Transcendental pairs of generic extensions*

Jindřich Zapletal
University of Florida

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Abstract

We isolate a new preservation class of Suslin forcings and prove several associated consistency results in the choiceless theory ZF+DC. For example, writing Γ_n for the hypergraph on $\mathcal{P}(\omega)$ consisting of n-tuples which modulo finite form a partition of ω, it is consistent with ZF+DC that the chromatic number of Γ_3 is countable, yet the chromatic number of Γ_4 is not.

1 Introduction

Following the initial work on geometric set theory [7], I isolate a new class of balanced forcings, the *transcendental balanced forcings*, verify that a number of partial orders belong to it, and show a number of preservation theorems for the extension of the Solovay model by forcings in this class. In this way, I obtain several independence results in the choiceless ZF+DC set theory regarding chromatic numbers of certain Borel graphs and hypergraphs. Recall that a *hypergraph* Γ of arity n on a set X is just a subset of $[X]^n$, its elements are its *hyperedges*. A (partial) function c on X is a Γ-*coloring* if c is not constant on any Γ-hyperedge. The chromatic number of Γ is the smallest cardinal κ such that there is a total Γ-coloring $c\colon X \to \kappa$. In the absence of the axiom of choice, we only discern between various finite values of the chromatic number and then countable and uncountable chromatic number. The study of chromatic numbers of Borel hypergraphs has long history [9, 6, 8, 1, 2]; in the choiceless ZF+DC theory, inequalities between chromatic numbers of such hypergraphs are subject to many consistency results [11, 7].

For a Polish group G one may consider hypergraphs on G associated with various equations, and their chromatic numbers. Consider the hypergraph $\Delta(G)$ of quadruples which solve the equation $g_0 g_1^{-1} g_2 g_3^{-1} = 1$. Note that in ZF, the countable chromatic number of $\Delta(G)$ is inherited by subgroups of G. I prove

Theorem 1.1. *Let G be a K_σ Polish group. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of $\Delta(G)$ is countable, yet the chromatic number of $\Delta(S_\infty)$ is uncountable.*

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This consistency result depends on the fact that S_∞ has (a small strengthening of) the ample generics property. It seems to be difficult to separate chromatic numbers of the hypergraphs $\Delta(G)$ for various other Polish groups, such as $G = S_\infty$ and $G =$the unitary group.

For a natural number $n \geq 2$ let Γ_n be the hypergraph on $\mathcal{P}(\omega)$ of arity n consisting of n-tuples of sets which modulo finite form a partition of ω. In the presence of a nonprincipal ultrafilter on ω, the chromatic number of each hypergraph Γ_n is two, as the membership in the ultrafilter constitutes a Γ_n-coloring. Without an ultrafilter, I learned how to separate chromatic numbers of Γ_n's:

Theorem 1.2. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ_3 is countable, while the chromatic number of Γ_4 is uncountable.*

Theorem 1.3. *It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ_4 is countable, while the chromatic number of Γ_5 is uncountable.*

It appears more challenging to separate the chromatic numbers of Γ_n for higher values of n, and I do not know how to do that.

For every number $n \geq 2$ let Θ_n be the hypergraph of arity n on $\mathcal{P}(\omega)$ consisting of sets d of size n such that $\bigcap d = 0$ and $\bigcup d = \omega$, both modulo finite. A membership in a nonprincipal ultrafilter on ω provides a Θ_n coloring with two colors. However, coloring Θ_n without an ultrafilter seems to be a great challenge for larger values of n. Θ_2 is locally countable and so can be colored by rather innocuous posets of [7, Section 6.4]. I do not know how to obtain a model where the chromatic number of Θ_3 is countable without an ultrafilter. For Θ_4, I have a preservation theorem.

Theorem 1.4. *Let κ be an inaccessible cardinal. In cofinally transcendentally balanced forcing extensions of the symmetric Solovay model derived from κ, the chromatic number of Θ_4 is uncountable.*

This rules out nonprincipal ultrafilters on ω in transcendentally balanced extensions. (Diffuse finitely additive probability measures on ω do not exist there either for a different chromatic number reason.) The important point here is that most known balanced Suslin posets which do not add an ultrafilter are transcendentally balanced. This includes for example the posets adding a transcendence basis for $\mathbb{R}$ over $\mathbb{Q}$, the coloring posets in [7, Section 8.2], and also the coloring posets introduced in the last section of the present paper. I do not know how to find a model of ZF+DC in which the chromatic number of Θ_4 is neither equal to 2 nor uncountable.

In Section 2, I introduce transcendence of pairs of generic extensions, a property weaker than mutual genericity, with several properties of mutually transcendental pairs of generic extensions. In Section 3 I provide a number of useful examples of mutually transcendental pairs. In Section 4, I define the notion of transcendental balance for Suslin forcing, and examples of Section 4 are used to prove a number of preservation theorems for generic extensions of the Solovay model obtained with transcendentally balanced forcings. Finally, in Section 5, I show that many known balanced forcings are transcendentally balanced, and build a new supply of coloring posets which are transcendentally balanced. These are in turn used to prove the theorems of this introduction.

Notation of the paper follows [3], and in matters of geometric set theory [7]. In particular, the calculus of virtual conditions in Suslin forcing of Section 4 is established in [7, Chapter 5].

2 Transcendental pairs of extensions

The key concept in this paper is a certain perpendicularity notion for pairs of generic extensions, which generalizes mutual genericity.

Definition 2.1. Let $V[G_0], V[G_1]$ be two generic extensions in an ambient generic extension. Say that $V[G_1]$ is *transcendental over* $V[G_0]$ if for every ordinal α and every open set $O \subset 2^\alpha$ in the model $V[G_1]$, if $2^\alpha \cap V \subset O$ then $2^\alpha \cap V[G_0] \subset O$. Say that the models $V[G_0]$, $V[G_1]$ are *mutually transcendental* if each of them is transcendental over the other one.

Here, the space 2^α is equipped with the usual compact product topology. For a finite partial function h from α to 2 write $[h] = \{x \in 2^\alpha \colon h \subset x\}$ The open set $O \subset 2^\alpha$ is then coded in $V[G_1]$ by a set $H \in V[G_1]$ of finite partial functions from α to 2 with the understanding that $O = \bigcup_{h \in H}[h]$, and in this way it is interpreted in $V[G_0][G_1]$. For the general theory of interpretations of topological spaces in generic extensions (unnecessary for this paper) see [10]. It is tempting to deal just with the Cantor space 2^ω instead of its non-metrizable generalizations, but the present definition has a number of small advantages and essentially no disadvantages as compared to the Cantor space treatment. On the other hand, extending the definition to all compact Hausdorff spaces is equivalent to the present form.

I first need to show that the notion of transcendence generalizes mutual genericity and in general behaves well with respect to product forcing. This is the content of the following proposition.

Proposition 2.2. *Let $V[G_0], V[G_1]$ be generic extensions such that $V[G_1]$ is transcendental over $V[G_0]$. Let $P_0 \in V[G_0]$ and $P_1 \in V[G_1]$ be posets. Let $H_0 \subset P_0$ and $H_1 \subset P_1$ be filters mutually generic over the model $V[G_0][G_1]$. Then $V[G_1][H_1]$ is transcendental over $V[G_0][H_0]$.*

Proof. Work in the model $V[G_0, G_1]$ and consider the product forcing $P_0 \times P_1$. Let α be an ordinal. Let $\langle p_0, p_1\rangle \in P_0 \times P_1$ be a condition, let $\tau \in V[G_1]$ be a P_1-name for an open set such that $p_1 \Vdash 2^\alpha \cap V \subset \dot{O}$, and let $\eta \in V[G_0]$ be a P_0-name for an element of 2^α. We need to find a stronger condition in the product which forces $\eta \in \tau$.

First, work in the model $V[G_0]$. Let $A_0 = \{h \colon h$ is a finite partial function from α to 2 and $p_0 \Vdash \check{h} \not\subset \eta\}$. By a compactness argument, the set $\bigcup_{h \in A_0}[h]$

must not cover the whole space 2^α; if it did, a finite subset of Q_0 would suffice to cover 2^α and there would be no space left for the point η. Let $y \in 2^\alpha \setminus \bigcup_{h \in A_0}[h]$ be an arbitrary point. Now work in the model $V[G_1]$ and let $A_1 = \{h \colon h$ is a finite partial function from α to 2 and there is a condition $q \leq p_0$ such that $q \Vdash [h] \subset \tau\}$; since $p_0 \Vdash 2^\alpha \cap V \subset \tau$, it must be the case that $2^\alpha \cap V \subset \bigcup_{h \in A_1}[h]$.

Now, use the transcendence of $V[G_1]$ over $V[G_0]$ to argue that $y \in Q$. It follows that there must be a finite partial function $h \in A_1$ such that $h \subset y$. It follows that there must be conditions $p_0' \leq p_0$ and $p_1' \leq p_1$ such that $p_0' \Vdash \check{h} \subset \eta$ and $p_1' \Vdash [h] \subset \tau$. Then the condition $\langle p_0', p_1' \rangle$ forces in the product $P_0 \times P_1$ that $\eta \in \tau$ as required. □

Corollary 2.3. *Mutually generic extensions are mutually transcendental.*

Proof. Just let $V[G_0] = V[G_1] = V$ in Proposition 2.2. □

In the remainder of this section, I isolate several properties of mutually transcendental extensions which will come handy later.

