Reducibility invariants in higher set theory[1]

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October 30, 2015

[1]2000 AMS subject classification 03E40.

[2]Partially supported by NSF grant DMS 1161078 and Institutional Research Plan No. AV0Z10190503 and grant IAA100190902 of GA AV ČR.

Preface

This book is a contribution to the classification theory of analytic equivalence relations. It shows that set theoretic techniques normally associated with the axiom of choice and combinatorics of uncountable cardinals can be efficiently used to prove new and difficult theorems about the structure of analytic equivalence relations. In many respects, this contradicts the conventional wisdom, which holds that the study of analytic equivalence relations is purely a matter of descriptive set theory and mathematical analysis and therefore impervious to efforts of combinatorial set theory. Thus, the resulting landscape is entirely unexpected and shows much promise for further investigation.

I developed the topic during my sabbatical year 2012-13. Before the summer 2012, almost no intuitions or results presented here existed. During the year, I gave a number of lectures (logic seminars at CalTech, UCLA, CTS Prague, as well as the set theory meeting in Luminy 2012) which documented the fast pace of development of the subject. In May 2013, I gave a minicourse at University of Münster that already contained many of the main themes present in this book.

I wish to thank the following individuals for insightful discussions on the subject of the book: Bohuslav Balcar, John Clemens, Vladimir Kanovei, Alexander Kechris, Andrew Marks, Benjamin Miller, Marcin Sabok, and Sławomir Solecki.

I gratefully acknowledge the financial support from the following sources: DFG SFB grant 878, NSF DMS grant 1001623, Institutional Research Plan No. AV0Z10190503 and grant IAA100190902 of Grant Agency of the Academy of Sciences of the Czech Republic, NSF DMS grant 1161078, and the generous sabbatical funding from University of Florida. Last but not least, I would like to thank the Fields Institute for Research in Mathematical Sciences for its support and hospitality during the Thematic Program on Forcing and its Applications in the fall of 2012.

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Chapter 1

Introduction

1.1 Outline of the subject

In the last quarter century, the theory of analytic equivalence relations traveled the road from non-existence to general acceptance and many successful applications [5, 12, 7, 9]. To outline its basic concern, for Polish spaces X, Y equipped with respective analytic equivalence relations E, F, say that E is *reducible to* Y if there is a Borel function $f: X \to Y$ such that for all $x_0, x_1 \in X$, $x_0\ E\ x_1 \leftrightarrow f(x_0)\ F\ f(x_1)$ holds. The theory analyzes this quasiorder as a natural measure of complexity of equivalence relations, establishes many reducibility and non-reducibility results, and places preexisting analytic equivalence relations from various areas of mathematics on the resulting map of the reducibility relation. The theory is highly relevant for all classification programs in contemporary mathematics. Among the recent successes, I will mention the evaluation of the complexity of the isomorphism relation for various classes of separable C^* algebras. It turned out that for the most general classes, the isomorphism equivalence relation is the most complex orbit equivalence relation. This development caused a fundamental reevaluation and redirection of the classification program for separable C^* algebras [3].

The methods of the theory of analytic equivalence relations typically do not reach far beyond the scope of classical descriptive set theory; all its basic concepts have been familiar to descriptive set theorists for a long time. The purpose of this book is to show that there is a natural connection between analytic equivalence relations and such notions of transfinite set theory as forcing, uncountable cardinals, or axiom of choice related combinatorics. The connection, in addition to being entirely unexpected, is also very fruitful: a number of important older results can be recategorized and broadly generalized, their proofs greatly streamlined. The new method is wide open for further applications. For example, the results of Section 4.1 clearly point to connections with another seemingly distant field: Shelah's classification theory of models.

To connect the theory of analytic equivalence relations with transfinite notions of set theory, consider the following natural expansion of a given equivalence relation on a Polish space to an equivalence on a proper class in the sense of set theory. Note that Polish spaces and their analytic subsets have canonical reinterpretations in all generic extensions.

Definition 1.1.1. Let E be an analytic equivalence relation on a Polish space X. Let P be a poset and τ a P-name for an element of X. Say that τ is and *E-symmetric name* if for any conditions $p_0, p_1 \in P$, in some generic extensions there are filters $G_0, G_1 \subset P$ which are separately generic over V, $p_0 \in G_0$, $p_1 \in G_1$, and $\tau/G_0 \mathrel{E} \tau/G_1$.

Definition 1.1.2. Let E be an analytic equivalence relation on a Polish space X. Let $\bar{X}$ be the proper class of all pairs $\langle P, \tau\rangle$ where P is a poset and τ is an E-symmetric P-name for an element of X. Let $\bar{E}$ be an equivalence relation on $\bar{X}$ defined by $\langle P, \tau\rangle \mathrel{\bar{E}} \langle Q, \sigma\rangle$ if in some generic extension there are filters $G \subset P$ and $H \subset Q$, separately generic over V, such that $\tau/G \mathrel{E} \sigma/H$.

Observe that the definition of $\bar{X}$ depends not only on X but also on the relation E. It is not difficult to see that $\bar{E}$ is an equivalence relation on $\bar{X}$. The equivalence relation E can be naturally embedded to $\bar{E}$ using the map $x \mapsto \langle P, \check{x}\rangle$ where $x \in X$, P is any poset (say, a trivial poset with one condition), and $\check{x}$ is the canonical P-name for x in the P-extension. This leads to the following definition.

Definition 1.1.3. An E-symmetric name τ on a poset P is *trivial* if it is $\bar{E}$-equivalent to the name $\check{x}$ on P for some element $x \in X$.

