Coloring triangles and rectangles*

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Abstract

It is consistent that ZF+DC holds, the hypergraph of rectangles on a given Euclidean space has countable chromatic number, while the hypergraph of equilateral triangles on $\mathbb{R}^2$ does not.

1 Introduction

This paper continues the study of algebraic hypergraphs on Euclidean spaces from the point of view of their chromatic numbers in choiceless context, started in [7, 8, 6]. In the context of ZFC, such hypergraphs were completely classified by Schmerl regarding their countable chromatic number [5]. In the choiceless context, the study becomes much more difficult and informative; in particular, the arity and dimension of the hypergraphs concerned begin to play much larger role. In this paper, I compare chromatic numbers of equilateral triangles with that of rectangles.

Definition 1.1. Δ denotes the hypergraph of equilateral triangles on $\mathbb{R}^2$. Let $n \geq 2$ be a number. Γ_n denotes the hypergraph of Euclidean rectangles on $\mathbb{R}^n$.

In the base theory ZFC, these hypergraphs are well-understood. By an old result of [1], Δ has countable chromatic number. On the other hand, the chromatic number of Γ_n is countable if and only if the Continuum Hypothesis holds [2], this for every $n \geq 2$. In the base theory ZF+DC, I present an independence result:

Theorem 1.2. *Let $n \geq 2$. It is consistent relative to an inaccessible cardinal that ZF+DC holds, the chromatic number of Γ_n is countable, yet the chromatic number of Δ is not.*

In fact, I prove much stronger statement than the theorem indicates. There is nothing much specific to Δ; it is only important that it is a nonempty algebraic hypergraph of arity three invariant under similarities of the underlying

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Euclidean metric space. It is possible to generalize further to replace Δ with many natural non-algebraic hypergraphs. At the same time, geometry of rectangles is exploited thoroughly and it is hard to see a meaningful generalization of the argument to hypergraphs different from Γ_n. The consistency result can be achieved simultaneously for all $n \geq 2$.

It is interesting to compare the proof of Theorem 1.2 with the techniques of [7] where I separate the chromatic number of Γ_n from that of Γ_{n+1}. While the general approach is very similar, the coloring posets used have different dimension characteristics and cannot be used interchangeably. It appears that there is no canonical poset for coloring the rectangle hypergraph in a given dimension.

The paper follows the set theoretic standard of [3]. The calculus of geometric set theory and balanced virtual conditions in Suslin forcings is developed in [4, Section 5.2].

2 A preservation theorem

In this section, I prove a preservation theorem for the chromatic number of Δ. In order to deal with Δ efficiently, I place it within the very general class of hypergraphs on Euclidean spaces invariant under similarities.

Definition 2.1. Let $n \geq 1$ be a number. A *similarity* is a permutation g of $\mathbb{R}^n$ such that for some constant $c > 0$, for all points $x, y \in \mathbb{R}^n$, $d(x, y) = c \cdot d(g(x), g(y))$, where d is the usual Euclidean distance on $\mathbb{R}^n$.

The following fact is a basic feature of Euclidean geometry; the proof is left to the reader.

Fact 2.2. *Let $n \geq 1$ be a number.*

1. *Similarities of $\mathbb{R}^n$ form a Polish group G_n under the pointwise convergence topology;*
2. *for any pair $\{x_0, x_1\} \in [\mathbb{R}^n]^2$, the map $g \mapsto \{g(x_0), g(x_1)\}$ is a continuous open surjection from G_n to $[\mathbb{R}^n]^2$.*

Every hypergraph on $\mathbb{R}^n$ which is defined by homogeneous equations on the distances between points in the hyperedges is invariant under similarities. This includes the hypergraphs of equilateral triangles, isosceles triangles, squares, rectangles, parallelograms etc. The following simple proposition is the only feature of this class of hypergraphs I use in this paper. Recall that for a Polish space X, the *Cohen poset* P_X consists of nonempty open subsets of X ordered by inclusion. It adds a single point of X, the unique point which belongs to all (reinterpretations of the) open sets in the generic filter. Such point is called a (Cohen) generic element of X.

Proposition 2.3. *Let $n, m \geq 1$ be numbers. Let $\Gamma \subset (\mathbb{R}^n)^m$ be a nonempty closed set invariant under similarities, in which injective m-tuples are dense. Let $\langle x_i : i \in m\rangle$ be a P_Γ-generic hyperedge.*

1. *For every index $i \in m$, the point x_i is a Cohen generic element of $\mathbb{R}^n$ over V;*
2. *for distinct indices $i, j \in m$, the models $V[x_i], V[x_j]$ are mutually generic.*

Proof. Let $i, j \in m$ be distinct indices. By [4, Proposition 3.1.1], it is enough to show that on the relatively open dense set $O \subset \Gamma$ consisting of injective n-tuples, the projection $\pi\colon O \to (\mathbb{R}^n)^2$ associated with coordinates i and j is an open function. To see this, let $P \subset O$ be a relatively open set and $\langle y_k \colon k \in m\rangle \in P$ a point. Let $U \subset G_n$ be an open neighborhood of the unit such that $\langle g(y_k)\colon k \in m\rangle \in P$ holds for all $g \in U$. Note that the tuple $\langle g(y_k)\colon k \in m\rangle$ automatically belongs to Γ as Γ is invariant under similarities. By Fact 2.2(2), the set $\{\langle g(y_i), g(y_j)\rangle \colon g \in W\}$ is an open subset of $(\mathbb{R}^n)^2$, and it is a subset of the projection of P containing the point $\langle y_i, y_j\rangle$. $\square$

To state a reasonably general preservation theorem for balanced extensions of the Solovay model, I recall the notion of $m, 2$-balance [4, Definition 13.1.1].