Proposition 2.4. *Let $V[G_0], V[G_1]$ be mutually transcendental generic extensions of V. Then $V[G_0] \cap V[G_1] = V$.*

Proof. It will be enough to show that $(2^\alpha \cap V[G_0]) \cap (2^\alpha \cap V[G_1]) = 2^\alpha \cap V$ holds for every ordinal α. Let $x \in 2^\alpha \cap V[G_0] \setminus V$ be an arbitrary point. The open set $O = 2^\alpha \setminus \{x\}$ in $V[G_0]$ covers $2^\alpha \cap V$. By the mutual transcendence, $2^\alpha \cap V[G_1] \subset O$ must hold as well. In particular, $x \notin V[G_1]$ as required. □

Proposition 2.5. *Let $V[G_0], V[G_1]$ be mutually transcendental generic extensions of V. Let X_0, X_1 be Polish spaces and $C \subset X_0 \times X_1$ be a K_σ-set. Let $x_0 \in X_0 \cap V[G_0]$ and $x_1 \in X_1 \cap V[G_1]$ be points such that $\langle x_0, x_1 \rangle \in C$. Then there is a point $x_0' \in X_0 \cap V$ such that $\langle x_0', x_1 \rangle \in C$.*

Proof. Since C is a countable union of compact sets, there is a compact set $K \subset C$ coded in V such that $\langle x_0, x_1 \rangle \in K$. Let $h \colon 2^\omega \to X_0$ be a continuous function onto the compact projection of the set K to X_0. Let $O \subset 2^\omega$ in $V[G_1]$ be the open set of all points $y \in 2^\omega$ such that $\langle f(y), x_1 \rangle \notin K$. The set O does not cover $2^\omega \cap V[G_0]$ since $h^{-1}x_0 \cap O = 0$. By the mutual transcendence, there must be a point $y \in 2^\omega \cap V \setminus O$. Let $x_0' = h(y)$ and observe that the point $x_0' \in X_0$ works as desired. □

Corollary 2.6. *Let X be a K_σ Polish field. Let $p(\bar{v}_0, \bar{v}_1)$ be a multivariate polynomial with coefficents in X and variables $\bar{v}_0, \bar{v}_1$. Let $V[G_0], V[G_1]$ be mutually transcendental generic extensions of V and let $\bar{x}_0 \in V[G_0]$, $\bar{x}_1 \in V[G_1]$ be strings of elements of X such that $p(\bar{x}_0, \bar{x}_1) = 0$. Then there is a string $\bar{x}_0' \in V$ arbitrarily close to $\bar{x}_0$ such that $p(\bar{x}_0', \bar{x}_1) = 0$.*

Proof. Apply the proposition with the additional insight that the spaces X^n for any natural number n are K_σ and solutions to a given polynomial form a closed set. □

Corollary 2.7. *Let E be a K_σ-equivalence relation on a Polish space X. Whenever $V[G_0], V[G_1]$ are mutually transcendental generic extensions of V and $x_0 \in X \cap V[G_0]$ and $x_1 \in X \cap V[G_1]$ are E-related points, then there is a point $x \in X \cap V$ E-related to them both.*

The last corollary can be generalized to some non-K_σ-equivalence relations as in the following proposition.

Proposition 2.8. *Let $\langle U_n, d_n : n \in \omega\rangle$ be a sequence of sets and metrics on each and let $X = \prod_n U_n$. Let $V[G_0], V[G_1]$ be mutually transcendental generic extensions of V and $x_0 \in X \cap V[G_0]$ and $x_1 \in X \cap V[G_1]$ be points such that $\lim_n d_n(x_0(n), x_1(n)) = 0$. Then there is a point $x \in X \cap V$ such that $\lim_n d_n(x(n), x_0(n)) = 0$.*

Proof. First argue that for every number $m \in \omega$ there is a point $y \in X \cap V$ such that $\forall n\ d_n(y(n), x_1(n)) \leq 2^{-m}$. To see this, fix a number $k \in \omega$ such that for all $n \geq k$, $d_n(x_0(n), x_1(n)) < 2^{-m-2}$. Let $A = \{\{u, v\} : \exists n \geq k\ u, v \in U_n$ and $d_n(u, v) > 2^{-m}\}$, and consider the space Z of all selectors on A, which is naturally homeomorphic to 2^A. In the model $V[G_0]$, let $O = \{z \in Z : \exists n \geq k\ \exists v \in U_n\ d_n(x_0(n), v) > 2^{-m}$ and $z(x_0(n), v) = v\}$. This is an open subset of the space Z. It does not cover $Z \cap V[G_1]$ as in the model $V[G_1]$, one can find a selector $z \in Z$ such that for all $n \geq k$ and all $\{u, v\} \in Z$ with $u, v \in U_n$, $z(u, v)$ is one of the points u, v which is not d_n-farther from $x_1(n)$ than the other. It is immediate from the definition of the set O and a triangle inequality argument that $z \notin O$. By a mutual transcendence argument, there is a selector $z' \in Z \cap V$ such that $z' \notin O$ holds.

Work in V. For each number $n \geq k$, let $B_n = \{u \in U_n : \forall v \in Y_n\ d_n(u, v) > 2^{-m} \to z'(u, v) = u\}$. The set B_n contains $x_0(n)$ by the choice of the selector z'. Moreover, for any two elements $u, v \in B_n$, $d_n(u, v) \leq 2^{-m}$ must hold: in the opposite case, the selector z' could not choose one element from the pair $\{u, v\}$ without contradicting the definition of the set B_n. Now consider any point $y \in X$ such that for all $n < k$, $y(n) = x_0(k)$ and for all $n \geq k$ $y(n) \in B_n$. Then $\forall n\ d_n(y(n), x_1(n)) \leq 2^{-m}$ as desired.

Now, let $C = \omega \times (X \cap V)$ and consider the set $B \subset C$ of all pairs $\langle m, y\rangle \in A$ such that $\limsup_n d_n(y(n), x_1(n)) \leq 2^{-m}$. As written, the set belongs to $V[G_1]$; however, it also belongs to $V[G_0]$ since replacing x_1 in its definition with x_0 results in the same set by the initial assumptions on x_0, x_1. By Proposition 2.4, $B \in V$ holds. By the work in the previous paragraph, for each $m \in \omega$ B contains some element whose first coordinate is m. Thus, in V there exists a sequence $\langle y_m : m \in \omega\rangle$ such that $\forall m\ \langle m, y_m\rangle \in B$. By a Mostowski absoluteness argument, there must be in V a point x such that for all $m \in \omega$, $\limsup_n (y_m(n), x(n)) \leq 2^{-m}$, since such a point, namely x_0, exists in $V[G_0]$. A triangular inequality argument then shows that $\lim_n (x(n), x_0(n)) = 0$ as desired. $\square$

I do not know whether further generalizations are possible. In particular, the following is open:

Question 2.9. Let E be a pinned Borel equivalence relation on a Polish space X. Let $V[G_0], V[G_1]$ be mutually transcendental generic extensions of V and $x_0 \in X \cap V[G_0]$ and $x_1 \in X \cap V[G_1]$ are E-related points. Must there be a point $x \in X \cap V$ E-related to them both?

3 Examples I

In this section, I provide several interesting pairs of mutually transcendental pairs of generic extensions. To set up the notation, for a Polish space X, write P_X for the Cohen poset of nonempty open subsets of X ordered by inclusion, with $\dot{x}_{gen}$ being its name for a generic element of X. If $f \colon X \to Y$ is a continuous open map then P_X forces $f(\dot{x}_{gen}) \in Y$ to be a point generic for P_Y [7, Proposition 3.1.1]. The first definition and proposition deal with Cohen elements of Polish spaces.

Definition 3.1. Let X, Y_0, Y_1 be compact Polish spaces and $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ be continuous open maps. Say that f_1 is *transcendental over* f_0 if for every nonempty open set $O \subset X$ there is a point $y_0 \in Y_0$ such that the set $f_1''(f_0^{-1}\{y_0\} \cap O) \subset Y_1$ has nonempty interior for every nonempty open set $O \subset X$.

Proposition 3.2. *Suppose that X, Y_0, Y_1 are Polish spaces, X is compact, and $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ are continuous open maps. The following are equivalent:*

1. *f_1 is transcendental over f_0;*
2. *P_X forces $V[f_1(\dot{x}_{gen})]$ to be transcendental over $V[f_0(\dot{x}_{gen})]$.*

Proof. To show that (1) implies (2), let α be an ordinal, let η be a P_{Y_0}-name for an element of 2^α, and let τ be a P_{Y_1}-name for an open subset of 2^α which is forced to contain $V \cap 2^\alpha$ as a subset. Let $O \subset X$ be a nonempty open set. To prove (2), I must find a strengthening $O' \subset O$ such that $O' \Vdash \eta/f_0(\dot{x}_{gen}) \in \tau/f_1(\dot{x}_{gen})$.

To this end, let $y_0 \in Y_0$ be a point such that the set $f_1''(f_0^{-1}\{y_0\} \cap O)$ has nonempty interior, and let $O_1 \subset Y_1$ denote that interior. Use the initial assumption on τ to find, for each $z \in 2^\alpha$, a condition $O_{1z} \subset O_1$ and a finite partial map $h_z \colon \alpha \to 2$ such that $h_z \subset z$ and $O_{1z} \Vdash [h_z] \subset \tau$. Use a compactness argument to find a finite set $a \subset 2^\alpha$ such that $2^\alpha = \bigcup_{z \in a}[h_z]$. The set $O_0 = \bigcap_{z \in a} f_0''(O \cap f_1^{-1}O_{1z} \subset Y_0$ is nonempty as it contains y_0, and it is open as the maps f_0, f_1 are continuous and open. Let $O_0' \subset O_0$ be a condition which decides the value $\eta(\check{\beta})$ for every ordinal $\beta \in \bigcup_{z \in a} \mathrm{dom}(h_z)$. Since $\bigcup_{z \in a}[h_z] = 0$, there must be a point $z \in a$ such that $O_0' \Vdash \check{h}_z \subset \eta$. The set $O' = O \cap f_0^{-1} O_0' \cap f^{-1} O_{1z}$ is nonempty and open, and it forces in P_X that $\eta/f_0(\dot{x}_{gen}) \in [h_z]$ and $[h_z] \subset \tau/f_1(\dot{x}_{gen})$.