The point of expanding the relation E to $\bar{E}$ is that there are many nontrivial E-symmetric names which can be investigated and classified. It is typically useful to restrict attention to some $\bar{E}$-invariant class of symmetric names. A natural $\bar{E}$-invariant class of E-symmetric names is closely connected with the notion of turbulence introduced by Greg Hjorth [7, 8]:

Definition 1.1.4. Let E be an analytic equivalence relation on a Polish space X. Let τ be an E-symmetric name on a poset P. Say that τ is *E-trim* if in some generic extension there are filters $G, H \subset P$ separately generic over V such that $V[G] \cap V[H] = V$, and $\tau/G \mathrel{E} \tau/H$. The equivalence relation E is P-trim if there are no nontrivial E-symmetric names on P.

It is often the case that common names such as a name for the generic or random element of the underlying space are trim.

Theorem 1.1.5. (Theorem 3.1.1) *Let $\Gamma \curvearrowright X$ be a continuous action of a Polish group on a Polish space, inducing an orbit equivalence relation E all of whose classes are meager and dense. The following are equivalent:*

1. *the action is generically turbulent;*

2. *the name for a generic element of X is E-trim.*

This restatement of turbulence has a number of advantages–it abstracts from the dynamical context, it opens the field to forcing methods, and it is much easier to use in applications provided that the basic forcing theory is taken for granted.

Corollary 1.1.6. (Corollary 3.1.3) *If E is an orbit equivalence relation of a generically turbulent action, then every Borel homomorphism of E to a Cohen-trim or treeable equivalence relation stabilizes on a comeager set.*

The class of Cohen-trim equivalence relations includes the relations classifiable by countable structures (Corollary 2.2.16), and so this theorem generalizes the motivational Hjorth's ergodicity result [12, Lemma 13.3.4]. However, this class includes many other relations, such as $=_J$ for the ideal J of nowhere dense subsets of the rational numbers (Theorem 3.2.28 and Example 3.2.27). Thus, we get a lot of new information. Another advantage of trimness over turbulence methods is based on the possibility of abstracting from the Baire category and group action context.
Most theorems of Chapter 3 deal with equivalence relations $=_J$, the equality modulo J on 2^ω, for various analytic ideals J on ω. All ideals in this book, by definition, contain all singletons and not ω. This is a really broad subject–a result of Rosendal [21] shows that equivalence relations of this type can be found arbitrarily high in the Borel reducibility hierarchy. I characterize various trimness properties of these equivalence relations through standard combinatorial properties of the ideals:

Recall that an ideal J is *ω-hitting* if for any countable collection of infinite subsets of ω ets there is a set in J with nonempty intersection with all of them.

Theorem 1.1.7. (Theorem 3.2.4) *Let J be an analytic ideal on ω. The name for a generic element of ω is $=_J$-trim if and only if the ideal J is ω-hitting.*

Corollary 1.1.8. *Let J be an ω-hitting analytic ideal on ω. Every Borel homomorphism of $=_J$ to a Cohen-trim equivalence relation stabilizes on a comeager set.*

The case of the name for the random element of 2^ω is considerably more complicated. Curiously, a connection with the well-known concept of concentration of measure appears:

Definition 1.1.9. A lower semicontinuous submeasure ϕ on ω *exhibits concentration of measure* if for ever positive real $\varepsilon > 0$ there is a positive real $\delta > 0$ such that for all but finitely many n, for all but finitely many m, for all sets $C_0, C_1 \subset 2^{m\setminus n}$ such that $|C_0|, |C_1| > (1-\delta)2^{m-n-1}$ there are $x_0 \in C_0$ and $x_1 \in C_1$ such that the set $\{k \in m \setminus n \colon x_0(k) \neq x_1(k)\}$ has ϕ-mass at most ε.

Theorem 1.1.10. (Theorem 3.2.10) *Let J be an analytic P-ideal on ω obtained from a lower semicontinuous submeasure ϕ on natural numbers. The name for a random element of 2^ω is $=_J$-trim if and only if ϕ exhibits the concentration of measure.*

The verification of the concentration of measure condition is in most cases nontrivial. Applications may be striking:

Corollary 1.1.11. (Corollary 3.2.16) *If E is the usual summable equivalence relation on 2^ω, then every Borel homomorphism of E to a trim or treeable equivalence relation stabilizes on a set of full measure.*

Recall that an ideal J on ω is *tall* if every infinite set contains an infinite subset in J.

Theorem 1.1.12. (Theorem 3.3.3) *Let J be an analytic ideal. The name for a generic element of ω is $=_J$-σ-trim if and only if the ideal J is tall.*

Corollary 1.1.13. *Let J be a tall analytic ideal on ω. Every Borel homomorphism of $=_J$ to an equivalence relation classifiable by countable structures stabilizes on a comeager set.*

Among further more sophisticated results of Chapter 3, I quote

Theorem 1.1.14. *Let J be the ideal of nowhere dense subsets of the rational numbers, and K be the ideal of subsets of rational numbers with Lebesgue null closure. Every Borel homomorphism from $=_J$ to $=_K$ stabilizes on a comeager subset of $2^{\mathbb{Q}}$ and also on a subset of $2^{\mathbb{Q}}$ of a full mass.*

Chapter 4 deals with another useful class of symmetric names which was introduced, in slightly different language, by Kanovei [12, Chapter 17]:

Definition 1.1.15. Let E be an analytic equivalence relation on a Polish space X. Let τ be an E-symmetric name on a poset P. Say that τ is *E-pinned* if in some generic extension there are mutually generic filters $G, H \subset P$ such that $\tau/G \; E \; \tau/H$. The equivalence relation is pinned if there are no nontrivial pinned names on any posets.

It turns out that for many equivalence relations E, the $\bar{E}$-classes of E-pinned names can be sensibly classified or connected to interesting model theoretic and combinatorial features of uncountable cardinals. The following cardinal invariant plays a significant role:

Definition 1.1.16. Let E be an analytic equivalence relation on a Polish space X.

