $\dot{P}$ which is selfgeneric at κ, and letting $^-\colon M \to \bar{M}$ be its transitive collapse and $k \subset \bar{\mathbb{Q}}_{<\kappa}$ be the $\bar{M}$-generic filter $k = \{\bar{a} : a \in \mathbb{Q}_{<\kappa} \cap M$ and $M \cap \bigcup a \in a\}$ we have that $d(\bar{\kappa})$ is a $\bar{M}[k]$-generic real for the poset $\bar{P}/k \in \bar{M}[k]\}$.

I will show that the set b is stationary, thus a condition in the poset $\mathbb{Q}_{<\lambda}$. Then a standard nonstationary tower argument can be applied to see that $b \Vdash \dot{h} \cap \mathbb{Q}_{<\kappa}$ is a V-generic subset of $\mathbb{Q}_{<\kappa}$ and the real $j_h(\check{d})(\check{\kappa})$ is $V[\dot{h} \cap \mathbb{Q}_{<\kappa}]$-generic via the poset $\dot{P}/\dot{h} \cap \mathbb{Q}_{<\kappa}$. Here $\dot{h}, j_h$ are the $\mathbb{Q}_{<\lambda}$-names for the generic filter and the generic ultrapower. So the external generic filter $h \subset \mathbb{Q}_{<\lambda}$ required in the Lemma can be obtained by forcing via the residue poset $\mathbb{Q}_{<\lambda} \restriction b/\dot{h} \cap \mathbb{Q}_{<\kappa}, j_h(\check{d})(\check{\kappa})(g, r)$.

So fix a function $f : H_\theta^{<\omega} \to H_\theta$; a closure point of f in the set b must be produced. Following the proof of Theorem 2.2. there is a submodel $M \prec \langle H_\theta, \in, f\rangle$ which is a continuous increasing union of countable submodels M_α for $\alpha \in \omega_1$ each of which is selfgeneric at κ and $\alpha \in \beta \in \omega_1$ implies that $M_\beta \cap V_\kappa$ end-extends $M_\alpha \cap V_\kappa$. Let $^-\colon M \to \bar{M}$ be the transitive collapse map, $\bar{M}_\alpha$ the image of M_α under the collapse map, and let $k \subset \bar{\mathbb{Q}}_{<\kappa}$ be the filter $k = \{\bar{a} : a \in \mathbb{Q}_{<\kappa}$ and $M \cap \bigcup a \in a\}$. By the assumptions, k is an $\bar{M}$-generic filter and also for each $\alpha \in \omega_1$ the set $k \cap \bar{M}_\alpha \subset \bar{\mathbb{Q}}_{<\kappa} \cap \bar{M}_\alpha$ is an $\bar{M}_\alpha$-generic filter and $\bar{M}[k]$ is a continuous increasing union of the models $\bar{M}_\alpha[k]$ for $\alpha \in \omega_1$. Coding the model $\bar{M}[k]$ as an ω_1-sequence of reals it is not hard to see that there must be an ordinal $\alpha \in \omega_1$ such that the real $d(\bar{M}_\alpha \cap \omega_1)$ is $\bar{M}_\alpha[k]$-generic for the poset $\bar{P}/k$. The model M_α is the required closure point of the function f in the set b. □

4. The homogeneous model

In the presentation of the homogeneous model, the author has three choices. Either, start with an optimal assumptions–the weakly compact cardinal in many cases–and construct the model using a heavy dose of Shelah's amalgamation [S1]. Or, start with a supercompact cardinal, which makes the forcing easier to describe and understand. Or, start with a model of ADℝ+θ regular and use the very simple forcing extension by the countable support product of θ many copies of $Add(1, \aleph_1)$. All of these three models have homogeneity properties for roughly the same notions. I choose the second option as the golden middle between what can be considered an extensive notational complexity or an exotic initial assumption.

4.1. The construction of the model.

Let κ be a supercompact cardinal with an inaccessible λ above it. Choose an elementary embedding $j : V \to N$ such that $j(\kappa) \geq \lambda$ and $j''\lambda \in N$. For every ordinal $\alpha \in \kappa$ define a measure U_α on $\mathcal{P}_{<\kappa}V_\alpha$ by $a \in U_\alpha \leftrightarrow j''V_\alpha \in j(a)$. These are normal measures which in a natural sense project onto each other: for $\alpha \in \beta \in \lambda$ we have $U_\alpha = \{a \subset \mathcal{P}_{<\kappa}V_\alpha : \exists b \in U_\beta\ a = \{x \cap V_\alpha : x \in b\}\}$.