The implication (2)→(1) is best proved by a contrapositive. Suppose that (1) fails, as witnessed by some open set $O \subset X$. Let $O' \subset O$ be some nonempty open set whose closure is a subset of O, and let $x \in O'$ be a point P_X-generic over the ground model. Write $y_0 = f_0(x) \in Y_0$ and $y_1 = f_1(x) \in Y_1$. Let $C = f_0''(f_1^{-1}\{y_1\} \cap \bar{O}')$. This is a closed subset of Y_0 coded in $V[y_1]$ which contains the point y_0. For the failure of (3), it is enough to show that C contains no ground model point. Indeed, if $y \in Y_0$ is a point in the ground model, then $D = f_1''(f_0^{-1}\{y\} \cap \bar{O}') \subset Y_1$ is a closed subset of Y_1 coded in the ground model which has empty interior by the choice of the set O; in particular, $D \subset Y_1$ is nowhere dense, and since $y_1 \in Y_1$ is a Cohen generic point, $y_1 \notin D$ holds. Comparing the definitions of the sets C and D, it is obvious that $y_0 \notin C$ as required. □

Example 3.3. Let b be a finite set and let $b = a_0 \cup a_1$ be a partition into two sets, each of cardinality at least two. Let X be the closed subset of $\mathcal{P}(\omega)^b$ consisting of those functions x such that $\bigcap_{i \in a} x(i) = 0$ and $\bigcup_{i \in a} x(i) = \omega$. Let $Y_0 = \mathcal{P}(\omega)^{a_0}$ and $Y_1 = \mathcal{P}(\omega)^{a_1}$. Let $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ be the projection functions. Then f_0, f_1 are continuous, open, and transcendental over each other.

Proof. The continuity and openness are left to the reader. To show that f_1 is transcendental over f_0, let $O \subset X$ be a nonempty relatively open set. Thinning the set O down if necessary, one can find a natural number $k \in \omega$ and sets $c_i \subset k$ for $i \in b$ such that $\bigcup_i c_i = k$ and $\bigcap_i c_i = 0$, and $O = \{x \in X \colon \forall i \in b\ x(i) \cap k = c_i\}$. Now let $y_0 \in Y_0$ be any point such that $\forall i \in a_0\ y_0(i) \cap k = c_i$ and for some $i \in a_0\ y_0(i) \subset k$, and for another $i \in a_0\ \omega \setminus k \subset y_0(i)$. There is such a point because $|a_0| \geq 2$ holds by the assumptions. Now, it is clear that the set $f_1''(f_0^{-1}\{y_0\} \cap O)$ is exactly the open set of all points $y_1 \in Y_1$ such that $\forall i \in a_1\ y_1(i) \cap k = c_i\}$. □

Example 3.4. Let b be a finite set and let $b = a_0 \cup a_1$ be a partition into nonempty sets. Let X, Y_0, Y_1 be the closed subsets of $\mathcal{P}(\omega)^b$, $\mathcal{P}(\omega)^{a_0}$, and $\mathcal{P}(\omega)^{a_1}$ consisting of tuples of pairwise disjoint subsets of ω respectively. Let $f_0 \colon X \to Y_0$ and $f_1 \colon X \to Y_1$ be the projection functions. Then f_0, f_1 are continuous, open, and mutually transcendental functions.

Proof. The continuity and openness is left to the reader. For the transcendental part, I will show that f_1 is transcendental over f_0. Let $O \subset X$ be a relatively open nonempty set. Find finite sets $c_i, d_i \subset \omega$ for each $i \in b$ such that $c_i \cap d_i = 0$ and the set $\{\langle z_i \colon i \in b\rangle \in \mathcal{P}(\omega) \colon \forall i \in b\ c_i \subset z_i \text{ and } d_i \cap z_i = 0\} \cap X$ is a nonempty subset of O. Note that the sets c_i for $i \in b$ must be pairwise disjoint, and we may arrange the sets d_i so that if $i, j \in b$ are distinct elements then $c_i \subset d_j$. Let $y_0 = \langle c_i \colon i \in a_0\rangle$ and let $O_1 = \{\langle z_i \colon i \in a_0\rangle \in \mathcal{P}(\omega) \colon \forall i \in a_0\ c_i \subset z_i$ and $d_i \cap z_i = 0\} \cap Y_1$; this is a nonempty open subset of Y_1. It is clear that for each point $y_1 \in O_1$, $\langle y_0, y_1\rangle \in O$ holds and the proof is complete. □

Another class of examples of transcendental pairs of generic extensions comes from actions of Polish groups with dense diagonal orbits [5]. I am going to need a local variant of this notion which appears to be satisfied in all natural actions with dense diagonal orbits. Recall that if a group G acts on a set X, then it also acts coordinatewise on the set X^n for every natural number n.

Definition 3.5. Let G be a Polish group acting continuously on a Polish space X. The action has

1. *dense diagonal orbits* if for every $n \in \omega$ there is a point $\vec{x} \in X^n$ such that $\{g \cdot \vec{x} \colon g \in G\}$ is dense in X^n.
2. *locally dense diagonal orbits* if for every open neighborhood $U \subset G$ of the unit and every nonempty open set $O \subset X$ there is a nonempty open set $O' \subset O$ such that for every $n \in \omega$ there is a point $\vec{x} \in X^n$ such that $\{g \cdot \vec{x} \colon g \in U\}$ is dense in $(O')^n$.

Proposition 3.6. *Let G be a Polish group acting on a Polish space X with locally dense diagonal orbits. Let $Y \subset G \times X^2$ be the closed set of all triples $\langle g, x_0, x_1\rangle$ such that $g \cdot x_0 = x_1$. Let P_Y be its associated Cohen forcing and $\langle \dot{g}, \dot{x}_0, \dot{x}_1\rangle$ its names for the generic triple. P_Y forces the following:*

1. *$\dot{g}$ is P_G-generic over V;*
2. *$\langle \dot{x}_0, \dot{x}_1\rangle$ is P_{X^2}-generic over V;*
3. *the model $V[\dot{x}_0, \dot{x}_1]$ is transcendental over $V[\dot{g}]$.*

Proof. For the first item, let $p \in P_Y$ be a condition and $D \subset G$ an open dense set. I must find a stronger condition which forces $\dot{g}$ into D. There are nonempty open neighborhoods $U \subset G$ and $O \subset X$ such that $\langle g, x, g \cdot x\rangle \in p$ whenever $g \in U$ and $x \in O$. Now, just note that the set $D \cap U$ is nonempty; therefore the set of all triples $\langle g, x, g \cdot x\rangle$ where $g \in U \cap D$ and $x \in O$ is a nonempty relatively open subset of Y which forces $\dot{g} \in D$ as desired.

For the second item, suppose first that $p \in P_Y$ is a condition and $D \subset X^2$ is an open dense set. I must find a stronger condition which forces the pair $\langle \dot{x}_0, \dot{x}_1\rangle$ into D. There is a point $g \in G$, an open neighborhood $U \subset G$ of the unit, and an open set $O \subset X$ such that $\langle gh, x_0, gh \cdot x_0\rangle \in p$ for all $h \in UU^{-1}$ and $x_0 \in O$. Use the dense orbit assumption to thin out the set O if necessary so that for every $n \in \omega$ there is a point $\vec{x} \in X^n$ such that $\{h \cdot \vec{x} \colon h \in U\}$ is dense in O^n. Since the set $D \subset X^2$ is open dense, there are open sets $P_0 \subset O$ and $P_1 \subset gO$ such that $P_0 \times P_1 \subset D$. By the choice of the set $O \subset X$, there must be a point $x \in P_0$ and a point $h \in UU^{-1}$ such that $hx \in g^{-1}P_1$, in other words $ghx \in P_1$. Now the relatively open set of all triples in p such that their second and third coordinates belong to P_0 and P_1 respectively is nonempty, and it forces $\langle \dot{x}_0, \dot{x}_1\rangle \in D$ as required.

For the third item, suppose that α is an ordinal, τ is a P_{X^2}-name for an open subset of 2^α which is forced to contain $V \cap 2^\alpha$, and η is a P_G-name for an element of 2^α. Suppose that $p \in P_Y$ is a condition. One can find an element $h \in G$, an open neighborhood $U \subset G$ of the unit, and a nonempty open set $O \subset X$ such that $\langle gh, x_0, gh \cdot x_0\rangle \in p$ for all $h \in UU^{-1}$ and $x_0 \in O$. Use the dense orbit assumption to thin out the set O if necessary so that for every $n \in \omega$ there is a point $\vec{x} \in X^n$ such that $\{h \cdot \vec{x} \colon h \in U\}$ is dense in O^n.

Now, use the initial assumption on the name τ to find, for each $z \in 2^\alpha$, a finite partial map $h_z \colon \alpha \to 2$ and a condition $O_{0z} \times O_{1z} \subset O \times gO$ such that $h_z \subset z$ and $O_{0z} \times O_{1z} \Vdash [h_z] \subset \tau$. Use a compactness argument to find a finite set $a \subset 2^\alpha$ such that $2^\alpha \subset \bigcup_{z \in a} [h_z]$. Now, the dense orbit assumption provides points $x_{0z} \in O_{0z}$ and a point $h \in UU^{-1}$ such that for each $z \in a$, $h \cdot x_{0z} \in g^{-1} O_{1z}$, or in other words $gh \cdot x_{0z} \in O_{1z}$. Let $U' \subset UU^{-1}$ be an open neighborhood such that for all $k \in U'$ and all $z \in a$, $gk \cdot x_{0z} \in O_{1z}$. Shrinking U' if necessary, assume that gU' decides the value of $\eta \restriction \bigcup_{z \in a} \operatorname{dom}(h_z)$. By the choice of the set a, there must be a point $z \in a$ such that $gU' \Vdash \check{h}_z \subset \eta$. Now the relatively open set of all triples in p whose coordinates belong to gU', O_{0z} and O_{1z} respectively is nonempty, and it forces $\eta \in \tau$ as desired. $\square$

Example 3.7. Let Y be the closed subset of S_∞^4 consisting of all quadruples $\langle g_0, g_1, g_2, g_3 \rangle$ such that $g_0 g_1^{-1} g_2 g_3^{-1} = 1$. The Cohen poset P_Y adds a generic quadruple $\langle \dot{g}_0, \dot{g}_1, \dot{g}_2, \dot{g}_3 \rangle$. It forces $V[\dot{g}_0, \dot{g}_2]$ and $V[\dot{g}_1, \dot{g}_3]$ to be mutually transcendental $P_{S_\infty^2}$-generic extensions of the ground model.