1. If τ is an E-symmetric name on a poset P, let $\kappa(\tau)$ be the minimum cardinal κ such that τ has a $\bar{E}$-equivalent on a poset of size κ.
2. $\kappa(E)$, the *pinned cardinal* of E, equals to $\sup\{\aleph_1, \kappa(\tau)^+ : \tau \text{ is an } E\text{-pinned name}\}$ or to ∞ if the aforementioned supremum does not exist.

The cardinal $\kappa(E)$ is an invariant of the Borel reducibility ordering and so it can be used for non-reducibility arguments. It is smaller than $\beth_{\omega_1}$ for Borel equivalence relations E (Theorem 4.2.3), and can attain curious values. For example, it is not difficult to find a Borel equivalence relation E such that $\kappa(E)$ is provably equal to $(\aleph_\omega^{\aleph_0})^+$, connecting the classification of analytic equivalence relations with singular cardinal combinatorics.

Example 1.1.17. (Example 4.3.6) There are Borel equivalence relations E, F classifiable by countable structures such that the natural proof of non-reducibility of E to F leads through the proof of independence of the Singular Cardinal Hypothesis on $\aleph_\omega$.

The classification of pinned names also yields ergodicity theorems similar to the ones obtained through generalizations of turbulence:

Theorem 1.1.18. (Theorem 4.5.12) *Suppose that E is the mutual domination equivalence relation on $(\omega^\omega)^\omega$. If h is a Borel homomorphism from E to an orbit equivalence relation, then h stabilizes on a set large with respect to the mutual domination ideal.*

The whole subject offers many sensible directions for further development, which can shed new light upon the classification of analytic equivalence relations. I state two open questions:

Question 1.1.19. Classify the F_σ ideals J on ω for which the relation $=_J$ is trim.

Question 1.1.20. Classify the pinned names for the measure equivalence.

1.2 Navigation

Chapter 2 contains results which frame the general context of the present subject. Section 2.1 contains basic observations about the class of E-symmetric names. Section 2.2 introduces a general pattern used for all variations of trimness in this book. The main result is Theorem 2.2.13 which shows that the class of equivalence relations exhibiting a given variation of trimness is closed under a great number of natural operations. Section 2.3 proves general ergodic theorems that are later applied to obtain many specific ergodicity results in this book.

Chapter 3 is the part of the book which is easiest to connect to previous work of Greg Hjorth. Section **??** shows the connection of the trimness concept with the turbulence concept. Sections 3.2 and 3.3 provide a number of characterization theorem for trimness in equivalence relations of the type $=_J$ where J is an ideal on ω. The combination of results of this kind together with Theorems 2.3.1 and 2.3.2 yields ergodicity results which are the raison d'etre of this book. Section 3.4 looks at more complicated variations of trimness in order to establish ergodicity results for ideals higher in the Katětov hierarchy.

Chapter 4 studies the notion of a pinned name and a pinned equivalence relation. Section 4.1 shows that in a good number of cases the $\bar{E}$-classes of E-pinned names can be naturally classified. I provide a new proof of Friedman–Stanley theorem and open the connection to model theory. Sections 4.2 and 4.3 deal with the concept of the pinned cardinal. I provide several upper bounds for its values, of which the most important is that the pinned cardinal of a Borel equivalence relation is less than $\beth_{\omega_1}$. There are several calculations of the values of the pinned cardinal which are associated with the combinatorics of small uncountable cardinals. For example, there are Borel equivalence relations whose pinned cardinal is (provably in ZFC) equal to $\aleph_\alpha$ for any countable nonzero ordinal α or, more strikingly, to $\max\{\mathfrak{c}, \aleph_\omega^{\aleph_0}\}^+$. Such results make it possible to prove non-reducibility results between Borel equivalence relations using consistency of various combinatorial statements about small uncountable cardinals. Section 4.4 proves several dichotomies for the values of the pinned cardinal. Section 4.5 shows that on a number of natural posets such as proper posets, no pinned names can exist. The most important result in this direction is Corollary 4.5.8 showing that in ZFC, there are no pinned names on $\aleph_1$-preserving posets for orbit equivalence relations. There is a natural equivalence relation which does have a pinned name on the Namba poset, and this leads to an ergodicity result, Theorem 4.5.12.

Chapter 5 is targeted at the part of the audience with pure set theoretic inclinations. The trimness variations introduced in this book serve as a vehicle for ergodicity results, which are descriptive set theoretic in nature. However, the trimness variations themselves are obtained by quantifying over chunks of the set theoretical universe far bigger than the part immediately relevant to descriptive set theory. This misalignment raises the question whether the trimness variations are absolute between forcing extensions. In the most important development in the chapter, I show that the statements "E is trim" and "E is pinned" are both absolute between generic extensions for Borel equivalence relations E.

The appendix contains basic information on forcing. Section 6.2 shows that finding suitable generic filters over countable models of ZF is a Borel procedure. This is important at several points in the book.

1.3 Notation

The book uses the standard set theoretic notation of [11] and the nomenclature of analytic equivalence relations of [5]. Thus, E_0 denotes the modulo finite equivalence relation on 2^ω, E_1 denotes the modulo finite equality on $(2^\omega)^\omega$, $=_+$ is the equality of ranges on $(2^\omega)^\omega$. E_{ω_1} is the analytic equivalence relation on the space X of binary relations on ω connecting x, y if they are isomorphic or if both fail to be wellorderings. E_{K_σ} is the equivalence relation on ω^ω connecting x, y if there is a number $n \in \omega$ such that for every $m \in \omega$, $|x(n) - y(m)| < n$ holds. The measure equivalence is the relation on the space of Borel probability measures on 2^ω connecting μ and λ if they share the same σ-ideal of null sets.