Definition 2.4. Let P be a Suslin forcing.

1. A virtual condition $\bar{p}$ in P is $m, 2$-balanced if for every collection $\langle V[G_i]\colon i \in m\rangle$ of pairwise muually generic extensions of V, every collection $\langle p_i \colon i \in m\rangle$ of conditions in P stronger than $\bar{p}$ in the respective generic extensions has a common lower bound;
2. P is $m, 2$-balanced if below every condition in p there is an $m, 2$-balanced virtual condition.

The following theorem is stated using the parlance of [4, Convention 1.7.18].

Theorem 2.5. *Let $n, m \geq 1$ be numbers. Let $\Gamma \subset (\mathbb{R}^n)^m$ be a nonempty closed hypergraph invariant under similarities of $\mathbb{R}^n$. In every forcing extension of the choiceless Solovay model by a cofinally $m, 2$-balanced Suslin forcing, every non-meager subset of $\mathbb{R}^n$ contains all vertices of an equilateral triangle.*

Proof. Let κ be an inaccessible cardinal. Let P be a Suslin forcing which is $m, 2$-balanced cofinally in κ. Let W be a choiceless Solovay model derived from κ. Work in W. Suppose that $p \in P$ is a condition and τ is a P-name for a non-null subset of $\mathbb{R}^n$. Both p, τ are definable from a ground model parameter and an additional parameter $z \in 2^\omega$. I must find a hyperedge in Γ and a condition in P stronger than p which forces the hyperedge to be a subset of τ. Let $V[K]$ be an intermediate extension obtained by a poset of cardinality smaller than κ such that $z \in V[K]$ and P is $m, 2$-balanced in $V[K]$. Work in $V[K]$.

Let $\bar{p} \leq p$ be an $m, 2$-balanced virtual condition in the poset P. Let Q be the usual Cohen poset of nonempty open subsets of X ordered by inclusion, adding an element $\dot{x}_{gen} \in X$. There must be a condition $O \in Q$, a poset R of cardinality smaller than κ and a $Q \times R$-name σ for a condition in P stronger than $\bar{p}$ such that $O \Vdash R \Vdash \mathrm{Coll}(\omega, < \kappa) \Vdash \sigma \Vdash \dot{x}_{gen} \in \tau$. Otherwise, $\bar{p}$ would force τ to be disjoint from the co-meager set of all elements of X Cohen generic over $V[K]$, However, the complement of this Borel set has λ-mass zero, and this would contradict the initial assumption on the name τ.

Now, use the invariance of the hypergraph Γ under similarities to see that $[O]^m \cap \Gamma$ is a nonempty set and therefore a condition in the poset P_Γ. Work in the model W. Find a hyperedge $\langle x_i : i \in m \rangle \in O^m \cap \Gamma$ generic over $V[K]$ for the poset Q. By Proposition 2.3, each of the points $x_i \in O$ is Q-generic over $V[K]$, and for distinct indices $i, j \in m$, the models $V[x_i], V[x_j]$ are mutually generic. Let $H_i \subset R$ for $i \in m$ be mutually generic filters over the model $V[K][x_0, x_1]$. The models $V[K][x_i][H_i]$ for $i \in m$ are pairwise mutually generic extensions of $V[K]$. For each $i \in m$ let $p_i = \sigma/x_i, H_i \leq \bar{p}$. By the balance assumption on the virtual condition $\bar{p}$, the conditions p_i for $i \in m$ have a common lower bound in the poset P. By the forcing theorem applied in the respective models $V[K][x_i][H_i]$, this common lower bound forces $\{x_i : i \in m\} \subset \tau$ as desired. $\square$

3 The coloring poset

To prove Theorem 1.2, I must produce a suitable Suslin poset adding a total Γ_n-coloring. The definition of the poset uses, as a technical parameter, a Borel ideal I on ω which contains all singletons which is not generated by countably many sets. Further properties of the ideal I seem to be irrelevant; the summable ideal will do.

Definition 3.1. Let $n \geq 2$ be a number. The poset P_n consists of partial functions $p \colon \mathbb{R}^n \to \omega$ such that there is a countable real closed subfield $\operatorname{supp}(p) \subset \mathbb{R}$ such that $\operatorname{dom}(p) = \operatorname{supp}(p)^n$, and p is a Γ_n-coloring. The ordering is defined by $p_1 \leq p_0$ if

1. $p_0 \subset p_1$;
2. for every hypersphere $S \subset \mathbb{R}^n$ visible in $\operatorname{supp}(p_0)$ and any two points $x, y \in \operatorname{dom}(p_1 \setminus p_0)$, if x, y are opposite points on S then $p_1(x) \neq p_1(y)$;
3. for any two parallel hyperplanes $P, Q \subset \mathbb{R}^n$ visible in $\operatorname{supp}(p_0)$ and any two points $x, y \in \operatorname{dom}(p_1 \setminus p_0)$, if x, y are opposite points on the respective hyperplanes P, Q then $p_1(x) \neq p_1(y)$;
4. if $a \subset \operatorname{supp}(p_1)$ is a finite set, then $p_1''\delta(p_0, p_1, a) \in I$ where $\delta(p_0, p_1, a) = \{x \in \operatorname{dom}(p_1 \setminus p_0) \colon x \text{ is algebraic over } \operatorname{supp}(p_0) \cup a\}$.