Let $V(\mathbb{R}^*)$ be a κ_{sym}-extension of V and work in $V(\mathbb{R}^*)$. For every ordinal $\alpha \in \lambda$ define a forcing Q_α by setting

(4.1.a) if α is a successor then $Q_\alpha = Add(1, \aleph_1 = \kappa)$. This forcing is σ-closed and adds a wellordering of the reals in ordertype κ.

(4.1.b) if α is limit then $Q_\alpha = \{\langle f, a\rangle : \exists\gamma \in \kappa\ f : \gamma \to (P_{<\kappa}V_\alpha)^V)$ is a $\subset$-continuous increasing function, $a \in U_\alpha\}$ ordered by $\langle f, a\rangle \geq \langle g, b\rangle$ if $f \subset$

$g, b \subset a$ and $\forall \delta \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)\ g(\delta) \in a$. This forcing adds a κ-sequence diagonalizing the measure U_α. In the definition of the forcing, note that while the functional values of f are in V, the function f itself is generally in $V(\mathbb{R}^*)$.

Define

$$P = \prod_{\alpha\in\lambda} Q_\alpha \text{ with countable support}$$
$$P_\beta = \prod_{\alpha\in\beta} Q_\alpha \text{ with countable support, and}$$
$$P^\beta = \prod_{\alpha=\beta\vee\beta\in\alpha\in\lambda} Q_\alpha \text{ with countable support}$$

Thus $P = P_\beta \times P^\beta$ for every ordinal $\beta \in \lambda$. The following can be routinely checked.

4.1.1. Lemma.

(1) *the posets* P, P_β, P^β *are all homogeneous for* $\beta \in \lambda$
(2) *every* P*-name for a subset of* $\mathbb{R}^*$ *is in fact a* P_β *name for some* $\beta \in \lambda$
(3) P *does not add reals and forces choice, diamond and* $\kappa = \aleph_1, \lambda = \aleph_2$.

Let $G \subset P$ be a $V(\mathbb{R}^*)$-generic filter. In the following subsections I will show that the model $V(\mathbb{R}^*)[G]$ has the homogeneity properties mentioned in the introduction.

4.2. Ramsey ultrafilters.

Work in the model $V(\mathbb{R}^*)$. Suppose ϕ is an ultrafilter formula and $\vec{F}$ is a P-name for a sequence of pairwise nonequivalent Ramsey ultrafilters. That the homogeneous model described above satisfies the clause (3) of Theorem 0.1 will be obvious from the following equivalence

(4.2.a) $P \Vdash \phi(\vec{F})$ if and only if for every large enough ordinal $\beta \in \lambda$ we have $P^\beta \times Add(1, \aleph_1) \times \prod \mathcal{P}(\omega)/fin \Vdash \phi(\vec{F}_{gen})$

since the right hand side does not depend on the choice of the sequence $\vec{F}$. The equivalence will be proved by showing that for each ordinal β such that $\kappa+\omega \in \beta \in \lambda$ and $\vec{F}$ is a P_β-name–see Lemma 4.1.1(2)–we have

(4.2.b) $P_\beta \Vdash \vec{F}$ is a $V(\mathbb{R}^*)$ generic filter on $\prod \mathcal{P}(\omega)/fin$
(4.2.c) $P^\beta \times \prod \mathcal{P}(\omega)/fin \Vdash P_\beta/\vec{F}(\vec{F}_{gen})$ is a nontrivial poset of size $\aleph_1$ with a σ-closed dense subset, therefore isomorphic to $Add(1, \aleph_1)$

It is not hard to argue from the homogeneity of the forcings $P^\beta, Add(1, \aleph_1)$ and $\prod \mathcal{P}(\omega)/fin$ that for the ultrafilter formula ϕ, all conditions in the forcing $P^\beta \times Add(1, \aleph_1) \times \prod \mathcal{P}(\omega)/fin$ must decide the validity of $\phi(\vec{F}_{gen})$ in the same way. Then the equivalence (4.2.a) follows from (4.2.c).