If E, F are equivalence relations on respective sets X, Y, then a function $h\colon X \to Y$ is called a *homomorphism* of E to F if for all $x_0, x_1 \in X$, $x_0 \mathrel{E} x_1 \to h(x_0) \mathrel{F} h(x_1)$. The function h is a *reduction* of E to F if the latter implication is in fact an equivalence. A homomorphism h *stabilizes* on a set $A \subset X$ if the image $h''A$ consists of pairwise F-related elements. I use a somewhat nonstandard, but very useful, notion of almost reduction and almost homomorphism:

Definition 1.3.1. Let E, F be analytic equivalence relations on respective Polish spaces X, Y. An *almost reduction* of E to F is a map $h\colon X \to Y$ such that there is a countable set $a \subset X$ such that for all $x_0, x_1 \in X \setminus [a]_E$ $x_0\ E\ x_1 \leftrightarrow h(x_0)\ F\ h(x_1)$. Similarly for almost homomorphism. We say that E is *Borel almost reducible* to F if there is a Borel almost reduction of E to F.

It is not difficult to see that Borel almost reducibility is a quasiorder coarser than Borel reducibility of analytic equivalence relations. The Borel reducibility of equivalence relations is denoted by $\leq_{\mathrm{B}}$, the Borel almost reducibility is denoted by $\leq_{\mathrm{aB}}$.

The nomenclature of analytic ideals is taken from [10]. All ideals in this book are assumed to contain all singletons and not ω. The Fréchet ideal is the ideal of finite sets. The *summable ideal* is the collection $J = \{a \subset \omega\colon \sum_{n\in a} \frac{1}{n+1} < \infty\}$. The *Rado graph ideal* is the ideal generated by the cliques and anticliques of the Rado graph on ω, where the Rado graph is the unique ultrahomogeneous and universal graph on ω. Ideals on ω are preordered by the *Katětov order* $\leq_K$, where $I \leq_K J$ if there is a function $f \in \omega^\omega$ such that f-preimages of I-small sets are in the ideal J; in particular, if $I \subset J$ then $I \leq_K J$.

Among other objects used in this book, I will mention the *nonempty intersection of ranges graph*. This is the Borel graph on $(2^\omega)^\omega$ connecting points x, y if $\mathrm{rng}(x) \cap \mathrm{rng}(y) = 0$.

Regarding forcing, all Polish spaces and their analytic subsets have canonical interpretations in forcing extensions. The interpretation theory is remarkably smooth, developed in detail in [13, Section 2.4]; in this book I will take it for granted as is customary. If P is a poset and X is a Polish space and $A \subset X$ is an analytic set, then $\dot{X}$ and $\dot{A}$ are P-names for the canonical interpretations of X and A respectively. If $h\colon X \to Y$ is a Borel function between two Polish spaces, P is a poset and τ is a P-name for an element of X then $h\tau$ denotes the P-name for an element of Y obtained by an application of (the interpretation of) the function h to τ in the P-extension. A point x in the (canonical interpretation of the) space X is called *generic* over V if it is obtained by the Cohen poset P_X of nonempty open subsets of X ordered by inclusion. If μ is a Borel probability measure on X then a point x in the (canonical interpretation of the) space X is called *random* over V if it is obtained by the Solovay poset of compact subsets of X of positive μ-mass, ordered by inclusion. In the special case where $X = 2^\omega$, the random point is always understood to be obtained from the usual product measure on 2^ω. A poset P is a *Suslin forcing* if it is an analytic subset of a Polish space and both compatibility and incompatibility of conditions in P are analytic relations. If P is a poset and $p \in P$ is a condition, then $P \restriction p$ denotes the poset of all elements of P which are $\leq p$.

Mostowski absoluteness states that transitive models of set theory agree on the truth value of $\mathbf{\Sigma}^1_1$ statements with parameters in the models. Shoenfield absoluteness states that transitive models of set theory containing ω_1^V agree on the truth value of $\mathbf{\Pi}^1_2$ statements with parameters in the models.

Chapter 2

General treatment

2.1 Symmetric names

The first concern of the book is the identification of some symmetric names on posets. In principle, the symmetric names form a very broad class; however, this book uses just a few constructions. First, observe that nontrivial E-symmetric names occur for a Borel equivalence relation E as soon as E is non-smooth.

Proposition 2.1.1. *Let E be a Borel equivalence relation on a Polish space X. Exactly one of the following occurs:*

1. *E is smooth;*
2. *there is a poset P and a nontrivial E-symmetric name on P.*

Proof. Start with the (1)→ ¬(2) implication. Suppose that E is smooth and fix a Borel reduction $h\colon X \to 2^\omega$ of E to the identity. By the Shoenfield absoluteness h remains such a reduction in all forcing extensions. If P is a poset and τ is a P-name for an element of X, then exactly one of the following occurs. Either, there are conditions $p_0, p_1 \in P$ and a natural number $n \in \omega$ such that $p_0 \Vdash h(\tau)(n) = 0$ and $p_1 \Vdash h(\tau)(n) = 1$. In such a case, τ is not an E-symmetric name, since in any generic extension, if $G_0, G_1 \subset P$ are filters generic over V such that $p_0 \in G_0$ and $p_1 \in G_1$, then $h(\tau/G_0) \neq h(\tau/G_1)$ and therefore $\tau/G_0 \mathrel{E} \tau/G_1$ fails. Or, there is $y \in 2^\omega$ such that $P \Vdash h(\tau) = y$. In this case, by the Mostowski absoluteness between V and its P-extension, there is a point $x \in X$ such that $h(x) = y$ and then $P \Vdash \tau \mathrel{E} \check{x}$ and τ is trivial. This proves the failure of (2).