Proposition 3.2. *$\leq$ is a σ-closed transitive relation.*

Proof. For the transitivity, suppose that $r \leq r \leq p$ are conditions in the poset P_n; I must show that $r \leq p$. Checking the items of Definition 3.1, (1) is obvious. For (2), suppose that S is a hypersphere visible in p and x, y are opposite points on it in $\operatorname{dom}(r \setminus p)$. By the closure properties of $\operatorname{dom}(q)$, either both x, y belong to $\operatorname{dom}(q)$ or both do not. In the former case (2) is confirmed by an application of (2) of $q \le p$, in the latter case (2) is confirmed by an application of (2) of $r \le q$. (3) is verified in a similar way. For (4), suppose that $a \subset \mathrm{supp}(r)$ is a finite set. Let $b \subset \mathrm{supp}(q)$ be an inclusion maximal set of points algebraic over $\mathrm{supp}(p) \cup a$ which is algebraically independent. Since finite algebraically independent sets over $\mathrm{supp}(p)$ form a matroid, it must be the case that $|b| \le |a|$ holds. Note that $\delta(p, r, a) \subseteq \delta(p, q, b) \cup \delta(q, r, a)$ and $r''\delta(p, r, a) \subseteq q''\delta(p, q, b) \cup r''(q, r, a)$. Thus, the set $r''\delta(p, r, a)$ belongs to I, since it is covered by two sets which are in I by an application of (4) of $q \le p$ and $r \le q$.

For the σ-closure, let $\langle p_i \colon i \in \omega\rangle$ be a descending sequence of conditions in P_n, and let $q = \bigcup_i p_i$; I will show that q is a common lower bound of the sequence. Let $i \in \omega$ be arbitrary and work to show $q \le p_i$. For brevity, I deal only with item (4) of Definition 3.1. Let $a \subset \mathrm{supp}(q)$ be a finite set. There must be $j \in \omega$ greater than i such that $a \subset \mathrm{supp}(p_j)$. By the closure properties of $\mathrm{dom}(p_j)$, it follows that $\delta(p_i, q, a) = \delta(p_i, p_j, a)$. Thus, $q''\delta(p_i, q, a) = p_j''\delta(p_i, p_j, a)$ and the latter set belongs to I by an application of (4) of $p_j \le p_i$. $\square$

Further analysis of the poset P_n depends on a characterization of compatibility of conditions.

Proposition 3.3. *Let $p_0, p_1 \in P_n$ be conditions. The following are equivalent:*

1. *p_0, p_1 are compatible;*
2. *for every point $x_0 \in \mathbb{R}^n$ there is a common lower bound of p_0, p_1 containing x in its domain;*
3. *the conjunction of the following:*
   - (a) *$p_0 \cup p_1$ is a function and a Γ_n-coloring;*
   - (b) *whenever S is a hypersphere visible from $\mathrm{supp}(p_0)$ and $x, y \in \mathrm{dom}(p_1 \setminus p_0)$ are opposite points on S, then $p_1(x) \ne p_1(y)$;*
   - (c) *whenever P, Q are parallel hyperplanes visible from $\mathrm{supp}(p_0)$ and $x, y \in \mathrm{dom}(p_1 \setminus p_0)$ are opposite points on them, then $p_1(x) \ne p_1(y)$;*
   - (d) *for every finite set $a \subset \mathrm{supp}(p_1)$, $p_1''\delta(p_0, p_1, a) \in I$;*
   - (e) *items above with subscripts $0, 1$ interchanged.*

Proof. (2) implies (1), which in turn implies (3) by Definition 3.1. The hard implication is the remaining one: (3) implies (2). Suppose that all items in (3) obtain and $x_0 \in \mathbb{R}^n$ is a point. To find a common lower bound of p_0, p_1 which contains x_0 in its domain, let $F \subset \mathbb{R}$ be a countable real closed field containing x_0 as an element and $\mathrm{supp}(p_0), \mathrm{supp}(p_1)$ as subsets. The common lower bound q will be constructed in such a way that $\mathrm{dom}(q) = F^n$. Write $d = F^n \setminus (\mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$. For every point $x \in d$ and every $i \in 2$, let $\alpha(x, i) = \{y \in \mathrm{dom}(p_i) \setminus \mathrm{dom}(p_{1-i}) \colon y$ and x are mutually algebraic over $\mathrm{supp}(p_{1-i})$.

Claim 3.4. *For each $x \in d$ and $i \in 2$, the set $p_i''\alpha(x, i)$ belongs to the ideal I.*

Proof. For definiteness set $i = 1$. The set $\alpha(x, 1)$ is a subset of $\delta(p_0, p_1, a)$ where a is the set of coordinates of any point in $\alpha(x, 1)$. The claim then follows from assumption (3)(d). $\square$

Now, use the claim to find a set $b \subset \omega$ in the ideal I which cannot be covered by finitely many elements of the form $p_i''\alpha(x, i)$ for $x \in d$ and $i \in 2$ and finitely many singletons. Let $q\colon F^n \to \omega$ be a function extending $p_0 \cup p_1$ such that $q \restriction d$ is an injection and for every $x \in d$, $q(x) \in b \setminus (p_0''\alpha(x, 0) \cup p_1''\alpha(x, 1))$. Such a function exists by the choice of the set b. I will show that $q \in P_n$ and q is a lower bound of p_0, p_1.

