Why is there only one definition of eigenvalue?

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Observation

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This is directly related to the fact that there is only one notion of graph quasirandomness.

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We usually define the eigenvalues of a matrix M to be the value λ so there is a vector v such that $Mv = \lambda v$. But this definition doesn't have a clear analog for a tensor: if v is a vector, Tv is a matrix, while if M is a matrix, $T \cdot M$ is a vector.

But there is an alternative definition of the eigenvalue of a matrix: an eigenvalue is a local maximum of the inner product

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This definition does make sense: we can define a *vector eigenvalue* of T to be a local maximum of

$$\lambda = \sum_{i,j,k} t_{ijk} v_i v_j v_k$$

over unit vectors v.

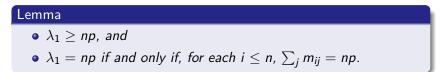
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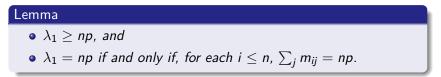
That is, suppose M is a large, square, symmetric, dense matrix:

- M is an $n \times n$ matrix with n big,
- $m_{ij} = m_{ji}$,
- the entries $m_{ij} \in [0,1]$,
- the average value $\sum_{i,j} m_{ij} = pn^2$.

Then the first eigenvalue measures the even distribution of M:

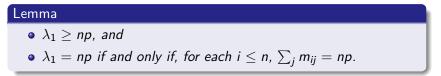


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Hint: consider the unit vector constantly equal to $1/\sqrt{n}$.

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Suppose \lambda_1 \approx np. We can then look at \lambda_2.
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Suppose we generate M randomly: for each ij, we flip a fair coin independently to decide whether m_{ij} is 0 or 1. Then, with high probability, $\lambda_1 \approx \frac{n}{2}$ and λ_2 is close to 0 (and therefore all other eigenvalues are also close to 0).

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$$\sum_i \lambda_i^4 = \operatorname{tr}(M^4) = \sum_{ijkl} m_{ij} m_{jk} m_{kl} m_{li}.$$

For most *ijkl*, these are four different pairs, so each quadruple has a $1/2^4$ chance of being present, so this sum is $\frac{1}{2^4}n^4$. If we think of M as the adjacency matrix of a graph, this amounts to counting the number of cycles of length 4.

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Since $\lambda_1 \approx \frac{n}{2}$, there is no room left for the other eigenvalues: $\sum_{1 < i \leq n} \lambda_i^4$ must be small. Furthermore, this is, suitably interpreted, an equivalence:

Theorem

There is a language \mathcal{L} for probabilistic matrices such that, for each sentence σ and each $p \in (0, 1)$, if σ is (with high probability) true in the random matrix with density p then there is an ϵ so that σ holds in any matrix with n large enough, $\lambda_1 - np < \epsilon$, and $|\lambda_2| < \epsilon$.

	The Second Eigenvalue	Other Notions
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For tensors, this doesn't work. λ_1 still measures even distribution:

Lemma

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$$\lambda_1 \geq n^{3/2}p$$
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but consider the tensor generated as follows:

• Generate a random matrix $B = [b_{ij}]$ by flipping a coin for each pair i, j.

• Set
$$t_{ijk} = b_{ij} + b_{ik} + b_{jk} \mod 2$$
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Then T is evenly distributed and has a small λ_2 , but is very different from a random tensor. For instance, in a random tensor we would have

$$\frac{1}{n^4} \sum_{ijkl} r_{ijk} r_{ijl} r_{ikl} r_{jkl} \approx 1/2^4$$

while

$$\frac{1}{n^4}\sum_{ijkl}t_{ijk}t_{ijl}t_{ikl}t_{jkl}\approx 1/2^3.$$

We could have defined a different kind of eigenvalue:

Definition

When T is a symmetric tensor, a matrix eigenvalue of T is a value λ which is a local maximum of

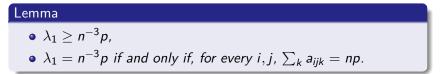
$$\sum_{i,j,k} t_{ijk} m_{ij} m_{ik} m_{jk}$$

among unit matrices M. We call such an M an eigenmatrix of T.

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The second *matrix* eigenvalue does actually measure deviation from quasirandomness.

But there are other notions:

• eigenvalues are local maxima of $\sum_{i,j,k} t_{ijk} m_{ij} v_k$ among unit matrices M and unit vectors v (this is the notion related to Tao's *slice rank*),

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• etc.

Suppose we have a *d*-tensor. There is a notion of eigenvalue corresponding to every antichain of proper subsets of $\{1, \ldots, d\}$, identifying antichains under permutations of $\{1, \ldots, d\}$.

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So for 3-tensors, we have antichains like:

- $\{\{1\},\{2\},\{3\}\}$ corresponding to vector eigenvalues,
- $\{\{1,2\},\{1,3\},\{2,3\}\}$ corresponding to matrix eigenvalues,
- {{1,2},{3}},
- $\{\{1,2\},\{1,3\}\},\$
- etc.

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The end.