The ¬(1)→(2) implication, suppose that E is not smooth. By the E_0 dichotomy, there is a Borel reduction $h\colon 2^\omega \to X$ of E_0 to E. Observe that the usual Cohen name τ for a generic element of 2^ω is a nontrivial E_0-symmetric name, and then $h\tau$ is a nontrivial E-symmetric name, confirming (2). □

Question 2.1.2. Does the conclusion of the proposition remain in force for analytic equivalence relations?

The most interesting symmetric names are constructed directly from the topological properties of the underlying Polish space. As the first class of examples, for every Polish space X consider the poset P_X of all nonempty open subsets of X ordered by inclusion. The poset adds a *Cohen generic* point of the space X, namely the unique point $\dot{x}_{gen} \in X$ which belongs to all sets in the generic filter. For many equivalence relations E on X, the name $\dot{x}_{gen}$ turns out to be E-symmetric:

Proposition 2.1.3. *Suppose that Γ is a Polish group continuously acting on a Polish space X so that all orbits are dense. Let E be the resulting orbit equivalence relation. The P_X-name for a Cohen generic element of X is E-symmetric.*

Proof. As a first observation, use the Shoenfield absoluteness to observe that in every generic extension, the orbits of the actions are going to be dense. Now, let $p_0, p_1 \in P_X$ be nonempty open sets; in some generic extension, I have to find P_X-generic points $x_0, x_1 \in X$ such that $x_0 \in p_0$, $x_1 \in p_1$, and $x_0 \mathrel{E} x_1$. To this end, pass to a P_X-generic extension with a generic point $x_0 \in p_0$. The orbit of x_0 is dense in X; in particular, it intersects the open set p_1. Since the action is continuous, and the ground model points of Γ are dense in $\Gamma \cap V[x_0]$, there is a group element $\gamma \in \Gamma$ in the ground model such that $x_1 = \gamma \cdot x_0 \in p_1$. Now, note that acting by γ naturally generates an automorphism of the poset P. It follows that the point $x_1 \in p_1$ is also P_X-generic over V. Thus, the points x_0, x_1 witness the symmetricity of the name for a Cohen generic element of X. □

As the second class of cases, for every Polish Borel probability space $\langle X, \mu \rangle$ consider the poset $P_{X\mu}$ of all Borel subsets of X of nonzero μ-mass. The poset adds a *random* point of the space X, namely the unique point $\dot{x}_{gen}$ which belongs to all sets in the generic filter. Again, the name $\dot{x}_{gen}$ turns out to be E-symmetric for many equivalence relations E:

Proposition 2.1.4. *Suppose that Γ is a Polish group acting in a measure preserving way on a Polish Borel probability space $\langle X, \mu \rangle$. Let E be the associated orbit equivalence relation. Suppose that there is a countable subgroup $\Delta \subset \Gamma$ whose action on $\langle X, \mu \rangle$ is ergodic. The $P_{X\mu}$-name for a random generic point is E-symmetric.*

Proof. Suppose that $p_0, p_1 \in P_{X\mu}$ are arbitrary conditions. By the ergodicity assumptions, there is a condition $q \subset p_0$ and an element $\gamma \in \Delta$ such that $g \cdot q \subset p_1$. Pass to a $P_{X\mu}$-generic extension with a random point $x_0 \in q$. Let $x_1 = \gamma \cdot x_1 \in p_1$. Since the action by Γ preserves the measure μ, the action by γ induces an automorphism of the poset $P_{X\mu}$ and so the point x_1 is also random over V. Thus, the points x_0, x_1 witness the symmetricity of the name for a random element of X. □

There is a great number of E-symmetric names which do not fall into these two basic cases. The possible constructions are not particularly well understood. I provide an example showing the wealth of possibilities. Define the *shifted* *domination* quasiorder on X =increasing functions in ω^ω by $x \leq y$ if for some $n \in \omega$, for all $m \in \omega$ $y(m+n) > x(m)$. Define the *shift domination* equivalence relation E on X as the intersection of $\leq$ and its inverse. The shift domination equivalence is bireducible to with the most complicated K_σ equivalence relation.

Example 2.1.5. Let P be the poset of Laver trees consisting of increasing sequences, $\dot{x}$ the P-name for a generic element of X. Let E be the shift domination equivalence. Then $\dot{x}$ is an E-symmetric name.

Proof. Let $p_0, p_1 \in P$ be arbitrary conditions; strengthening one of them if necessary, it can be assumed that both have a trunk of the same length, say $n \in \omega$. Pass to a generic extension in which $\mathcal{P}(\omega^\omega) \cap V$ is countable. By a usual fusion argument, it is possible to find Laver conditions $q_0 \leq p_0$ and $q_1 \leq p_1$ such that they both have trunks of length n, and all their branches are P-generic over V. It is easy to find branches x_0 through q_0 and x_1 through q_1 such that for every $m \geq n$, $x_0(m), x_1(m)$ are both greater than all numbers in the ranges of $x_0 \restriction m$ and $x_1 \restriction m$. The points $x_0, x_1 \in X$ are E-equivalent, P-generic, and they meet the conditions p_0, p_1 respectively. This proves that $\dot{x}$ is an E-symmetric name. $\square$

A great number of canonization results of [13] can be interpreted as saying that certain posets do not carry E-symmetric names for certain equivalence relations E. For example, in [13, Section 6.3] it is shown that for every equivalence relation F classifiable by countable structures on all branches of a Laver tree, there is a smaller Laver tree whose branches are either all pairwise F-inequivalent, or otherwise all pairwise F-equivalent. It is easy to see that this result says that there are no F-symmetric names on the Laver forcing for equivalence relations F classifiable by countable structures.

To wrap up the introductory investigation into the class of symmetric names, I provide two easy propositions. One of them shows that everything about an E-symmetric name depends only on the name and not really on the poset surrounding it; the other implies that $\bar{E}$ is an equivalence relation.