To see that $q \in P_n$, let $R \subset \mathrm{dom}(q)$ be a rectangle and work to show that R is not monochromatic. The treatment splits into cases.

Case 1. $R \subset \mathrm{dom}(p_0) \cup \mathrm{dom}(p_1)$. By the closure properties of the sets $\mathrm{dom}(p_0)$ and $\mathrm{dom}(p_1)$, there are two subcases.

Case 1.1. R is entirely contained in one of the two conditions. Then R is not monochromatic as both p_0, p_1 are Γ_n-colorings.

Case 1.2. There are exactly two vertices of R in $\mathrm{dom}(p_0 \setminus p_1)$ and exactly two vertices of R in $\mathrm{dom}(p_1 \setminus p_0)$. There are again two subcases.

Case 1.2.1 If the two vertices in $\mathrm{dom}(p_0 \setminus p_1)$ are opposite on the rectangle R, then they determine a hypersphere visible from $\mathrm{supp}(p_0)$ on which the other two vertices are opposite as well. Then the other two vertices receive distinct p_1-colors by assumption (3)(b).

Case 1.2.2. If the two vertices in $\mathrm{dom}(p_0 \setminus p_1)$ are next to each other on the rectangle R, then they determine parallel hyperplanes visible from $\mathrm{supp}(p_0)$ on which the other two vertices are opposite as well. Then the other two vertices receive distinct p_1-colors by assumption (3)(c).

Case 2. R contains exactly one vertex in the set d; call it x. By the closure properties of the sets $\mathrm{dom}(p_0)$ and $\mathrm{dom}(p_1)$, the remaining three vertices of R cannot all belong to the same condition. Thus, there must be two vertices contained in (say) $\mathrm{dom}(p_0)$ and one vertex, call it y, in $\mathrm{dom}(p_1 \setminus p_0)$. Then y, x are mutually algebraic over $\mathrm{dom}(p_0)$. Thus $y \in \alpha(x, 1)$ and $q(x) \neq q(y)$ by the initial assumptions on the function q. In conclusion, the rectangle R is not monochromatic.

Case 3. R contains more than one vertex in the set d. Then R is not monochromatic as $q \restriction d$ is an injection.

This shows that $q \in P_n$ holds. I must show that $q \leq p_1$; the proof of $q \leq p_0$ is symmetric. To verify Definition 3.1 (2), suppose that S is a hypersphere visible in $\mathrm{dom}(p_0)$ and $x, y \in \mathrm{dom}(q \setminus p_0)$ are opposite points on S. If $x, y \in \mathrm{dom}(p_1)$ then item (3)(b) shows that $q(x) \neq q(y)$. If $x \in d$ and $y \in \mathrm{dom}(p_0)$ (or vice versa) then $y \in \alpha(x, 0)$ and $q(x) \neq q(y)$ by the choice of the color $q(x)$. Finally, if $x, y \in d$ then $q(x) \neq q(y)$ as $q \restriction d$ is an injection.

Definition 3.1 (3) is verified in the same way. For item (4) of Definition 3.1, let $a \subset F$ be a finite set. Let $a' \subset \mathrm{supp}(p_0)$ be a maximal set in $\mathrm{supp}(p_0)$ which is algebraically free over $\mathrm{supp}(p_1)$. Since algebraically free sets over $\mathrm{supp}(p_0)$ form a matroid, $|a'| \leq |a|$ holds, in particular a' is finite. Now, $\delta(q, p_1, a) \subset$

$\delta(p_1, p_0, a') \cup b$, the first set on the right belongs to I by assumption (3)(d), so the whole union belongs to I as required. $\square$

Corollary 3.5. *P_n is a Suslin poset.*

Proof. It is clear from Definition 3.1 that the underlying set and the ordering of the poset P_n are Borel. Proposition 3.3 shows that the (in)compatibility relation is Borel as well. $\square$

Corollary 3.6. *P_n forces the union of the generic filter to be a total Γ_n-coloring.*

Proof. By a genericity argument, it is enough to show that for every condition $p \in P_n$ and every point $x_0 \in \mathbb{R}^n$ there is a stronger condition containing x_0 in its domain. This follows from Proposition 3.3 with $p = p_0 = p_1$. $\square$

It is time for the balance proofs. They use the following fact, proved in greater generality in [8].

Fact 3.7. *Let $n \geq 2$ be a number. Let $V[G_0]$, $V[G_1]$ be mutually generic extensions.*

1. *Let $C \subset \mathbb{R}^n$ be an affine set definable from parameters in $V[G_0]$. Suppose that $x_1 \in V[G_1] \cap \mathbb{R}^n$ is a point in C. Then there is an affine set $D \subseteq C$ definable in parameters from V such that $x_1 \in D$;*
2. *same as (1) except for algebraic sets;*
3. *same as (1) except for semialgebraic sets.*

In addition, if $a \subset \mathbb{R} \cap V[G_1]$ is a finite set and $r \in \mathbb{R} \cap V[G_1]$ is a real algebraic over $(\mathbb{R} \cap V[G_0]) \cup a$, then r is algebraic over $(\mathbb{R} \cap V) \cup a$.