Proof. Consider the continuous action of S_∞^2 on S_∞ given by $(h_0, h_2) \cdot h_1 = h_0 h_1 h_2^{-1}$. It has locally dense diagonal orbits: if $U \subset (S_\infty)^2$ is an open neighborhood of the unit and $O \subset S_\infty$ is a nonempty open set, then thinning down one may assume that there is a number $n \in \omega$ such that $U = \{\langle h_0, h_2 \rangle \colon h_0 \restriction n = h_1 \restriction n \text{ is the identity}\}$ and $O = \{h_1 \colon [v] \subset h_1\}$ for some permutation v of n. Then the action of U on O is naturally homeomorphic to the whole action of S_∞^2 on S_∞. That action though has dense diagonal orbits because already the conjugation action of S_∞ on S_∞ has them [5].

Now, to show for example that P_Y forces $V[\dot{g}_1, \dot{g}_3]$ to be transcendental over $V[\dot{x}_0, \dot{x}_1]$, consider the self-homeomorphism of S_∞^4 which takes inverses of the second and third coordinates. Note that $\langle g_0, g_1, g_2, g_3 \rangle \in Y$ iff $(g_0, g_2^{-1}) \cdot g_1^{-1} = g_3$ and apply Proposition 3.6. $\square$

The last class of examples in this section deals with posets other than the Cohen poset.

Definition 3.8. Let P be a Suslin partial order. Say that P is *Suslin-σ-centered* if $P = \bigcup_{n \in \omega} A_n$ where each set $A_n \subset P$ is analytic and centered.

Proposition 3.9. *Let P_1 be a Suslin poset which is Suslin-σ-centered. Let $V[G_0]$ be an arbitrary generic extension, let $H \subset P_1$ be a filter generic over $V[G_0]$ and let $G_1 = H \cap V$. Then the extensions $V[G_0], V[G_1]$ are mutually transcendental.*

Proof. I use an apparently novel abstract combinatorial property of σ-linked posets encapsulated in the following claim.

Claim 3.10. *Let Q be a σ-linked poset, and $B \subset [Q]^{<\aleph_0}$ be a family of finite subsets of Q such that for every condition $q \in Q$ there is a set in B consisting* *only of conditions stronger than q. Then there is a countable set $C \subset B$ such that every condition in Q is compatible with every condition in some set in C.*

Proof. Let $Q = \bigcup_n D_n$ be a partition of Q into linked subsets. Let M be a countable elementary submodel of a large structure containing this partition and the set B; we claim that $C = B \cap M$ works. Suppose that $q \in Q$ is a condition and find a set $b \in B$ which consists solely of conditions stronger than q. Let $e = \{n \in \omega\colon b \cap D_n \neq 0\}$; this is a finite set and therefore $e \in M$. By elementarity of the model M, there is a set $c \in C$ such that $e = \{n \in \omega\colon c \cap D_n \neq 0\}$. Since the sets D_n for $n \in \omega$ are linked, it follows that q is compatible with each element of c as desired. □

Let P_0 be the poset generating the $V[G_0]$ extension and work in V. Fix a cover $P_1 = \bigcup_n A_n$ consisting of analytic centered sets. To see that $V[G_1]$ is transcendental over $V[G_0]$, work in V and fix and ordinal α. Suppose that τ is a P_1-name for an open subset of 2^α such that $P_1 \Vdash 2^\omega \cap V \subset \tau$. We will argue that $P_0 * \dot{P}_1 \Vdash 2^\omega \cap V[\dot{G}_0] \subset \tau/\dot{G}_1$. To this end, for each condition $p_1 \in P_1$ let O_{p_1} be the union of all basic open subsets of 2^α which p_1 forces to be subsets of τ; in particular, $p_1 \Vdash O_{p_1} \subset \tau$. Let $B \subset P_1^{<\aleph_0}$ be the set of all finite sets $b \subset P_1$ such that $\bigcup_{p_1 \in b} O_{p_1} = 2^\alpha$. Now, fix a condition $q \in P_1$. As $q \Vdash 2^\alpha \cap V \subset \tau$, it must be the case that $2^\alpha = \bigcup_{p_1 \leq q} O_{p_1}$ holds, and by a compactness argument, there is a set $b \in B$ consisting only of conditions stronger than q. Thus, assumptions of the claim are satisfied, and there must be a countable set $C \subset B$ such that every condition in P_1 is compatible with all conditions in some set in C. Note that this property of the set C persists to the model $V[G_0]$ by the Shoenfield absoluteness.

Now, move to the model $V[G_0]$, pick a point $x \in 2^\alpha$, and a condition $p_2 \in P_1$; we must find a strengthening of p_2 which forces $\check{x} \in \tau/\dot{G}_1$. By the work in the previous paragraph, there is a set $c \in C$ such that p_2 is compatible with all conditions in it. Since $\bigcup\{O_{p_1}\colon p_1 \in c\} = 2^\alpha$, there is a condition $p_1 \in c$ such that $x \in O_{p_1}$. Any common lower bound of p_2 and p_1 forces $\check{x} \in \tau/\dot{G}_1$ as required.

To see that $V[G_0]$ is transcendental over $V[G_1]$, let τ be a P_0-name for an open subset of 2^α such that $P_0 \Vdash 2^\alpha \cap V \subset \dot{O}$. Similarly to the previous work, for each condition $p_0 \in P_0$ let O_{p_0} be the union of all basic open subsets of 2^α which p_0 forces to be subsets of τ and let $B \subset P_1^{<\aleph_0}$ be the set of all finite sets $b \subset P_1$ such that $\bigcup_{p_1 \in b} O_{p_1} = 2^\alpha$. Thus, for every condition $p_0 \in P_0$ there is a set $b \in B$ consisting of conditions stronger than p_0. Now, let η be a P_1-name for an element of 2^α, and let $\langle p_0, \dot{p}_1\rangle$ be a condition in the iteration $P_0 * \dot{P}_1$. We must find a strengthening which forces $\eta \in \tau/\dot{G}_1$. To this end, first use the centeredness assumption again to strengthen p_0 if necessary to find a number $n \in \omega$ such that $p_0 \Vdash \dot{p}_1 \in \dot{A}_n$. Let $b \in B$ be a set consisting of conditions stronger than p_0, and choose an enumeration $b = \{p_0^i\colon i \in m\}$. Let $\{G_0^i\colon i \in m\}$ be a mutually generic collection of filters on P_0 such that $p_0^i \in G_0^i$ holds for all $i \in m$. In the model $V[G_0^i\colon i \in m]$, note that the conditions $\dot{p}_1/G_0^i \in A_n$ have a common lower bound in the poset P_1; let $H \subset P_1$ be a filter generic over the model $V[G_0^i \colon i \in m]$ containing them all. Since $\bigcup_i O_{p_0^i} = 2^\alpha$, there must be a number $i \in m$ such that $\eta/H \cap V \in \tau/G_0^i$. This membership relation is a statement of the model $V[G_0^i][H \cap V[G_0^i]]$ which is a $P_0 * \dot{P}_1$ extension of V. Thus, there has to be a condition stronger than $\langle p_0, \dot{p}_1 \rangle$ forcing it as required. $\square$

Example 3.11. Let $x_0 \in \omega^\omega$ be a point Hechler-generic over the ground model, and $x_1 \in \omega^\omega$ a point Hechler generic over $V[x_0]$. Then $V[x_0]$ and $V[x_1]$ are mutually transcendental extensions of V.

Example 3.12. One cannot weaken the σ-centeredness assumption in Example 3.9 to σ-linkedness. To see this, let X be a compact metric space with a Borel probability measure μ. Let $V[G]$ be a generic extension of V in which there is a compact μ-positive set $C \subset X$ containing no points of $V \cap X$. Let P be the random forcing associated with μ and let $x \in C$ be a point P-generic over $V[G]$. Then $V[G]$ is not compactly transcendental to $V[x]$ since the complement of C is an open set covering $X \cap V$ but not $X \cap V[x]$. Note that $V[x]$ is transcendental to $V[G]$ by the first half of Proposition 3.9, so transcendence is not a symmetric property of pairs of extensions.

4 Preservation theorems

As with all similar notions of perpendicularity of generic extensions, transcendence gives rise to a natural companion: a preservation property for Suslin forcings.

Definition 4.1. Let P be a Suslin forcing. We say that a virtual condition $\bar{p}$ in P is *transcendentally balanced* if for every pair of mutually transcendental generic extensions $V[G_0], V[G_1]$ inside some ambient forcing extension, and for all conditions $p_0 \in V[G_0]$ and $p_1 \in V[G_1]$ stronger than $\bar{p}$, p_0 and p_1 have a common lower bound.

I now state and several preservation theorems for transcendentally balanced extensions of the symmetric Solovay model.

Theorem 4.2. *Let κ be an inaccessible cardinal. In cofinally transcendentally balanced forcing extensions of the symmetric Solovay model derived from κ, every nonmeager subset of $\mathcal{P}(\omega)$ contains a collection d of cardinality four such that $\bigcap d = 0$ and $\bigcup d = \omega$, both modulo finite.*

Proof. Let P be a Suslin forcing which is cofinally transcendentally balanced below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Suppose that $p \in P$ is a condition, τ is a P-name, and $p \Vdash \tau \subset \mathcal{P}(\omega)$ is a nonmeager set. I must find a set $d \subset \mathcal{P}(\omega)$ of size four such that $\bigcap d = 0$ and $\bigcup d = \omega$, both modulo finite, and a strengthening of the condition p which forces $\check{d} \subset \tau$.