To prove (4.2.b) fix the name $\vec{F}$ and the ordinal $\beta \in \lambda$. Choose a large regular cardinal $\theta \in \lambda$ and let $M \prec V_\theta$ be an elementary submodel such that

(4.2.d) $|M| = \aleph_0$ and M contains all the relevant objects. In particular, the name $\vec{F}$ is definable from an element of V and a real in M.
(4.2.e) $M \cap V \in V$ and $\forall \alpha \in \beta \cap M\ M \cap V \cap V_\alpha \in \bigcap(U_\alpha \cap M)$

and suppose $g \subset P_\beta \cap M$ is an M-generic filter. Since we are working in a choiceless environment, it is perhaps worth it to see that such models and filters are plentiful.

4.2.1. Claim. *The set of models as in (4.2.d,e) is stationary.*

Proof. Let $f : V_\theta^{<\omega} \to V_\theta$ be a function; we must produce a suitable closure point for it. The function f is definable from some real r and an element x of $V_\lambda \cap V$ and the real r is V-generic for some poset $P \in V_\kappa \cap V$. By the normality of the measures U_α it is possible to choose in V an elementary submodel $Z \prec V_\lambda$ of size $< \kappa$ such that $P, x \in Z$ and $\forall \alpha \in \beta \cap Z\ Z \cap V_\alpha \in \bigcap(U_\alpha \cap Z)$. By Theorem 1.2 there is an external V-generic filter $H \subset Coll(\omega, <\kappa)$ such that $\mathbb{R}^* = \mathbb{R} \cap V[H]$ and $r \in Z[H]$. We have

(4.2.f) $Z[H] \cap V = Z \in V$ by the κ-c.c. of the poset $Coll(\omega, <\kappa)$

(4.2.g) $Z[H] \cap V(\mathbb{R}^*) \in V(\mathbb{R}^*)$ since the model is countable in $V[H]$ and $V[H] \models V(\mathbb{R}^*)$ is closed under countable sequences

(4.2.h) the model $Z[H] \cap V_\theta \cap V(\mathbb{R}^*)$ is closed under the function f since $f \in Z[H]$.

Thus the model $Z[H] \cap V_\theta \cap V(\mathbb{R}^*)$ is the required closure point of the function f. $\square$

Also,

4.2.2. Claim. *The filter $g \subset P_\beta$ has a lower bound in P_β.*

Proof. For a successor ordinal $\alpha \in \beta \cap M$ let $p_\alpha = \bigcup(g \cap Q_\alpha)$, and for a limit $\alpha \in \beta \cap M$ let $p_\alpha = \bigcup(g \cap Q_\alpha) \cup \{M \cap \kappa \mapsto M \cap V_\alpha \cap V\}$. (4.2.e) and the definition of the forcing P then can be used that the function $\{\alpha \mapsto p_\alpha : \alpha \in \beta \cap M\}$ is a lower bound of the filter $g \subset P_\beta$. $\square$

Let $\vec{f} = \vec{F}/g \cap M$ coordinatewise. I will prove that

(4.2.i) for every sequence $\vec{x}$ of subsets of ω diagonalizing the sequence $\vec{f}$ of filters coordinatewise, there is a lower bound $p \in P_\beta$ of the filter g such that $p \Vdash \vec{x} \in \vec{F}$ coordinatewise.

This will show that $P_\beta \Vdash \vec{F}$ is a $V(\mathbb{R}^*)$-generic filter on $\prod \mathcal{P}(\omega)/fin$ and $\bigwedge \vec{f}$ is the projection of the condition $\bigwedge g \in P_\beta$ into $\prod \mathcal{P}(\omega)/fin$ associated with the name $\vec{F}$. (4.2.b) immediately follows. To prove (4.2.c) note that $P^\beta \Vdash |P_\beta| = \aleph_1$ and so $P_\beta \times \prod \mathcal{P}(\omega)/fin \Vdash |P_\beta/\vec{F}(\vec{F}_{gen})| = \aleph_1$. Using the result of [Fo], for the σ-closedness it is then enough to show that in the generic extension by $P^\beta \times \prod \mathcal{P}(\omega)/fin$ the player II has a winning strategy in the descending chain game on the poset $P_\beta/\vec{F}(\vec{F}_{gen})$:

I	p_0		p_1		$\dots$
II		q_0		q_1	$\dots$

where $p_0 \geq q_0 \geq p_1 \geq q_1 \geq \dots$ are conditions in the poset $P_\beta/\vec{F}(\vec{F}_{gen})$ and II wins if the chain of these conditions has a lower bound. The following claim is instrumental for this. It also may explain the intuition behind the choice of the forcing P.