Theorem 3.8. *Let $n \geq 2$ be a number. In the poset P_n,*

1. *for every total Γ_n-coloring $c \colon \mathbb{R}^n \to \omega$, the pair $\langle \mathrm{Coll}(\omega, \mathbb{R}), \check{c} \rangle$ is balanced;*
2. *every balanced pair is equivalent to one as in item (1);*
3. *distinct colorings yield inequivalent balanced pairs.*

In particular, the poset P_n is balanced if and only if the Continuum Hypothesis holds.

Proof. For (1), let $V[G_0], V[G_1]$ be mutually generic extensions and $p_0, p_1 \leq c$ be conditions in P_n in the corresponding extensions; I must show that p_0, p_1 are compatible. This is done using Proposition 3.3.

For item (3)(a), it is clear that $V[G_0] \cap V[G_1] = V$, therefore $p_0 \cup p_1$ is a function. To show that it is a Γ_n-coloring, suppose that $R \subset \mathrm{dom}(p_0 \cup p_1)$ is a rectangle. By the closure properties of $\mathrm{dom}(p_0)$ and $\mathrm{dom}(p_1)$, R is either wholly included in one of the conditions, or it contains two points in $\mathrm{dom}(p_0 \setminus V)$ and two points in $\mathrm{dom}(p_1 \setminus V)$. In the former case R is not monochromatic as p_0, p_1

are Γ_n-coloring. In the latter case, there are two subcases. If the two points in $\mathrm{dom}(p_0) \cap R$ are opposite on the rectangle R, then they are on a hypersphere which is visible in both $\mathrm{supp}(p_0)$ and $\mathrm{supp}(p_1)$, therefore in the ground model and they receive distinct colors as $p_0 \leq c$. If the two points in $\mathrm{dom}(p_0) \cap R$ are adjacent on the rectangle R, then they are on parallel hyperplanes which are visible in both $\mathrm{supp}(p_0)$ and $\mathrm{supp}(p_1)$, so visible in V and they receive distinct colors as $p_0 \leq c$.

Items (3)(b) and (c) are verified in a similar way. For item (3)(d), use Fact 3.7 to show that for any finite set $a \subset \mathrm{supp}(p_1)$, $\delta(p_0, p_1, a) = \delta(c, p_1, a)$. Then, the p_1-image of this set belongs to the ideal I by Definition 3.1(4) applied to $p_1 \leq c$.

For (2), let $\langle Q, \sigma \rangle$ be a balanced pair. Strengthening the poset Q and/or the name σ, I may assume that $Q \Vdash \sigma \in P_n$ is a condition and $\mathbb{R}^n \cap V \subset \mathrm{dom}(\sigma)$. A balance argument shows that for every point $x \in \mathbb{R}^n$, there must be a specific number $c(x) \in \omega$ such that $Q \Vdash \sigma(\check{x}) = c(x)$. It is clear that the function c is a total Γ_n-coloring. By [4, Proposition 5.2.6], it is enough to show that $Q \Vdash \sigma \leq \check{c}$. Suppose towards a contradiction that this fails; then, there has to be a condition $q \in Q$ which forces a specific item of Definition 3.1 to fail. Let $G_0, G_1 \subset$ be mutually generic filters containing the condition q, and let $p_0 = \sigma/G_0$ and $p_1 = \sigma/G_1$. It is clear that the corresponding item of Proposition 3.3(3) has to fail. In conclusion, the conditions p_0, p_1 are incompatible, violating the balance assumption on the pair $\langle Q, \sigma \rangle$.

(3) is immediate. For the last sentence, if CH fails, then there is no total Γ_n-coloring by the result of [2] and balance fails by item (2). On the other hand, if CH holds and $p \in P_n$ is a condition, choose an enumeration $\langle x_\alpha \colon \alpha \in \omega_1 \rangle$ of $\mathbb{R}^n$ and by recursion on $\alpha \in \omega_1$ build conditions $p_\alpha \in P_n$ so that

- $p = p_0 \geq p_1 \geq \dots$;
- $x_\alpha \in \mathrm{dom}(p_{\alpha+1})$;
- $p_\alpha = \bigcup_{\beta \in \alpha} p_\beta$ for limit ordinals α.

The successor step is possible by Corollary 3.6 and the limit step by Proposition 3.2. In the end, let $c = \bigcup_\alpha p_\alpha$ and observe that c is a total Γ_n-coloring and $c \leq p$. $\square$

Theorem 3.9. *Every balanced virtual condition in P_n is $3, 2$-balanced.*

The fine details of this proof are the reason behind the rather mysterious Definition 3.1.

Proof. Let $c \colon \mathbb{R}^n \to \omega$ be a total Γ_n-coloring. Let $V[G_i]$ for $i \in 3$ be pairwise mutually generic extensions. Suppose that $p_i \leq c$ is a condition in P_n for each $i \in 3$; I must find a common lower bound of all p_i for $i \in 3$.