To this end, let $z \in 2^\omega$ be a point such that p, τ are both definable from the parameter z and some parameters in the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of cardinality less than κ such that $z \in V[K]$ and $V[K] \models P$ is transcendentally balanced. Work in $V[K]$. Let $\bar{p} \leq p$ be a transcendentally balanced virtual condition. Let Q be the Cohen poset of nonempty open subsets of $\mathcal{P}(\omega)$, adding a single generic point $\dot{z}$. There must be a condition $q \in Q$ and a poset R of cardinality smaller than κ and an $Q \times R$-name σ for a condition in P stronger than $\bar{p}$ such that $q \Vdash_Q R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \dot{z} \in \tau$. Otherwise, in the model W the condition $\bar{p}$ would force τ to be disjoint from the co-meager set of elements of $\mathcal{P}(\omega)$ which are Cohen-generic over $V[K]$, contradicting the initial assumption on τ.

Now, let $X = \{\langle x \rangle \in \mathcal{P}(\omega)^4 \colon \bigcup_{i \in 4} x(i) = \omega \text{ and } \bigcap_{i \in 4} x(i) = 0\}$ with the topology inherited from $\mathcal{P}(\omega)^4$. Let $x \in X$ be a point generic over $V[K]$ for the Cohen poset with X. By Example 3.3, $x(0), x(1)$ are mutually Cohen-generic elements of $\mathcal{P}(\omega)$, so are $x(2), x(3)$, and the models $V[K][x(0), x(1)]$ and $V[K][x(2), x(3)]$ are mutually transcendental. Choose finite modifications z_i of x_i such that $z_i \in q$ holds for all $i \in 4$; each of these points is still Q-generic over $V[K]$ and meets the condition q. Let $H_i \colon i \in 4$ be filters on R mutually generic over the model $V[K][x]$ and let $p_i = \sigma/z_i, H_i$. By Proposition 2.2, conclude that the models $V[K][z_0][H_0]$ and $V[K][z_1][H_1]$ are mutually generic extensions of $V[K]$, so are $V[K][z_2][H_2]$ and $V[K][z_3][H_3]$, and the models $V[K][z_0, z_1][H_0, H_1]$ and $V[K][z_2, z_3][H_2, H_3]$ are mutually transcendental extensions of $V[K]$. Now, the balance assumption on the virtual condition $\bar{p}$, we see that the conditions p_0, p_1 have a common lower bound p_{01} in the model $V[K][z_0, z_1][H_0, H_1]$, the conditions p_2 and p_3 have a common lower bound p_{23} in the model $V[K][z_2, z_3][H_2, H_3]$, and finally the conditions p_{01} and p_{23} have a common lower bound as well. The forcing theorem then shows that such a lower bound then forces in the model W that $\check{z}_i \in \tau$ holds for all $i \in 4$. The proof is complete. $\square$

For every number $n \geq 2$ let Θ_n be the hypergraph of arity n on $\mathcal{P}(\omega)$ consisting of sets d of size n such that $\bigcap d = 0$ and $\bigcup d = \omega$, both modulo finite.

Corollary 4.3. *Let κ be an inaccessible cardinal. In cofinally transcendentally balanced forcing extensions of the symmetric Solovay model derived from κ, the chromatic number of Θ_4 is uncountable.*

Theorem 4.4. *Let κ be an inaccessible cardinal. In cofinally transcendentally balanced forcing extensions of the symmetric Solovay model derived from κ, every nonmeager subset of S_∞ contains a quadruple of distinct points solving the equation $g_0 g_1^{-1} g_2 g_3^{-1} = 1$.*

Proof. Let P be a Suslin forcing which is cofinally transcendentally balanced below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Suppose that $p \in P$ is a condition, τ is a P-name, and $p \Vdash \tau \subset S_\infty$ is a nonmeager set. I must find distinct points $z_0, z_1, z_2, z_3 \in S_\infty$ such that $z_0 z_1^{-1} z_2 z_3^{-1} = 1$ and a strengthening of the condition p which forces all four of these points into τ.

To this end, let $z \in 2^\omega$ be a point such that p, τ are both definable from the parameter z and some parameters in the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of cardinality less than κ such that $z \in V[K]$ and $V[K] \models P$ is transcendentally balanced. Work in $V[K]$. Let $\bar{p} \leq p$ be a transcendentally balanced virtual condition. Let Q be the Cohen poset of nonempty open subsets of S_∞, adding a single generic point $\dot{g}$. There must be a condition $q \in Q$ and a poset R of cardinality smaller than κ and an $Q \times R$-name σ for a condition in P stronger than $\bar{p}$ such that $q \Vdash_Q R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash_P \dot{g} \in \tau$. Otherwise, in the model W the condition $\bar{p}$ would force τ to be disjoint from the co-meager set of elements of S_∞ which are Cohen-generic over $V[K]$, contradicting the initial assumption on τ.

Now, let $X = \{x \in S_\infty^4 \colon x(0)x(1)^{-1}x(2)x(3)^{-1} = 1\}$ with the topology inherited from S_∞^4. Consider the nonempty relatively open set $O \subset X$ given by $O = q^4 \cap X$. Note that the set O is indeed nonempty because any constant quadruple in S_∞^4 belongs to X. Let $\langle z_i \colon i \in 4\rangle \in O$ be a tuple generic over $V[K]$ for the Cohen poset with X. By Example 3.7, z_0, z_2 are mutually Cohen-generic elements of S_∞ below the condition q, so are z_1, z_3, and the models $V[K][z_0, z_2]$ and $V[K][z_1, z_3]$ are mutually transcendental. Let $H_i \colon i \in 4$ be filters on R mutually generic over the model $V[K][z_0, z_1, z_2, z_3]$ and let $p_i = \sigma/g_i, H_i$. By Proposition 2.2, conclude that the models $V[K][z_0][H_0]$ and $V[K][z_2][H_2]$ are mutually generic extensions of $V[K]$, so are $V[K][z_1][H_1]$ and $V[K][z_3][H_3]$, and the models $V[K][z_0, z_2][H_0, H_2]$ and $V[K][z_1, z_3][H_1, H_3]$ are mutually transcendental extensions of $V[K]$. Now, the balance assumption on the virtual condition $\bar{p}$, we see that the conditions p_0, p_2 have a common lower bound p_{02} in the model $V[K][z_0, z_2][H_0, H_2]$, the conditions p_1 and p_3 have a common lower bound p_{13} in the model $V[K][z_1, z_3][H_1, H_3]$, and finally the conditions p_{02} and p_{13} have a common lower bound as well. The forcing theorem then shows that such a lower bound then forces in the model W that $\check{z}_i \in \tau$ holds for all $i \in 4$. The proof is complete. $\square$

Corollary 4.5. *Let κ be an inaccessible cardinal. In cofinally transcendentally balanced forcing extensions of the symmetric Solovay model derived from κ, the chromatic number of the hypergraph on S_∞ consisting of solutions to the equation $g_0 g_1^{-1} g_2 g_3^{-1} = 1$ is uncountable.*

Certain consistency results require amalgamation diagrams with multiple forcing extensions. The following definitions and a theorem show one such possibility.

Definition 4.6. A finite collection $\{V[G_i] \colon i \in a\}$ of generic extensions is *mutually transcendental* if for every index $j \in a$, the models $V[G_j]$ and $V[G_i \colon i \in a, i \neq j]$ are mutually transcendental. The collection is *in n-tuples mutually transcendental* if every subcollection of size n is mutually transcendental.

Definition 4.7. Let $m > n$ be natural numbers. Let P be a Suslin forcing.

1. A virtual condition $\bar{p}$ in P is *m, n-transcendentally balanced* if for every tuple $\langle V[G_i] \colon i \in m\rangle$ of generic extensions, mutually transcendental in n-tuples, and conditions $p_i \leq \bar{p}$ in the respective models $V[G_i]$, the conditions p_i for $i \in m$ have a common lower bound.

2. The poset P is *m, n-transcendentally balanced* if below every condition $p \in P$ there is an m, n-transcendentally balanced virtual condition.

Theorem 4.8. *Let κ be an inaccessible cardinal and $n \geq 2$ be a number. In cofinally $n+1, n$-transcendentally balanced forcing extensions of the symmetric Solovay model derived from κ, every nonmeager subset of $\mathcal{P}(\omega)$ contains $n+1$ many sets which modulo finite form a partition of ω.*

Proof. Let P be a Suslin forcing which is cofinally $n+1, n$-transcendentally balanced below κ. Let W be the symmetric Solovay model derived from κ and work in the model W. Suppose that $p \in P$ is a condition, τ is a P-name, and $p \Vdash \tau \subset \mathcal{P}(\omega)$ is a nonmeager set. I must find a collection $\{a_i : i \in n+1\}$ which is modulo finite a partition of ω and a condition stronger than p which forces every element of this collection into τ.

To this end, let $z \in 2^\omega$ be a point such that p, τ are both definable from the parameter z and some parameters in the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of cardinality less than κ such that $z \in V[K]$ and $V[K] \models P$ is $n+1, n$-transcendentally balanced. Work in $V[K]$. Let $\bar{p} \leq p$ be a $n+1, n$-transcendentally balanced virtual condition. Let Q be the Cohen poset of nonempty open subsets of $\mathcal{P}(\omega)$, adding a single generic point $\dot{a}$. There must be a condition $q \in Q$ and a poset R of cardinality smaller than κ and an $Q \times R$-name σ for a condition in P stronger than $\bar{p}$ such that $q \Vdash_Q R \Vdash \mathrm{Coll}(\omega, <\kappa) \Vdash \sigma \Vdash_P \dot{a} \in \tau$. Otherwise, in the model W the condition $\bar{p}$ would force τ to be disjoint from the co-meager set of elements of $\mathcal{P}(\omega)$ which are Cohen-generic over $V[K]$, contradicting the initial assumption on τ.