4.2.3. Claim. $P^\beta \Vdash$ *there is a closed unbounded set of models M as in (4.2.d,e).*

Proof. Fix a generic filter $G^\beta \subset P^\beta$ and work in $V(\mathbb{R}^*)[G^\beta]$. Let $Z \prec H_\lambda = H_{\aleph_2}$ be a countable submodel containing all important objects. It is enough to show that $M = Z \cap V_\theta \cap V(\mathbb{R}^*)$ satisfies all of (4.2.d,e). (4.2.d) is immediate. For (4.2.e) observe that Z contains the club subset of V_θ^V added by the filter $G^\beta \cap Q_\theta$ as an element whence $Z \cap V_\theta \cap V \in V$ and $Z \cap V_\theta \cap V \in \bigcap(M \cap U_\theta)$. (4.2.e) for the model M follows. $\square$

Now let $G^\beta, \vec{F}_{gen}$ be mutually generic filters on P^β and $\prod \mathcal{P}(\omega)/fin$ respectively and let $C \in V(\mathbb{R}^*)[G^\beta][\vec{F}_{gen}]$ be a closed unbounded set of models from the previous claim. In the model $V(\mathbb{R}^*)[G^\beta][\vec{F}_{gen}]$ the player II wins the descending chain game in the following way. At round $n \in \omega$ he first plays on the side a model $M_n \prec H_\theta$ and a filter $g_n \subset P_\beta \cap M_n$ so that

(4.2.j) $M_n \in C$ and $M_0 \subset M_1 \subset \dots$

(4.2.k) g_n is an M_n-generic filter containing the condition p_n, $g_0 \subset g_1 \subset \dots$ and $\vec{F}_{gen} \cap M_n = \vec{F}/g_n \cap M_n$

and then plays $q_n = \bigwedge g_n$. This is easy to do and by the remark after his moves really are in the poset $P_\beta/\vec{F}(\vec{F}_{gen})$. In the end let $M = \bigcup_n M_n$ and $g = \bigcup_n g_n$. Then $M \in C$ by the closedness of the set C, $\vec{F}_{gen} \cap M = \vec{F}/g \cap M$ by (4.2.k) and therefore the condition $\bigwedge g \in P_\beta/\vec{F}(\vec{F}_{gen})$ is a lower bound of all the conditions played in the run of the game. The player II has won.

Thus the only thing left to prove is (4.2.i). Fix the model M and the M-generic filter $g \subset P_\beta \cap M$. Use (4.2.e) and the supercompactness of the measures to see that writing $\bar{M}$ for the transitive collapse of M and $^-: M \to \bar{M}$ for the collapse map,

(4.2.l) $\bar{M} \cap V \in V$, the bar map restricted to $M \cap V$ is in V and $V_{\bar{\beta}+\omega} \subset \bar{M}$

(4.2.m) $V(\mathbb{R}^* \cap M)$ is a $\bar{\kappa}_{sym}$-extension of V

(4.2.n) let $\bar{g}$ be the pointwise image of the filter $g \subset P_\beta$ under the bar map. Then $\bar{g} \subset \bar{P}_\beta$ is a $V(\mathbb{R}^* \cap M)$-generic filter, $\vec{f} = \bar{F}/g$ and $V(\mathbb{R}^* \cap M)[\bar{g}] \models \vec{f}$ is a sequence of pairwise nonequivalent Ramsey ultrafilters.