Work in the model $V[G_i \colon i \in 3]$. Let $F \subset \mathbb{R}$ be a countable real closed field containing $\mathrm{supp}(p_i)$ for $i \in 3$. I will construct a lower bound q such that $F = \mathrm{supp}(q)$. Write $d = F^n \setminus \bigcup_i \mathrm{dom}(p_i)$. For each point $x \in d$ and for each pair

$i, j \in 3$ of distinct indices, define sets $\alpha(x, i, j), \beta(x, i, j)$ and $\gamma(x, i, j) \subset \mathrm{dom}(p_i)$ as follows:

- $\alpha(x, i, j) = \{y \in \mathrm{dom}(p_i \setminus c)$: for some hypersphere $S \subset \mathbb{R}^n$ visible in $\mathrm{supp}(p_j)$ such that x, y are opposite points on $S\}$;
- $\beta(x, i, j) = \{y \in \mathrm{dom}(p_i \setminus c)$: there are parallel hyperplanes $P, Q \subset \mathbb{R}^n$ visible in $\mathrm{supp}(p_j)$ such that x, y are opposite points on P, Q respectively$\}$;
- $\gamma(x, i, j) = \{y \in \mathrm{dom}(p_i \setminus c)$: there are points $x_j \in \mathrm{dom}(p_j \setminus c)$ and $x_k \in \mathrm{dom}(p_k \setminus c)$ such that x, y, x_j, x_k are four vertices of a rectangle listed in a clockwise or counterclockwise order$\}$. Here $k \in 3$ is the index distinct from i and j.

Claim 3.10. *There is a finite set $a \subset \mathrm{supp}(p_i)$ such that $\alpha(x, i, j)$ consists of points algebraic over $(\mathbb{R} \cap V) \cup a$.*

Proof. This is clear if $\alpha(x, i, j) = 0$. Otherwise, let $y \in \alpha(x, i, j)$ be any point and argue that all other points in $\alpha(x, i, j)$ are algebraic over $(\mathbb{R} \cap V) \cup y$. To see this, suppose that $z \in \alpha(x, i, j)$ is any other point. Let S_y, S_z be hyperspheres visible in $\mathrm{supp}(p_j)$ such that x is opposite of y on S_y and opposite of z on S_z. It follows that z is algebraic over $\mathrm{supp}(p_j) \cup y$: one first derives x from y and then z from x. By Fact 3.7 z is algebraic over $(\mathbb{R} \cap V) \cup y$ as desired. □

Claim 3.11. *There is a finite set $a \subset \mathrm{supp}(p_i)$ such that $\beta(x, i, j)$ consists of points algebraic over $(\mathbb{R} \cap V) \cup a$.*

Proof. This is parallel to the previous argument. □

Claim 3.12. *There is a finite set $a \subset \mathrm{supp}(p_i)$ such that $\gamma(x, i, j)$ consists of points algebraic over $(\mathbb{R} \cap V) \cup a$.*

Proof. This is the heart of the whole construction and the reason why item (4) appears in Definition 3.1. For each point $y \in \gamma(x, i, j)$ choose points $x_j(y) \in \mathrm{dom}(p_k \setminus c)$ and $x_k \in \mathrm{dom}(p_k \setminus c)$ witnessing the membership relation. Let $H(y) \subset \mathbb{R}^n$ be the hyperplane passing through y and perpendicular to the vector $y - x_j(y)$; thus, $x \in H(y)$. Write $H = \bigcap_{y \in \gamma(x,i,j)} H(y)$. Let $a \subset \gamma(x, i, j)$ be a set of minimum cardinality such that $H = \bigcap_{y \in a} H(y)$; the set a is finite. I will show that every point $y \in \gamma(x, i, j)$ is algebraic over $(\mathbb{R} \cap V) \cup a$. This will prove the claim.

Let $y \in \gamma(x, i, j)$ be an arbitrary point. Consider the set $A = \{u \in (\mathbb{R}^n)^{m+1} \colon \forall z \in \mathbb{R}^n\ (\forall l \in m\ (x_j(a(l)) - u(l)) \cdot (z - u(l)) = 0) \to (x_j(y) - u(m)) \cdot (z - u(l)) = 0\}$. The set A is semi-algebraic in parameters from $\mathrm{supp}(p_j)$ and contains the tuple $a^\frown y$. By Fact 3.7 and the mutual genericity assumption between $V[G_i]$ and $V[G_j]$, there is a set $B \subset A$ semialgebraic in parameters from $\mathbb{R} \cap V$ such that $a^\frown y \in B$. Note that B_a is a subset of the hypersphere of which the segment between $x_j(y)$ and x, and also the segment between $x_k(y)$ and y, is a diameter. Let $C = \{u \in B \colon u(m)$ is the farthest point of $B_{u \restriction m}$ from

$x_i(y)\}$. This is a semi-algebraic set in parameters from $\mathrm{supp}(p_k)$. By Fact 3.7 and the mutual genericity assumption between $V[G_i]$ and $V[G_k]$, there is a semi-algebraic set $D \subseteq C$ definable from parameters in $\mathbb{R} \cap V$ such that $a^\frown y \in D$. Clearly, $D_a = \{y\}$. It follows that y is algebraic over $(\mathbb{R} \cap V) \cup a$ as desired. $\square$

Now, define the set $f(x) \subset \omega$ of forbidden colors by setting it to the union of $p_i''(\alpha(x,i,j) \cup \beta(x,i,j) \cup \gamma(x,i,j)$ for all choices of distinct indices $i, j \in 3$. By the claims and Definition 3.1(4) applied to $p_i \leq c$, $f(x) \in I$. Let $b \subset \omega$ be a set in the ideal I which cannot be covered by finitely many sets of the form $f(x)$ for $x \in d$, and finitely many singletons. Let $q \colon F^n \to \omega$ be any map extending $\bigcup_i p_i$ and such that $q \restriction d$ is an injection such that $q(x) \in b \setminus f(x)$ holds for every $x \in d$. I claim that q is the requested common lower bound of the conditions p_i for $i \in 3$.