Let X be the closed subset of $\mathcal{P}(\omega)^{n+1}$ consisting of tuples of sets which form a partition of ω. For every set $b \subset n+1$ of cardinality n, consider the closed subset Y_b of $\mathcal{P}(\omega)^b$ consisting of pairwise disjoint sets. It is easy to check that the projection map from X to Y_b is open. Thus, the poset P_X of nonempty relatively open subsets of X adds a generic tuple $\dot{x}$, and the restriction $\dot{x} \restriction b$ is generic for the poset P_{Y_b} of nonempty relatively open subsets of Y by [7, Proposition 3.1.1]. By Example 3.4, P_X forces the tuple $\{V[K][\dot{x}(i)] : i \in n+1\}$ to be in n-tuples mutually transcendental over $V[K]$.

Now, move to W and find an $n+1$-tuple x which is generic over the model $V[K]$ for the poset P_X. Let $H_i \subset R$ for $i \in n+1$ be a collection of filters mutually generic over the model $V[K][x]$; use Proposition 2.2 and the work in the previous paragraph to argue that the tuple $\{V[K][x_i][H_i] : i \in n+1\}$ is in n-tuples mutually transcendental. For each $i \in n+1$, make a finite adjustment to $x(i)$ so that the resulting set $a_i \subset \omega$ meets the condition q; note that a_i is Q-generic over $V[K]$. Let $p_i = \sigma/a_i, H_i$. The balance assumption on the virtual condition $\bar{p}$ shows that the conditions p_i for $i \in n+1$ have a common lower bound. That lower bound forces each set a_i for $i \in \omega$ into τ as required. $\square$

Corollary 4.9. *Let $n \geq 2$ be a natural number and Γ_n be the hypergraph on $\mathcal{P}(\omega)$ of $n+1$-tuples which form a modulo finite partition of ω. In cofinally*

$n+1, n$*-transcendentally balanced forcing extensions of the symmetric Solovay model derived from* κ*, the chromatic number of* Γ_{n+1} *is uncountable.*

One can also prove a preservation theorem of a different type:

Theorem 4.10. *Let* κ *be an inaccessible cardinal. In cofinally transcendentally balanced forcing extensions of the symmetric Solovay model derived from* κ*, for every uncountable set* $A \subset \omega^\omega$ *there is a function in* ω^ω *which pointwise dominates uncountably many elements of* A*.*

Proof. Let P be a Suslin forcing which is cofinally transcendentally balanced below κ. Let W be the symmetric Solovay model derived from κ, let $n > 1$ be a number, and work in the model W. Suppose that $p \in P$ is a condition and τ is a P-name such that $p \Vdash \tau \subset \omega^\omega$ is an uncountable set. I must find a function $h \in \omega^\omega$ and a condition stronger than p which forces that $\check{h}$ dominates uncountably many elements of τ.

To this end, let $z \in 2^\omega$ be a point such that p, τ are both definable from the parameter z and some parameters in the ground model. Let $V[K]$ be an intermediate forcing extension obtained by a poset of cardinality less than κ such that $z \in V[K]$ and $V[K] \models P$ is transcendentally balanced. Work in $V[K]$. Let $\bar{p} \leq p$ be a transcendentally balanced virtual condition. Since the set τ is forced to be uncountable, there must be a poset Q_0 and Q_0-names η_0 for an element of ω^ω which is not in $V[K]$ and σ_0 for a condition in P stronger than $\bar{p}$ such that $Q_0 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_0 \Vdash \eta_0 \in \tau$. Now, let $H_0 \subset Q_0$ be a filter generic over the model $V[K]$, let $p_0 = \sigma_0/H_0$ and let $x = \eta_0/H_0$. Let $h \in \omega^\omega$ be a function Hechler generic over the model $V[K][H_0]$ which pointwise dominates x. By Proposition 3.9, $V[K][H_0]$ and $V[K][h]$ are mutually transcendental extensions of $V[K]$.

We claim that in the model W, $\bar{p} \Vdash \check{h}$ pointwise dominates uncountably many elements of τ. This will complete the proof. Suppose towards contradiction that this fails. Work in the model $V[K][h]$. There must be a poset Q_1 of cardinality less than κ, a Q_1-name η_1 for a countable sequence of elements of ω^ω, and a Q_1-name σ_1 for a condition in P stronger than $\bar{p}$ such that $Q_1 \Vdash \text{Coll}(\omega, < \kappa) \Vdash \sigma_1 \Vdash_P \eta_1$ enumerates all elements of τ pointwise dominated by $\check{h}$.

Now, in the model W find a filter $H_1 \subset Q_1$ generic over $V[K][H_0][h]$. By Proposition 2.2 $V[K][H_0]$ and $V[K][h][H_1]$ are mutually transcendental extensions of $V[K]$. Let $p_1 = \sigma_1/H_1$ and $y = \eta_1/H_1$; these objects belong to the model $V[K][h][H_1]$. By the mutual transcendenceof the models $V[K][H_0]$ and $V[K][h][H_1]$, $x \notin V[K][h][H_1]$; in particular, $x \notin \text{rng}(y)$. By the mutual transcendence and the balance of the condition $\bar{p}$, the conditions p_0, p_1 are compatible in P. Their common lower bound forces $\check{x} \in \tau$ as well as $\check{y}$ to enumerate all elements of τ pointwise dominated by $\check{h}$. This is impossible as $\check{x}$ is pointwise dominated by h and does not belong to the range of y. ☐

5 Examples II

The whole enterprise in the previous sections would be pointless if there were no substantial transcendentally balanced posets. In this section, I will produce or point out a number of examples in this direction. At first, I consider posets or classes of posets known from previous work.

Proposition 5.1. *Every placid Suslin poset is transcendentally balanced.*

This class of examples is very broad: it includes among others posets adding a Hamel basis for a Polish space over a countable field, posets adding maximal acyclic subsets to Borel graphs, or posets adding a selector to pinned Borel equivalence relations classifiable by countable structures.

Proof. Recall [7, Definition 9.3.1] that a poset P is placid if below every condition $p \in P$ there is a virtual balanced condition $\bar{p} \leq p$ which is placid: whenever $V[G_0]$ and $V[G_1]$ are generic extensions such that $V[G_0] \cap V[G_1] = V$ and $p_0 \in V[G_0]$ and $p_1 \in V[G_1]$ are conditions stronger than $\bar{p}$, then p_0, p_1 are compatible. Now, if $V[G_0], V[G_1]$ are mutually transcendental extensions of the ground model, then $V[G_0] \cap V[G_1] = V$ by Proposition 2.4, and therefore a placid virtual condition also transcendentally balanced. The proposition follows. $\square$

Proposition 5.2. *Let X be a K_σ Polish field with a countable subfield F. The poset adding a transcendence basis to X over F is transcendentally balanced.*

Proof. Reviewing the proof of [7, Theorem 6.3.9] it becomes clear that the only feature of mutually generic extensions $V[G_0]$ and $V[G_1]$ there is that if p is a multivariate polynomial with coefficients in F, $\vec{x}_0 \in X \cap V[G_0]$ and $\vec{x}_1 \in X \cap V[G_1]$ are tuples such that $p(\vec{x}_0, \vec{x}_1) = 0$, then there are tuples $\vec{x}_0', \vec{x}_1'$ in the ground model such that $p(\vec{x}_0', \vec{x}_1) = p(\vec{x}_0, \vec{x}_1') = 0$. However, this is satisfied for mutually transcendental extensions $V[G_0], V[G_1]$ as well by Corollary 2.6. This completes the proof. $\square$

Proposition 5.3. *Let E be an equivalence relation on a Polish space of one of the following types:*

1. *E is K_σ;*
2. *for some sequence $\langle Y_n, d_n \colon n \in \omega \rangle$ of countable metric spaces, E is the equivalence relation on $X = \prod_n Y_n$ connecting points x_0, x_1 if the distances $d_n(x_0(n), x_1(n))$ tend to zero as n tends to infinity.*

The poset adding a selector to E is transcendentally balanced.

Proof. Note that the equivalence relation E is pinned ([4, Chapter 17], but it follows directly from Corollary 2.7 or Proposition 2.8) and therefore [7, Theorem 6.4.5] applies. The only feature of mutually generic extensions $V[G_0]$ and $V[G_1]$ in the proof of the balance of P is that every E-class represented both in $V[G_0]$ and $V[G_1]$ is represented in V. However, this feature holds true for mutually transcendental extensions by Corollary 2.7 or Proposition 2.8. $\square$

Now it is time to produce transcendentally balanced posets for some new and more difficult tasks. I will only look at coloring posets for hypergraphs of a certain type.

Definition 5.4. Let $n \geq 2$ be a natural number, X a Polish space, and Γ a hypergraph on X of arity n.

1. Γ is *redundant* if for every set $a \subset X$ of cardinality $n-1$, the set $\{x \in X : a \cup \{x\} \in \Gamma\}$ is countable.

If Γ is redundant, then

2. a set $b \subset X$ is Γ-*closed* if for every set $a \subset b$ of cardinality $n-1$ the countable set $\{x \in X : a \cup \{x\} \in \Gamma\}$ is a subset of b;
3. if a set $b \subset X$ is Γ-closed, define the equivalence relation $E(b, \Gamma)$ on $X \setminus b$ as the smallest equivalence containing all pairs $\{x_0, x_1\}$ such that for some set $a \subset b$ of cardinality $n-2$, $a \cup \{x_0, x_1\} \in \Gamma$.