Now $P_\beta \Vdash \vec{F}$ is a sequence of P-points and $\bigwedge g \Vdash \vec{f} \subset \vec{F}$ coordinatewise. Therefore there must be a condition $p \in P_\beta$ below $\bigwedge g$ and a sequence $\vec{x}$ of subsets of ω such that $\vec{x}$ diagonalizes $\vec{f}$ and $p \Vdash \vec{x} \in \vec{F}$ coordinatewise. By Lemma 3.1.2, the sequence $\vec{x}$ represents a $V(\mathbb{R}^* \cap M)[\bar{g}]$-generic filter on the poset $P_{\vec{f}}$. By standard factoring facts about κ_{sym}, the model $V(\mathbb{R}^*)$ is a κ_{sym}-generic extension of the model $V(\mathbb{R}^* \cap M)[\bar{g}][\vec{x}]$. The homogeneity of the poset $P_{\vec{f}}$ and of the name κ_{sym} and the last three sentences give

(4.2.o) $V(\mathbb{R}^* \cap M)[\bar{g}] \models P_{\vec{f}} \Vdash \kappa_{sym} \Vdash$ there is a condition $p \in P_\beta$ below $\bigwedge g$ such that $p \Vdash \vec{x}_{gen} \in \vec{F}$ coordinatewise. Here P_β and $\vec{F}$ really stand for the definitions of P_β and $\vec{F}$ using elements of the model $V(\mathbb{R}^* \cap M)$.

Finally, (4.2.i) follows. If $\vec{x}$ is any sequence diagonalizing the filters in $\vec{f}$ coordinatewise then $\vec{x}$ represents a $V(\mathbb{R}^* \cap M)[\bar{g}]$-generic filter on the poset $P_{\vec{f}}$ and $V(\mathbb{R}^*)$ is a κ_{sym}-extension of the model $V(\mathbb{R}^* \cap M)[\bar{g}][\vec{x}]$. Thus (4.2.o) applies directly to give a lower bound $p \in P_\beta$ of the filter g such that $p \Vdash \vec{x} \in \vec{F}$ coordinatewise as desired.

4.3. Free trees. The case of free trees needs an additional trick. Work in a general ZFC context for a while.

4.3.1. Definition. [J1] Let S be the forcing adding a generic Souslin tree $\dot{T}_{gen}$ via countable approximations. That is, S consists of sets a such that for some ordinal $\alpha \in \omega_1$ called dom(a) a is a countable subset of ${}^{\alpha}\omega$, ordered by $a \geq b$ if dom$(a) \in$ dom(b) and $a = \{f \restriction \text{dom}(a) : f \in b\}$. The generic ω_1-tree $\dot{T}_{gen}$ is obtained as the union of all conditions in the generic filter.

4.3.2. Definition. Given an ω_1-tree T let P_T be a forcing for adding a *generic isomorph* of the tree T, that is, an S-generic tree $\dot{T}_{gen}$ and a function $\dot{F}_{gen} : \check{T} \to \dot{T}_{gen}$ which is an isomorphism of the two trees on a club. The poset P_T consists of triples $\langle a, c, f\rangle$ where $a \in S$, $c \subset$ dom(a) is a closed set and f is a level-preserving isomorphism of the trees $\{t \in T : \exists \alpha \in c\ t \in T_\alpha\}$ and $\{g \restriction \alpha : g \in a \text{ and } \alpha \in c\}$, the latter ordered by inclusion. The ordering is defined by $\langle a_0, c_0, f_0\rangle \geq \langle a_1, c_1, f_1\rangle$ if $a_0 \geq a_1$ in S, c_1 endextends c_0 and $f_0 \subset f_1$. The tree $\dot{T}_{gen}$ is obtained as the union of first coordinates of conditions in the P_T-generic filter and the isomorphism $\dot{F}_{gen}$ is the union of third coordinates of conditions in the filter. The domain of $\dot{F}_{gen}$ is the union of second coordinates of conditions in the generic filter denoted by $\dot{C}_{gen}$.

4.3.3. Claim.

(1) *The posets S, P_T are σ-closed of the size of continuum.*
(2) $P_T \Vdash \dot{T}_{gen}$ *is an S-generic tree and $\dot{F}_{gen}$ is an isomorphism of the trees $\check{T}$ and $\dot{T}_{gen}$ on the club $\dot{C}_{gen}$.*
(3) *If ϕ is a forcing formula then all conditions in the poset S decide the validity of $\phi(\dot{T}_{gen})$ in the same way.*

Now back to the model $V(\mathbb{R}^*)$. Suppose ϕ is a forcing formula and $P \Vdash \dot{T}$ is a free tree. I will show

(4.3.a) $P \Vdash \phi(\dot{T})$ just in case for every sufficiently large ordinal $\beta \in \lambda$, $P^{\beta+1} \times Add(1, \aleph_1) \times S \Vdash \phi(\dot{T}_{gen})$

which will prove Theorem 0.2(3) since the right hand side of the equivalence is independent of the choice of the name $\dot{T}$.