Claim 3.13. *q is a Γ_n-coloring.*

Proof. Let $R \subset F^n$ be a rectangle; I must show that q is not constant on it. The proof breaks into numerous cases and subcases.

Case 1. R contains no elements of the set d. Let $a \subset 3$ be an inclusion minimal set such that $R \subset \bigcup_{i \in a} \mathrm{dom}(p_i)$.

Case 1.1. $|a| = 1$. Here, R is not monochromatic because p_i is a Γ_n-coloring where i is the unique element of a.

Case 1.2. $|a| = 2$, containing indices $i, j \in 3$. The closure properties of the domains of p_i and p_j imply that each set $\mathrm{dom}(p_i \setminus c)$ and $\mathrm{dom}(p_j \setminus c)$ contains exactly two points of R.

Case 1.2.1. The two points in $\mathrm{dom}(p_i \setminus C) \cap R$ are adjacent in R. Then the hyperplanes containing the two respective points and perpendicular to their connector are visible in both $V[g_i]$ and $V[G_j]$, so in V. The two points are opposite on these planes and therefore they receive distinct p_i colors by Definition 3.1(3). Therefore, R is not monochromatic.

Case 1.2.2. The two points in $\mathrm{dom}(p_i \setminus C) \cap R$ are opposite in R. Then both the center of the rectangle R and the real number which is half of the length of the rectangle diagonal belong to both $V[G_i]$ and $V[G_j]$, so to V. The hypersphere S they determine is visible from V, and the two points of $\mathrm{dom}(p_i \setminus C) \cap R$ are opposite on S. Applying Definition 3.1(2) to $p_i \leq c$, it is clear that the two points receive distinct p_i colors and R is not monochromatic.

Case 1.3. $|a| = 3$. Then there must be index $i \in 3$ such that $\mathrm{dom}(p_i \setminus c)$ contains exactly two points of R and $\mathrm{dom}(p_j \setminus c)$ contains exactly one point of R for each index $j \neq i$. I will show that this case cannot occur regardless of the colors on the rectangle R. For an index $j \neq i$, write x_j for the unique point in $R \cap \mathrm{dom}(p_j \setminus c)$.

Case 1.3.1. The two points in $\mathrm{dom}(p_i \setminus C) \cap R$ are adjacent in R. Consider the two hyperplanes Q_j, Q_k containing these two points respectively and perpendicular to their connecting segment, indexed by $j, k \neq i$. Reindexing if necessary, $x_j \in Q_j$ and $x_k \in Q_k$ holds. By Fact 3.7, there must be algebraic (even affine) sets $Q'_j \subseteq Q_j$ and $Q'_k \subseteq Q_k$ visible from the ground model and still containing

x_j and x_k. This means that x_k can be recovered in $V[G_j]$ as the closest point to x_j in Q'_k. This is impossible as $V[G_j] \cap V[G_k] = V$.
Case 1.3.2. The two points in $\mathrm{dom}(p_i \setminus C) \cap R$ are opposite in R. Consider the hypersphere S in which these two points are opposite. S then contains x_j and x_k and these two points are opposite in S. By Fact 3.7, there must be algebraic sets $S_j \subseteq S$ and $S_k \subseteq S$ visible from the ground model and still containing x_j and x_k. This means that x_k can be recovered in $V[G_j]$ as the farthest point to x_j in S_k. This is impossible as $V[G_j] \cap V[G_k] = V$.
Case 2. R contains exactly one point in the set d; call this unique point x. Let $a \subset 3$ be an inclusion minimal set such that $R \setminus \{x\} \subset \bigcup_{i \in a} \mathrm{dom}(p_i)$.
Case 2.1. $|a| = 1$. This cannot occur since $\mathrm{dom}(p_i)$ would contain x with the other three vertices of R, where $i \in 3$ is the only element of a.
Case 2.2. $|a| = 2$, containing indices $i, j \in 3$. Here, for one of the indices (say j) $\mathrm{dom}(p_j)$ has to contain two elements of R while $\mathrm{dom}(p_i \setminus c)$ contains just one; denote the latter point by x_i.
Case 2.2.1. The points x_i and x are opposite on the rectangle R. Then $x_i \in \alpha(x, i, j)$ as the hypersphere on which x_i, x are opposite points is the same as the one on which the other two points are opposite, and therefore is visible in $\mathrm{supp}(p_j)$. The choice of the map q shows that $q(x) \neq p_i(x_i)$, so R is not monochromatic.
Case 2.2.1. The points x_i and x are opposite on the rectangle R. Then $x_i \in \beta(x, i, j)$ as x_i, x are opposite points on the hyperplanes passing through the other two points and perpendicular to their connecting segment, and these are visible in $\mathrm{supp}(p_j)$. The choice of the map q shows that $q(x) \neq p_i(x_i)$, so R is not monochromatic.
Case 2.3. $|a| = 3$. For each index $i \in 3$ let $x_i \in R$ be the unique point in $\mathrm{dom}(p_i \setminus c)$. Let $i, j, k \in 3$ be indices such that the sequence x, x_i, x_j, x_k goes around the rectangle R. Then $x_i \in \gamma(x, i, j)$ holds. The choice of the map q shows that $q(x) \neq p_i(x_i)$, so R is not monochromatic.
Case 3. R contains more than one point in the set d. Then R is not monochromatic as $d \restriction d$ is an injection. $\square$