Proposition 2.1.6. *Let E be an equivalence relation on a Polish space X. Suppose that P is a poset regularly embedded into Q, and τ is a P-name for an element of X. Then*

1. *τ is E-symmetric as a name on P if and only if it is E-symmetric as a name on Q;*
2. *$\langle P, \tau \rangle \; \bar{E} \; \langle Q, \tau \rangle$.*

Proof. For the right-to-left direction of (1), suppose that τ is E-symmetric as a name on Q, and $p_0, p_1 \in P \subset Q$ are conditions. In some generic extension, let $G_0, G_1 \subset Q$ be filters generic over V such that $p_0 \in G_0$, $p_1 \in G_1$, and $\tau/G_0 \; E \; \tau/G_1$. Then, $G_0 \cap P$ and $G_1 \cap P$ are filters on P generic over V, $\tau/G_0 = \tau/G_0 \cap P$ and $\tau/G_1 = \tau/G_1 \cap P$ are E-related points, and so the filters $G_0 \cap P$ and $G_1 \cap P$ witness the fact that τ is an E-symmetric name on P.

For the left-to-right direction of (1), suppose that τ is E-symmetric as a name on P, and $q_0, q_1 \in Q$ are conditions. As P is a regular subposet of Q, there must be $p_0 \in P$ such that every $p \leq p_0$ in P is compatible with q_0, and there also must be $p_1 \in P$ such that every $p \leq p_1$ in P is compatible with q_1 in Q. In some generic extension, find filters $G_0, G_1 \subset P$ generic over V such that $p_0 \in G_0$ and $p_1 \in G_1$ and $\tau/G_0\ E\ \tau/G_1$. By Fact 6.1.6, in a further generic extension there are filters $H_0, H_1 \subset Q$ generic over V such that $G_0 \subset H_0$, $G_1 \subset H_1$ and $q_0 \in H_0$ and $q_1 \in H_1$. Clearly, $\tau/G_0 = \tau/H_0$ and $\tau/G_1 = \tau/H_1$ are E-related points, and so the filters H_0 and H_1 witness the fact that τ is an E-symmetric name on P.

(2) is immediate: if $G \subset Q$ is a filter generic over V and $H = G \cap P$, then H is a filter generic over V as well and the points $\tau/G, \tau/H \in X$ are equal, and therefore E-related. $\square$

Proposition 2.1.7. *Let E be an equivalence relation on a Polish space X. Let $\langle P, \tau\rangle$ and $\langle Q, \sigma\rangle$ be E-symmetric names. The following are equivalent:*

1. *$\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$;*
2. *$P \Vdash$ for every condition $q \in Q$, in some further forcing extension there is a filter $H \subset Q$ containing q, generic over V, such that $\tau/\dot{G}\ E\ \sigma/H$, whener $\dot{G}$ is the canonical P-name for a generic filter on P.*

Proof. (2)→(1) follows immediately from the definition of the relation $\bar{E}$. (1)→(2) is a little harder and I will first prove it in the case $P = Q$ and $\tau = \sigma$. Suppose towards a contradiction that there are conditions $p_0, p_1 \in P$ such that $p_0 \Vdash$in no further forcing extension there is a filter $H \subset P$ containing p_1, generic over V, such that $\tau/\dot{G}\ E\ \tau/H$. Use the fact that τ is an E-symmetric name to find, in some generic extension, filters $G_0, G_1 \subset P$, separately generic over V such that $\tau/G_0\ E\ \tau/G_1$ and $p_0 \in G_0$ and $p_1 \in G_1$. In the model $V[G_0]$, the sentence "in some generic extension there is a filter $H \subset P$ containing p_1, generic over V, such that $\tau/G_0\ E\ \tau/H$" holds, as witnessed by H. This contradicts the choice of the conditions p_0, p_1 and the forcing theorem.

Now, treat the case of a general pair $\langle Q, \sigma\rangle$ $\bar{E}$-related to $\langle P, \tau\rangle$. Suppose towards a contradiction that there are conditions $p \in P$ and $q \in Q$ such that $p \Vdash$ in no further forcing extension there is a filter $H \subset Q$ containing q, generic over V, such that $\tau/\dot{G}\ E\ \sigma/H$. Use the definition of $\bar{E}$ to pass to a forcing extension in which there are filters $G_0 \subset P$ and $H_0 \subset Q$ such that $\tau/G_0\ E\ \sigma/H_0$. Use the first paragraph on P and Q and the forcing theorem to see that in some further forcing extension, there are filters $G \subset P$ and $H \subset Q$ separately generic over V such that $p \in G$, $q \in H$, $\tau/G_0\ E\ \tau/G$ and $\sigma/H_0\ E\ \sigma/H$. By the transitivity of the equivalence relation E, $\tau/G\ E\ \sigma/H$, and so the filters G, H violate the assumptions on p, q and the forcing theorem. $\square$

Corollary 2.1.8. *$\bar{E}$ is an equivalence relation. Moreover, $\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$ is equivalent to the statement that $\tau \dot{\cup} \sigma$ is an E-symmetric name on $P \dot{\cup} Q$.*

Here, $P \dot\cup Q$ is the disjoint union of the posets P, Q, and $\tau \dot\cup \sigma$ is a $P \dot\cup Q$-name which is equal to τ on P and to σ on Q.