So fix an ordinal β such that $\kappa + \omega \in \beta \in \lambda$ and $\dot{T}$ is a P_β-name. Then $P_\beta \Vdash$CH+AC and so by Claim 4.3.3(1)$P_\beta \Vdash$ the posets $Q_\beta, Add(1, \aleph_1)$ and $P_{\dot{T}}$ are isomorphic. Therefore the poset $P_{\beta+1} = P_\beta \times Q_\beta$ is naturally isomorphic to $P_\beta * P_{\dot{T}}$ and for the sake of simplicity I will assume that $P_{\beta+1}$ is actually equal to $P_\beta * P_{\dot{T}}$. Let $\dot{T}_{gen}, \dot{C}_{gen}, \dot{F}_{gen}$ be names for the generic objects added by the $P_{\dot{T}}$ stage of that iteration and note that $P_{\beta+1} \Vdash \dot{T}_{gen}$ represents a $V(\mathbb{R}^*)$-generic filter on the poset S.

I will show that

(4.3.b) for every model $M \prec V_\theta$ as in (4.2.d,e), every M-generic filter $g \subset P_{\beta+1} \cap M$, and every condition $a \in S$ such that $\dot{T}_{gen}/g = \{f \restriction \alpha : f \in a \text{ and } \alpha \in M \cap \kappa\}$ there is a condition $p \in P_{\beta+1}$, $p \leq \bigwedge(g \cap P_{\beta+1})$ such that $p \Vdash \check{a}$ is in the S-generic filter determined by the tree $\dot{T}_{gen}$.

The rest of the proof can be essentially copied from the previous subsection. Namely, one shows that $P^{\beta+1} \times S \Vdash P_{\beta+1}/\dot{T}_{gen}$ is a poset of size $\aleph_1$ with a σ-closed dense subset, and so isomorphic to $Add(1, \aleph_1)$. Consequently, $P \Vdash \phi(\dot{T}_{gen})$ if and only if $P^{\beta+1} \times Add(1, \aleph_1) \times S \Vdash \phi(\dot{T}_{gen})$; also $P \Vdash \phi(\dot{T}_{gen}) \leftrightarrow \phi(\dot{T})$ since the two trees are isomorphic on a club via the map $\dot{F}_{gen}$ and ϕ is a forcing formula. This will complete the proof.

So fix the model M and an M-generic filter $g \subset P_{\beta+1} \cap M$ as in (4.2.d,e), and let $^-: M \to \bar{M}$ be the transitive collapse and $\bar{g}$ the image of g under the collapse. As in the previous subsection it can be verified that

(4.3.c) $\bar{g} \subset \bar{P}$ is an $\bar{M}$-generic filter

(4.3.d) $V_{\bar{\beta}+\omega} \subset \bar{M}$ and so $V(\mathbb{R}^* \cap M)$ is a $\bar{\kappa}_{sym}$-extension of V and $\bar{g} \subset \bar{P}$ is a $V(\mathbb{R}^* \cap M)$-generic filter

(4.3.e) $V(\mathbb{R}^* \cap M)[\bar{g}] \models \bar{T}/\bar{g}$ is a free tree.

(4.3.f) the filter $g \subset P_{\beta+1}$ has a lower bound in the poset $P_{\beta+1}$.