Finally, let $i \in 3$ be an index; I must prove that $q \leq p_i$ holds. It is clear that $p_i \subset q$ holds. The following claims verify other items of Definition 3.1,

Claim 3.14. *If $S \subset \mathbb{R}^n$ is a hypersphere visible in $\mathrm{supp}(p_i)$ and $x, y \in \mathrm{dom}(q \setminus p_i)$ are opposite points on it, then $q(x) \neq q(y)$.*

Proof. The arguments splits into cases.
Case 1. If x, y both belong to the set d, then $q(x) \neq q(y)$ as $q \restriction d$ is an injection.
Case 2. If $x \in d$ and $y \notin d$, let $j \in 3$ be an index distinct from i such that $y \in \mathrm{dom}(p_j \setminus c)$. Then, $y \in \alpha(x, j, i)$ holds and therefore $q(x) \neq p_j(y)$ as $q(x) \notin p''_j \alpha(x, j, i)$.
Case 3. If neither of the points x, y belongs to d, then there are two subcases.
Case 3.1. There is $j \in 3$ such that both x, y belong to $\mathrm{dom}(p_j) \setminus c$. In such a case, the hypersphere S is also visible in $\mathrm{supp}(p_j)$. By the mutual genericity of the models $V[G_i]$ and $V[G_j]$, the hypersphere S is visible in the ground model.

It follows that $q(x) = p_j(x) \neq p_j(y) = q(y)$ by Definition 3.1 (2) applied to $p_j \leq c$.

Case 3.2. $x \in \mathrm{dom}(p_j \setminus c)$ and $y \in \mathrm{dom}(p_k \setminus c)$ for distinct indices j, k. By the mutual genericity of the models $V[G_i]$ and $V[G_j]$ and Fact 3.7, there is an algebraic set $T \subset S$ coded in the ground model such that $x \in T$. Then x can be recovered in $V[G_k]$ as the point on T farthest away from y, contradicting the fact that $V[G_j] \cap V[G_k] = 0$. □

Claim 3.15. *If $P, Q \subset \mathbb{R}^n$ are parallel hyperplanes visible in* $\mathrm{supp}(p_i)$ *and $x, y \in \mathrm{dom}(q \setminus p_i)$ are opposite points on them, then $q(x) \neq q(y)$.*

Proof. The argument is similar to that for Claim 3.14. □

Claim 3.16. *If $a \subset F^n$ is a finite set, then $q''\delta(p_i, q, a) \in I$.*

Proof. For each index $j \in 3$ distinct from i, let $a_j \subset \delta(p_i, q, a) \cap \mathrm{dom}(p_j)$ be an inclusion-maximal set which is algebraically free over $\mathrm{supp}(p_i)$. Since sets algebraically free over $\mathrm{supp}(p_i)$ form a matroid, $|a_j| \leq |a|$. By Fact 3.7 $\delta(p_i, q, a) \cap \mathrm{dom}(p_j) = \delta(c, p_j, a_j)$ holds. This means that $\delta(p_i, q, a) = \delta(c, p_j, a_j) \cup \delta(c, p_k, a_k) \cup d$ where $j, k \in 3$ are the two indices distinct from i. Now, $p_j''(c, p_j, a_j) \in I$ by Definition 3.1(4) applied to $p_j \leq c$, $p_k''(c, p_k, a_j) \in I$ by Definition 3.1(4) applied to $p_k \leq c$, and $q''b \in I$ as this set is a subset of b. As the ideal I is closed under unions and subsets, $q''\delta(p_i, q, a) \in I$ as desired. □

This concludes the proof of the proposition. □

Finally, I can complete the proof of Theorem 1.2. Let $n \geq 2$ be a number. Let κ be an inaccessible cardinal. Let W be the choiceless Solovay model derived from κ. Let P_n be the Suslin poset of Definition 3.1, and let $G \subset P_n$ be a filter generic over W. $W[G]$ is a model of ZF+DC since it is a σ-closed extension of a model of ZF+DC. I claim that in $W[G]$, the chromatic number of Γ_n is countable while the chromatic number of Δ is not. The former assertion follows immediately from Corollary 3.6. The latter assertion follows from the conjunction of Theorem 2.5 and Theorem 3.9. The proof is complete.

References

[1] Jack Ceder. Finite subsets and countable decompositions of Euclidean spaces. *Rev. Roumaine Math. Pures Appl.*, 14:1247–1251, 1969.

[2] Paul Erdős and Péter Komjáth. Countable decompositions of $\mathbb{R}^2$ and $\mathbb{R}^3$. *Discrete and Computational Geometry*, 5:325–331, 1990.

[3] Thomas Jech. *Set Theory.* Springer Verlag, New York, 2002.

[4] Paul Larson and Jindřich Zapletal. *Geometric set theory.* AMS Surveys and Monographs. American Mathematical Society, Providence, 2020.

[5] James H. Schmerl. Avoidable algebraic subsets of Euclidean space. *Trans. Amer. Math. Soc.*, 352:2479–2489, 1999.

[6] Jindrich Zapletal. Coloring the distance graphs in three dimensions. 2021. submitted.

[7] Jindrich Zapletal. Krull dimension in set theory. 2021. submitted.

[8] Jindrich Zapletal. Noetherian spaces in choiceless set theory. 2021. in preparation.