Proof. For the first sentence, it is only necessary to show that $\bar{E}$ is transitive. Suppose that $\langle P_0, \tau_0\rangle \ \bar{E} \ \langle P_1, \tau_1\rangle \ \bar{E} \ \langle P_2, \tau_2\rangle$ and work to prove that $\langle P_0, \tau_0\rangle \ \bar{E} \ \langle P_2, \tau_2\rangle$. Let $G_0 \subset P_0$ be a generic filter. By the proposition applied to P_0, in some generic extension there is $G_1 \subset P_1$, a filter generic over V such that $\tau_0/G_0 \ E \ \tau_1/G_1$. By the proposition applied to P_1, in some further generic extension there is a filter $G_2 \subset P_2$ generic over V such that $\tau_1/G_1 \ E \ \tau_2/G_2$. The transitivity of the relation E now shows that the filters G_0, G_2 witness the statement $\langle P_0, \tau_0\rangle \ \bar{E} \ \langle P_2, \tau_2\rangle$ as desired.

The second sentence follows immediately from the proposition. $\square$

A final note in this section shows that realizations of E-symmetric names form an E-invariant class in a suitable sense.

Proposition 2.1.9. *Let E be an analytic equivalence relation on a Polish space X. Let τ be an E-symmetric name on a poset P and let σ be a name for an element of X on a poset Q. If in some generic extension there are filters $G \subset P$ and $H \subset Q$ separately generic over V such that $\tau/G \ E \ \sigma/H$, then there is a condition $q \in H$ such that σ is E-symmetric on $Q \restriction q$ and $\bar{E}$-related to τ.*

Proof. Stay in the ground model. The assumptions imply that there is a poset R and R-names $\dot{G}$ and $\dot{H}$ for filters on P and Q respectively, separately generic over V such that $\tau/\dot{G} \ E \ \sigma/\dot{H}$. By Fact 6.1.6(1), there must be a condition $q \in Q$ such that for every $q' \leq q$ there is a condition $r \in R$ forcing $\check{q}' \in \dot{H}$. I claim that the name σ is E-symmetric on $Q \restriction q$.

To this end, suppose that $q_0, q_1 \leq q$ are conditions in Q. Find conditions $r_0, r_1 \in R$ such that $r_0 \Vdash \check{q}_0 \in \dot{H}$ and $r_1 \Vdash \check{q}_1 \in \dot{H}$. Use Fact 6.1.6(1) to find conditions $p_0, p_1 \in P$ such that for every $p_0' \leq p_0$ and $p_1' \leq p_1$ there are conditions $r_0' \leq r_0$ and $r_1' \leq r_1$ in the poset R forcing $\check{r}_0' \in \dot{G}$ and $\check{r}_1' \in \dot{G}$ respectively. Use the E-symmetricity of the name τ to find a generic extension in which there are filters $G_0, G_1 \subset P$ generic over V such that $p_0 \in G_0$, $p_1 \in G_1$, and $\tau/G_0 \ E \ \tau/G_1$. Use Fact 6.1.6(2) to pass to a further generic extension in which there are filters $K_0, K_1 \subset R$ such that $r_0 \in K_0$, $r_1 \in K_1$, and $G_0 = \dot{G}/K_0$ and $G_1 = \dot{G}/K_1$ holds. Let $H_0 = \dot{H}/K_0$ and $H_1 = \dot{H}/K_1$. These are two filters on Q generic over V, contain the conditions q_0 and q_1, by the forcing theorem $\sigma/H_0 \ E \ \tau/G_0$ and $\sigma/H_1 \ E \ \sigma/G_1$, and by the transitivity of the relation E, $\alpha/H_0 \ E \ \sigma/H_1$ as desired.

Thus, the name σ is E-symmetric on $Q \restriction q$. It is $\bar{E}$-related to τ by the definitions. $\square$

2.2 Trimness variations

The methodology developed in this book depends on many variations of trimness, each of them useful for a different purpose. Each variation is defined using a certain system of generic filters. It is worthwhile to establish a very general pattern of these variations and treat them at this very general level. The following definition will be helpful.

Definition 2.2.1. Let $n \leq \omega$. A formula ϕ taking as arguments n-tuples of generic extensions of the ground model is a *perpendicularity relation* if

1. $\phi(V, V, V, \dots)$ holds;
2. the validity of $\phi(V[G_i] \colon i \in n)$ depends only on the models and not on the particular choice of the generic filters;
3. if $\phi(V[G_i] \colon i \in n)$ holds, $P_i \in V$ are posets and $H_i \subset P_i$ are separately generic filters over V in $V[G_i]$ for $i \in n$, then $\phi(V[H_i] \colon i \in n)$ holds;
4. if $\phi(V[G_i] \colon i \in n)$ holds and $P_i \in V[G_i]$ are posets for $i \in n$ then in some further generic extension there are filters $H_i \subset P_i$ respectively generic over $V[G_i]$ for $i \in n$ so that $\phi(V[G_i][H_i] \colon i \in n)$ holds;
5. $\phi(V[G_i] \colon i \in n)$ implies $\bigcap_i V[G_i] = V$.

The demands (1) and (2) flow naturally from the context, (3) and (4) are motivated by the desire (and necessity) to make the resulting notions $\bar{E}$-invariant as in Proposition 2.2.3. The justification of demand (5) is trickier. It occurs naturally in all particular cases considered in this book, and it is necessary in the arguments showing that the resulting notions are preserved under natural operations on equivalence relations as in Claim 2.2.14. It is in general not enough to demand that $\bigcap_i V[G_i] \cap 2^\omega = V \cap 2^\omega$, even though in important cases this weaker property implies $\bigcap_i V[G_i] = V$.

Definition 2.2.2. Let E be an analytic equivalence relation on a Polish space X. Suppose that ϕ is a perpendicularity relation with arity $n \leq \omega$. Let P be a poset.

1. An E-symmetric name τ is *E-ϕ-trim* if in some generic extension, there are filters $G_i \subset P$ for $i \in n$ such that $\phi(V[G_i] \colon i \in n)$ holds, and for all $i, j \in n$ $\tau/G_i \mathrel{E} \tau/G_j$ holds.
2. The equivalence relation E is P-ϕ-trim if for no condition $p \in P$ there is a nontrivial E-ϕ-trim name on $P \restriction p$.