First let me consider the following general situation. Choose any lower bound $p \in P_\beta$ of the filter $g \subset P_\beta$ such that it decides the $\bar{\kappa}$-th level of the tree $\dot{T}$ to be some b, naturally identified with some countable collection of cofinal branches through the tree $\bar{T}/\bar{g}$. By Example 1.7, $V(\mathbb{R}^* \cap M)[\bar{g}](b)$ is a $(\bar{T}/\bar{g})_{sym}$-extension of the model $V(\mathbb{R}^* \cap M)[\bar{g}]$. By Claim 1.6 $V(\mathbb{R}^*)$ is a $Coll(b) * \kappa_{sym}$-extension of the model $V(\mathbb{R}^* \cap M)[\bar{g}](b)$. By the forcing theorem and Claim 1.6,

(4.3.g) $V(\mathbb{R}^* \cap M)[\bar{g}] \models (\bar{T}/\bar{g})_{sym} \Vdash Coll((\bar{T}/\bar{g})_{sym}) \Vdash \check{\kappa}_{sym} \Vdash$ there is a condition $p \in \dot{P}_\beta$ which is a lower bound of the filter $\check{g} \subset \dot{P}_\beta$ and $p \Vdash (\bar{T}/\bar{g})_{sym}$ is the $\bar{\kappa}$-th level of the tree $\dot{T}$.

Now (4.3.b) follows. Suppose $a \in S$ is a condition such that $\bar{T}_{gen}/\bar{g} = (\dot{T}_{gen}/g) \cap M = \{f \restriction \alpha : f \in a \text{ and } \alpha \in M \cap \kappa\}$. Consider the set b of all those cofinal branches x of the tree $\bar{T}/\bar{g}$ for which there is some function $d \in a$ such that $(\bar{F}_{gen}/\bar{g})''x \subset d$ holds. By Example 1.7 $V(\mathbb{R}^* \cap M)[\bar{g}](b)$ is a $(\bar{T}/\bar{g})_{sym}$-extension of the model $V(\mathbb{R}^* \cap M)[\bar{g}]$ and so by (4.3.g) there must be a condition $p \in P_\beta$ which is a lower bound of the filter $g \cap P_\beta$ and forces b to be the $\bar{\kappa}$-th level of the tree $\dot{T}$. Now let $c = \bar{C}_{gen}/\bar{g} \cup \{\bar{\kappa}\}$ and let $f = (\bar{F}_{gen}/\bar{g}) \cup \{x \mapsto \bigcup((\bar{F}_{gen}/\bar{g})''x) : x \in b\}$. It is now not hard to verify that $p^\frown\langle a, c, f\rangle$ is the lower bound of the filter g required for (4.3.b).

4.4. Lusin sets.

At this stage it should not be unreasonable to assume that the reader can prove Theorem 0.3(3) on his own along the lines of subsections 3.3 and 4.2.

4.5. Diamond sequences.

Here I have to assume that the embedding $j : V \to N$ the construction of the model is almost huge, that is, $j(\kappa) = \lambda$. Note that this implies that λ is a limit of inaccessible cardinals and more.

So work in the model $V(\mathbb{R}^*)$ and fix a P-name $\dot{d}$ for a good diamond sequence. I will prove two combinatorial facts about the name–Lemma 4.5.1 and 4.5.2–after which the proof is almost the same as the one in subsection 4.3, with forcing a generic isomorph of the sequence $\dot{d}$ and so on.

Fix an ordinal $\beta \in \lambda$ such that $\dot{d}$ is a P_β-name and choose a large inaccessible cardinal θ_0 between β and λ. Furthermore let $M_0 \prec V_{\theta_0}$ be an elementary submodel such that

(4.5.a) M_0 is countable, contains all relevant objects and $M_0 \cap V \in V$

(4.5.b) for every ordinal $\gamma \in \kappa$ there is an inaccessible cardinal θ_1 between θ_0 and λ and an elementary submodel $M_1 \prec V_{\theta_1^+}$ so that M_1 endextends M_0, for each $\alpha \in M_1 \cap \theta_1 + 1$ $M_1 \cap V_\alpha \cap V \in \bigcap(U_\alpha \cap M_1)$ and $V_\gamma \cap V$ is in the transitive collapse of the model $M_1 \cap V$.