Example 5.5. The hypergraph Γ on $\mathbb{R}$ of arity 3 consisting of solutions to the equation $x^3 + y^3 + z^3 - 3xyz = 0$ is redundant. Every real closed subfield of $\mathbb{R}$ is Γ-closed.

Example 5.6. The hypergraph Γ on $\mathbb{R}^2$ consisting of vertices of equilateral triangles is redundant. A similar hypergraph on $\mathbb{R}^3$ is not redundant.

Example 5.7. Let $n \geq 2$ be a number. The hypergraph Γ_n on $\mathcal{P}(\omega)$ consisting of n-tuples which modulo finite partition ω is redundant. Every Boolean subalgebra of $\mathcal{P}(\omega)$ containing all singletons is Γ_n-closed.

Example 5.8. Let G be a Polish group with a countable dense subset $d \subset G$. Let $n \geq 2$ be a natural number. The hypergraph $\Sigma(G, n)$ consisting of all n-tuples whose product belongs to d is redundant. Note that if G is not abelian, then the product depends on the order of the elements, so one must say "the product of all elements in some order belongs to d".

Example 5.9. Let G be a Polish group and $n \geq 2$ be a natural number. The hypergraph $\Theta(G, n)$ of all $2n$-tuples whose alternating product $g_0 g_1^{-1} g_2 g_3^{-1} \dots$ in some order is equal to 1 is redundant.

In the common case that the set $b \subset X$ is countable, the equivalence relation $E(b, \Gamma)$ has all classes countable since it is the path connectedness equivalence of a locally countable graph. If the redundant hypergraph Γ is Borel, then the equivalence relation $E(b, \Gamma)$ is Borel as well. The complexity of $E(b, \Gamma)$ never enters the considerations of this paper; it can be an arbitrarily complex countable Borel equivalence relation, but in the natural examples it is hyperfinite. For this paper, the following feature of these equivalences is much more relevant.

Proposition 5.10. *Let X be a K_σ Polish space and Γ a redundant F_σ hypergraph of arity $n \geq 2$ on X. Let $V[G_0], V[G_1]$ be mutually transcendental generic extensions of the ground model. Then on $X \cap V[G_0]$, $E(V \cap X, \Gamma) = E(V[G_1] \cap X, \Gamma)$.*

Proof. The left-to-right inclusion is obvious as increasing the set b increases the equivalence relation $E(b, \Gamma)$. The right-to-left inclusion is the heart of the matter. Suppose that $x, x' \in X \cap V[G_0]$ are two points in $X \cap V[G_0]$ which are $E(X \cap V[G_1], \Gamma)$ equivalent. Then there must be a number $m \in \omega$, points $x_i \in X$ for $i \leq m$ and sets $y_i \in [X \cap V[G_1]]^{n-2}$ for $i < m$ such that $x = x_0$, $x' = x_m$, and $\forall i < m\ y_i \cup \{x_i, x_{i+1}\} \in \Gamma$. The tuple $\langle x_i : i \leq m, y_i : i < m\rangle$ will be called a *walk* from x to x'.

Let $K \subset X$ be a compact set coded in the ground model containing all points mentioned in the walk. Let d be a complete metric on X and let $\varepsilon > 0$ be a positive rational such that for any two points mentioned in the walk, if they are distinct then they have d-distance at least ε. Let $\Delta \subset \Gamma$ be a ground model coded compact set such that all hyperedges in the walk belong to Δ.

Now, consider the space $Y = \{\langle z_i : i \in m\rangle : z_i \in [K]^{n-2}$ and distinct points in each z_i have a distance at least $\varepsilon\}$; this is a compact subspace of $([K]^{n-2})^m$ in the ground model. Consider the set $C \subset Y$ consisting of tuples $\langle z_i : i \in m\rangle$ which can serve in a walk from x to x' which uses only points in K, whose hyperedges belong to Δ, and in which any two distinct points have distance at least ε. The set $C \subset Y$ is compact, as it is a projection of a compact set of walks. The set C is coded in $V[G_0]$, and the sequence $\langle y_i : i \in m\rangle \in V[G_1]$ belongs to it. By the mutual transcendence of the models $V[G_0]$ and $V[G_1]$, the set C contains a ground model element. A review of definitions reveals that this means that x, x' are $E(X \cap V, \Gamma)$-related. □

F_σ redundant hypergraphs of arity three or four on K_σ spaces can be colored by a transcendentally balanced Suslin forcing. The following definition and theorem provide a general treatment. However, in certain special cases it is possible to find posets which have stronger preservation properties, as is done in [7, Section 8.2]. Arity five (and higher) presents challenges that I do not know how to overcome in this generality, as the discussion of configurations in the proof of Claim 5.13 becomes untenable.

Definition 5.11. Let X be a K_σ Polish space and Γ be a redundant F_σ hypergraph of arity three or four on X. The *coloring poset* P_Γ consists of all partial Γ colorings $p : X \to \omega \times \omega$ whose domain is a countable Γ-closed subset of X. The ordering is defined by $p_1 \leq p_0$ if $p_0 \subset p_1$ and for every $E(\Gamma, \mathrm{dom}(p_0))$-class $a \subset \mathrm{dom}(p_1)$, $p_1'' a \subset \omega \times \omega$ has all vertical sections finite (and, if the arity of Γ is four, the function $p_1 \restriction a$ is an injection).

Theorem 5.12. *Let Γ be a redundant F_σ hypergraph of arity three on a K_σ Polish space X. The poset P_Γ is Suslin and σ-closed, and it forces the union of the generic filter to be a total Γ-coloring on X. Moreover,*

1. *for every total* Γ*-coloring* $c\colon X \to \omega \times \omega$*, the pair* $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ *is transcendentally balanced;*

2. *for every balanced pair* $\langle Q, \tau\rangle$ *there is a total coloring* $c\colon X \to \omega \times \omega$ *such that the pairs* $\langle Q, \tau\rangle$ *and* $\langle \mathrm{Coll}(\omega, X), \check{c}\rangle$ *are equivalent;*

3. *distinct total* Γ*-colorings provide inequivalent balanced pairs.*

In particular, under CH, the poset P_Γ *is transcendentally balanced.*

Proof. The easiest part is the σ-closure. If $\langle p_i \colon i \in \omega\rangle$ is a decreasing sequence of conditions in P_Γ, then $\bigcup_i p_i$ is their common lower bound. For the property of the generic filter, I need to show that for every condition p_0 and a point $x \in X$ there is a condition $p_1 \leq p_0$ such that $x \in p_1$. To see that, just find a countable Γ-closed set $b \subset X$ such that $\mathrm{dom}(p_0) \cup \{x\} \subset b$, enumerate the set $b \setminus \mathrm{dom}(p_0)$ by $\{x_n \colon n \in \omega\}$, and define a function $p_1 \colon b \to \omega \times \omega$ by the demands $p_0 \subset p_1$ and $p_1(x_n) = (n, n)$. It is not difficult to see that $p_1 \in P_\Gamma$ and $p_1 \leq p_0$ is as required.

It is clear that the ordering on P_Γ is a Borel relation. Borelness of the compatibility relation of the poset P_Γ follows immediately from the following claim, which is used later as well.

Claim 5.13. *Let* $p_0, p_1 \in P_\Gamma$ *be conditions. Then* p_0 *is compatible with* p_1 *if and only if the conjunction of the following item occurs:*

1. $p_0 \cup p_1$ *is a function;*

2. *for every* $E(\mathrm{dom}(p_0), \Gamma)$*-class* a*,* $p_1''a \subset \omega \times \omega$ *has all vertical sections finite (and, if the arity of* Γ *is four, the function* $p_1 \restriction a$ *is injective);*

3. *the same demand as in (2) except with* $0, 1$ *interchanged.*

Proof. The failure of any of the items excludes the existence of the lower common bound by the definition of the ordering. Now, suppose that the items are satisfied. Let $b \subset X$ be a countable Γ-closed set such that $\mathrm{dom}(p_0) \cup \mathrm{dom}(p_1) \subset b$, write $c = b \setminus (\mathrm{dom}(p_0) \cup \mathrm{dom}(p_1))$, enumerate c as $\{x_n \colon n \in \omega\}$ and choose a function $p \colon b \to \omega \times \omega$ so that $p_0 \cup p_1 \subset p$ and for every $n \in \omega$, $p(x_n) = (n, m)$ so that the point (n, m) does not belong to any of the sets $p_1''a$ where a is the $E(\mathrm{dom}(p_0), \Gamma)$-class such that $x_n \in a$, or $p_0''a$ where a is the $E(\mathrm{dom}(p_1), \Gamma)$-class such that $x_n \in a$. This is possible by items (2) and (3). I will show that $p \in P_\Gamma$ and p is a lower bound of p_0, p_1.

To show that $p \in P_\Gamma$, it is enough to verify that p is a Γ-coloring. Suppose that e is a Γ-hyperedge, and work to show that $p \restriction e$ is not constant. We first deal with the case where the arity of Γ is three.

Case 1. $e \cap c$ contains more than one element. Then $p \restriction e$ is not constant since $p \restriction c$ is injective.

Case 2. $e \cap \mathrm{dom}(p_0)$ contains more than one element. By the Γ-closure of $\mathrm{dom}(p_0)$ this means that $e \subset \mathrm{dom}(p_0)$ and so $p \restriction e$ is not constant since p_0 is a Γ-coloring.

Case 3. $e \cap \mathrm{dom}(p_1)$ contains more than one element. This case is symmetric to Case 2.
Case 4. The last configuration is that the sets c, $\mathrm{dom}(p_0)$, and $\mathrm{dom}(p_1)$ contain one point of e each. Call these points y_0, y_1, y_2 respectively. Then y_0 is $E(\mathrm{dom}(p_1), \Gamma)$-related to y_1. By the description of p then, $p(y_0) \neq p(y_1)$ holds, and $p \restriction e$ is not constant as desired.

If the arity of Γ is four, there are more configurations to discuss, and in one of them the injectivity demand will play a key role.
Case 1. $e \cap c$ contains more than one element. Then $p \restriction e$ is not constant since $p \restriction c$ is injective.
Case 2. $e \cap \mathrm{dom}(p_0)$ contains more than two elements. By the Γ-closure of $\mathrm{dom}(p_0)$ this means that $e \subset \mathrm{dom}(p_0)$ and so $p \restriction e$ is not constant since p_0 is a Γ-coloring.
Case 3. $e \cap \mathrm{dom}(p_1)$ contains more than two elements. This case is symmetric to Case 2.
Case 4. $e \cap \mathrm{dom}(p_0)$ and $e \cap \mathrm{dom}(p_1)$ both contain exactly two elements. Writing $\{y_0, y_1\} = e \cap \mathrm{dom}(p_1)$, it follows that y_0, y_1 are $E(\mathrm{dom}(p_0), \Gamma)$-related and therefore by item (2) they receive distinct p_1-colors. Thus $p \restriction e$ is not constant.
Case 5. Both $e \cap c$ and $e \cap \mathrm{dom}(p_0)$ contain exactly one element. Denoting these points by y_0, y_1 respectively, it is clear that they are $E(\mathrm{dom}(p_1), \Gamma)$-related. Thus, $p(y_0) \neq p_0(y_1)$ by the choice of the function p, and $p \restriction e$ is not constant.
Case 6. Both $e \cap c$ and $e \cap \mathrm{dom}(p_1)$ contain exactly one element. This case is symmetric to Case 5.

Finally, I have to show that p is a common lower bound of p_0, p_1. By symmetry, it is enough to show that $p \leq p_0$ holds. To verify that, note that for every $E(\mathrm{dom}(p_0), \Gamma)$ class $a \subset b$, $p_1'' a$ has all vertical sections finite by item (2) and moreover, $p''(a \setminus \mathrm{dom}(p_0))$ has all vertical sections of cardinality at most one by the choice of p. In total, $p'' a$ has all vertical sections finite. If the arity of Γ is four, then the function $p_1 \restriction a$ is injective by item (2) and $p \restriction a \setminus \mathrm{dom}(p_1)$ is an injection and uses no values that $p_1 \restriction a$ uses by the choice of p. In total, the function $p \restriction a$ is an injection. This shows that $p \leq p_0$ holds as required. $\square$

Now, for the first item of the theorem. Suppose that $c \colon X \to \omega \times \omega$ is a total coloring. Clearly, $\mathrm{Coll}(\omega, X) \Vdash \check{c} \in P_\Gamma$ holds. Now suppose that $V[G_0], V[G_1]$ are mutually transcendental generic extensions of the ground model and $p_0 \in V[G_0]$, $p_1 \in V[G_1]$ are conditions stronger than c. I must prove that the conditions are compatible. To do this, use Claim 5.13. It is clear that $p_0 \cap p_1$ is a function, since the domains of p_0 and p_1 intersect in $X \cap V$, and on that set both p_0, p_1 are equal to c. To verify the demand (2) of Claim 5.13, just note that $E(\mathrm{dom}(p_0), \Gamma) \restriction \mathrm{dom}(p_1)$ is equal to $E(X \cap V, \Gamma) \restriction \mathrm{dom}(p_1)$ by Proposition 5.10 and use the fact that $p_1 \leq c$ holds. Demand (3) of Claim 5.13 is verified in a symmetric way.

For the second item of the theorem, let $\langle Q, \tau \rangle$ be a balanced pair. Strengthening τ if necessary, we may assume that $Q \Vdash X \cap V \subset \mathrm{dom}(\tau)$. A balance argument then shows that for every ground model point $x \in X$ there is a pair

$c(x) \in \omega \times \omega$ such that $Q \Vdash \tau(\check{x}) = c(x)$. I claim that the Γ-coloring c works as in (2). It will be enough to show that $Q \Vdash \tau \leq \check{c}$. If this failed, then there must be a condition $q \in Q$ which forces that there is some $E(X \cap V, \Gamma)$-class $a \subset \mathrm{dom}(\tau)$ such that $\tau \restriction a$ has an infinite vertical section. Let $G_0, G_1 \subset Q$ be mutually generic filters containing the condition q and let $p_0 = \tau/G_0$ and $p_1 = \tau/G_1$. The two conditions p_0, p_1 should be compatible in P_Γ, but the contradictory assumption together with Claim 5.13 shows that they are not.

The third item of the theorem is immediate. For the last sentence, suppose that the continuuum hypothesis holds and let $p \in P_\Gamma$ be a condition. I must find a total coloring $c \colon X \to \omega \times \omega$ which is stronger than p in the sense of the ordering on the poset P_Γ. To do this, use the CH assumption to find a continuous increasing sequence $\langle M_\alpha \colon \alpha \in \omega_1 \rangle$ of countable elementary submodels of some large structure such that $\Gamma, p \in M_0$ and $X = \bigcup_\alpha M_\alpha$. Find a total map $c \colon X \to \omega \times \omega$ such that $p \subset c$ and on each of the sets $X \cap M_0 \setminus \mathrm{dom}(p)$, $X \cap M_{\alpha+1} \setminus M_\alpha$ it is an injection with the range included in the diagonal on $\omega \times \omega$. Note that each set $X \cap M_\alpha$ is Γ-closed by elementarity, so any Γ-hyperedge which is not a subset of $\mathrm{dom}(p_0)$ must have two elements coming from the same set on the above list; in conclusion, the map c is a Γ-coloring. It is easy to check that $c \leq p$ as required. $\square$

In arity four, there are many closed redundant hypergraphs such that their coloring number being equal to $\aleph_0$ is equivalent to the continuum hypothesis–for example, the hypergraph of rectangles in $\mathbb{R}^2$ [2]. Thus, in arity four the CH assumption in the theorem is to some extent necessary. I do not know if the assumption is also necessary in arity three in general. There are many specific hypergraphs in which the requisite colorings have been proved to exist in ZFC.

The poset P_Γ in arity three or four has an important additional preservation property.

Theorem 5.14. *(ZFC+CH) Let X be a K_σ Polish space and Γ a redundant F_σ-hypergraph on it of arity three (or four). The poset P_Γ is $4,3$-transcendentally balanced (or $5,4$-transcendentally balanced).*

Proof. I present the proof for arity three. In view of Theorem 5.12, it will be enough to show the following. If $c \colon X \to \omega \times \omega$ is a total Γ coloring, and $\{V[G_i] \colon i \in 4\}$ is a collection or extensions in triples mutually transcendental, $p_i \in V[G_i]$ for $i \in 4$ are conditions stronger than c, then the conditions $\{p_i \colon i \in 4\}$ have a common lower bound.

To do this, work in the model $V[G_i \colon i \in 4]$. Let $b \subset X$ be a Γ-closed subset of X such that $\bigcup_i p_i \subset b$. Let $d = b \setminus \bigcup_i \mathrm{dom}(p_i)$ and enumerate the elements of d as $\{x_n \colon n \in \omega\}$. Now, define a lower bound p of the conditions $\{p_i \colon i \in 4\}$ as a function with domain b such that $\bigcup_i p_i \subset p$ and for each $n \in \omega$, $p(x_n)$ is some pair (n, m) such that for no pair $i, j \in 4$ of distinct indices, writing a for the $E(\mathrm{dom}(p_i), \Gamma)$-class to which x_n belongs, (n, m) does not belong to $p_j''a$.

First of all, observe that the requirements on p can be met. For any pair $i, j \in 4$ of distinct indices, the models $V[G_i]$ and $V[G_j]$ are mutually transcendental. By Proposition 5.10, the equivalences $E(\mathrm{dom}(p_i), \Gamma)$ and $E(X \cap V, \Gamma)$ coincide on dom(p_j), and since $p_j \leq c$, it is the case that $p''_j a$ has all vertical sections finite. Thus the function p can be found as required.

Now, I have to prove that $p \in P_\Gamma$ holds and p is a common lower bound of all conditions p_i for $i \in 4$. This follows nearly literally the proof of Claim 5.13, with the consideration of one extra (impossible) configuration. Namely, it is impossible for there to be distinct indices i_0, i_1, i_2 and a hyperedge $e \subset b$ such that each set dom(p_{i_k}) for $k \in 3$ contains exactly one element of e. This is so because the redundancy of Γ together with a Mostowski absoluteness argument shows that $e \subset V[G_{i_0}, G_{i_1}]$ and $V[G_{i_2}] \cap V[G_{i_0}, G_{i_1}] = V$ by the mutual transcendence assumption. □

Finally, we are in position to prove the theorems in the introduction. Start in the symmetric Solovay model. For Theorem 1.1, let G be a K_σ Polish group and let $\Delta(G)$ be the closed hypergraph of all solutions to $g_0 g_1^{-1} g_2 g_3^{-1} = 1$. It is immediate that $\Delta(G)$ is a redundant hypergraph of arity four. The poset $P_{\Delta(G)}$ of Definition 5.11 is transcendentally balanced under CH by Theorem 5.12. By Corollary 4.5, in the $P_{\Delta(G)}$ extension of the Solovay model, $\Delta(G)$ has countable chromatic number while $\Delta(S_\infty)$ does not.

For Theorem 1.2, consider the hypergraph Γ_3 on $\mathcal{P}(\omega)$ of arity three consisting of triples which modulo finite form a partition of ω. It is clearly a redundant hypergraph. The poset P_{Γ_3} of Definition 5.11 is (under CH, but in fact also in good old ZFC) $4,3$-transcendentally balanced by Theorem 5.14. By Corollary 4.9, in the P_{Γ_3}-extension of the Solovay model, the chromatic number of Γ_3 is countable while that of Γ_4 is uncountable. Theorem 1.3 is proved in the same way. Finally, Theorem 1.4 just restates Corollary 4.3.

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