It is an easy exercise to use the almost hugeness of the embedding j along the lines of Claims 4.2.1, 4.2.3 to get

4.5.1. Lemma. *The set of models M_0 as above is stationary and $P^\beta \Vdash$ this set actually contains a club.*

The key point is

4.5.2. Lemma. *Suppose M_0 is a model satisfying (4.5.a,b), $g \subset P_\beta \cap M_0$ is a M_0-generic filter and $r \in \mathbb{R}^*$ is a real. Then there is a lower bound of the filter g in the poset P_β such that it forces $\check{r} = \dot{d}(\check{M}_0 \cap \check{\kappa})$.*

Proof. Suppose this fails for some M_0, g, r. Let $^-\colon M_0 \to \bar{M}_0$ be the transitive collapse and $\bar{g}$ the pointwise image of the filter g under the bar map. As in (4.2.l,m,n), $V(\mathbb{R}^* \cap M_0)$ is a $\bar{\kappa}_{sym}$-extension of V and $\bar{g} \subset \bar{P}_\beta$ is a $V(\mathbb{R}^* \cap M_0)$-generic filter. By Claim 1.6, there is an inaccessible cardinal $\gamma \in \kappa$ of $V(\mathbb{R}^* \cap M_0)[\bar{g}]$ and a poset Y of size $< \gamma$ such that the real r is $V(\mathbb{R}^*)[\bar{g}]$-generic for the poset Y.

Now a little forcing theory. The $\kappa_{sym} * P_\beta$ extension of V satisfies the axiom of choice and therefore [J3, Lemma 25.3] it is a Z-extension for some poset $Z \in M_0 \cap V$. Then $V(\mathbb{R}^* \cap M_0)[\bar{g}]$ is a $\bar{Z}$-extension of V and there must be a $\bar{Z}$-name $\bar{\tau} \in V$ whose realization is Y. By the forcing theorem, there must be a condition $z \in Z$ such that

(4.5.c) $V \models \bar{z} \Vdash_{\bar{Z}} \kappa_{sym} \Vdash$ no lower bound of the filter $\dot{g}$ forces in $\dot{P}_\beta$ the real $\dot{d}(\bar{\kappa})$ to be $V^{\bar{Z}}$-generic via the poset $\bar{\tau}$.

By the assumptions (4.5.b) on the model M_0 there are an inaccessible cardinal θ_1 and a model $M_1 \prec V_{\theta_1^+}$ so that (4.5.b) holds with our γ. Let $\tau \in M_1 \cap V$ be a name such that its image under the transitive collapse of the model M_1 is $\bar{\tau}$. By the properties of the model M_1 in (4.5.b) and by (4.5.c) above,

(4.5.d) $V \models X = \{M \prec V_{\theta_1} \cap V :$ writing$\widetilde{\ }\colon M \to \widetilde{M}$ for the transitive collapse of the model M and $\dot{\mathbb{R}}, \dot{g}$ for the generic objects added by the forcing $\widetilde{Z}$, we have in $\widetilde{Z}$ that $\widetilde{z} \Vdash \check{\kappa}_{sym} \Vdash$ in $\dot{P}_\beta$, $\bigwedge \dot{g} \Vdash$ the real $\dot{d}(\widetilde{\kappa})$ is not $V(\dot{\mathbb{R}})[\dot{g}]$-generic for the poset $\widetilde{\tau}\} \in U_{\theta_1}$.

Now let $V(\mathbb{R}^*)[G_\beta][G^\beta]$ be an extension such that $G_\beta \subset P_\beta$ and $G^\beta \subset P^\beta$ are mutually $V(\mathbb{R}^*)$-generic filters and the V-generic filter on the poset Z given by $\mathbb{R}^*$ and G_β contains the condition z. Work in the model $V(\mathbb{R}^*)[G_\beta][G^\beta]$. Looking at the structure $V_{\theta_1} \cap V(\mathbb{R}^*)[G_\beta]$ we see that it can be decomposed as an increasing continuous union of models $\langle M_\alpha : \alpha \in \omega_1 = \kappa\rangle$ with $M_\alpha \cap V \in X$ since it has size $\aleph_1$ and the poset Q_{θ_1} adds a club through the set X. Coding this structure into an ω_1 sequence of reals and using the definitory property of the good diamond sequence $d = \dot{d}/G_\beta \times G^\beta$, there must be an ordinal $\alpha \in \kappa$ such that the real $d(M_\alpha \cap \kappa)$ is M_α-generic for the poset $\tau/\mathbb{R}^*, G_\beta$. However this contradicts the definition of the set X from which the model $M_\alpha \cap V$ comes. $\square$

The rest of the proof follows closely subsection 4.3.

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