Proposition 2.2.3. *Suppose that E is an analytic equivalence relation on a Polish space X and ϕ is a perpendicularity relation. Then the class of E-ϕ-trim names is $\bar{E}$-invariant.*

Proof. Let $n \leq \omega$ be the arity of the relation ϕ. Suppose that $\langle P, \tau \rangle$ and $\langle Q, \sigma \rangle$ are posets and E-symmetric names for an element of X respectively, and τ is a ϕ-trim name on P. To conclude that σ is a ϕ-trim name, first select a sequence $\langle G_i \colon i \in n \rangle$ of filters on P generic over V such that $\forall i, j \in n \ \tau/G_i \mathrel{E} \tau/G_j$ and

$\phi(V[G_i]\colon i \in n)$ both hold. Pick posets $R_i \in V[G_i]$ which force in the respective models that there is a filter $K_i \subset Q$ generic over V such that $\sigma/K_i \mathrel{E} \tau/G_i$. Use (4) of Definition 2.2.1 pass to a further generic extension in which there are filters $H_i \subset R_i$ respectively generic over $V[G_i]$ for $i \in n$ such that $\phi(V[G_i, H_i]\colon i \in n)$ holds. In each of the models $V[G_i][H_i]$, pick a filter $K_i \subset Q_i$ generic over V such that $\sigma/K_i \mathrel{E} \tau/G_i$ holds. Now, $\phi(V[K_i]\colon i \in n)$ holds by Definition 2.2.1(3). It follows that the filters $K_i \subset Q$ for $i \in n$ witness the fact that the name σ is ϕ-trim. □

It is now time to discuss the perpendicularity relations central for this book.

Example 2.2.4. $\phi(V[G], V[H]) =$"$V[G]$ and $V[H]$ are mutually generic extensions" is a perpendicularity relation.

Proof. For (2) and (3) of Definition 2.2.1, it is exactly necessary to show that if P_0, P_1, Q_0, Q_1 are posets, τ_0 and τ_1 are P_0- and P_1-names for generic filters on Q_0 and Q_1 respectively, then $P_0 \times P_1 \Vdash \tau_0 \times \tau_1 \subset Q_0 \times Q_1$ is generic over the ground model. However, this is exactly the contents of Lemma 6.1.9.

For (4) of Definition 2.2.1, suppose that P_0, P_1 are posets, $G_0 \times G_1 \subset P_0 \times P_1$ is a filter generic over V, and $Q_0 \in V[G_0]$ and $Q_1 \in V[G_1]$ are posets. Let $H_0 \times H_1 \subset Q_0 \times Q_1$ be a filter generic over $V[G_0 \times G_1]$. It is not difficult to argue that the filters $G_0 * H_0 \subset P_0 * \dot{Q}_0$ and $G_1 * H_1 \subset P_1 * \dot{Q}_1$ are mutually generic over V, yielding (4). □

Definition 2.2.5. Let E be an analytic equivalence relation on a Polish space X. Let τ be an E-symmetric name on a poset P. Call τ *E-pinned* if it is E-ϕ-trim for the perpendicularity relation $\phi(V[G], V[H]) =$"$V[G], V[H]$ are mutually generic extensions of V". Call the equivalence relation E *P-pinned* if all E-pinned names on P are trivial, and *pinned* if it is P-pinned for all posets P.

Example 2.2.6. Suppose $n \leq \omega$. $\phi(V[G_i]\colon i \in n) =$"$\bigcap_{i\in n} V[G_i] = V$" is a perpendicularity relation.

Proof. It is only necessary to verify (4) of Definition 2.2.1. Suppose that, in some generic extension $V[G]$, $V[G_i]$ for $i \in n$ are generic extensions of V such that $\bigcap_{i\in n} V[G_i] = V$, and $Q_i \in V[G_i]$ are posets. Let $\prod_i H_i \subset \prod_i Q_i$ be a filter on the finite support product, generic over $V[G]$. It is enough to check that $\bigcap_i V[G_i][H_i] = V$.

Suppose towards a contradiction that $\tau \in V[G_0]$ is a Q_0-name for a set of ordinals and $q \in \prod_i Q_i$ is a condition forcing that $\tau/\dot{H}_0 \in \bigcap_i V[G_i][\dot{H}_i] \setminus V$. Since $\bigcap_i V[G_i] = V$, the condition can be strengthened first to identify a specific index $j \in n$ such that $\tau_0/\dot{H}_0 \notin V[G_j]$ and then to identify a specific Q_j-name $\sigma \in V[G_j]$ such that $\tau/\dot{H}_0 = \sigma/\dot{H}_j$, and then to include j in $\mathrm{dom}(q)$. Now, the condition $q(j) \in Q$ does not decide the membership of all ordinals in σ since it forces σ not to be an element of $V[G_j]$. Thus, it is possible to strengthen the condition q further on 0-th and j-th coordinates to find an ordinal α such that $q(0)$ and $q(j)$ disagree on the truth values of the statements $\check{\alpha} \in \tau$ and $\check{\alpha} \in \sigma$. This contradicts the assumption that $q \Vdash \tau/\dot{H}_0 = \sigma/\dot{H}_j$. □

Definition 2.2.7. Let E be an analytic equivalence relation on a Polish space X. Let τ be an E-symmetric name on a poset P. Call τ *E-trim* if it is E-ϕ-trim for the perpendicularity relation $\phi(V[G], V[H]) =$"$V[G] \cap V[H] = V$". Call the equivalence relation E *P-trim* if all E-trim names on P are trivial, and *trim* if it is P-trim for all posets P.

Definition 2.2.8. Let E be an analytic equivalence relation on a Polish space X. Let τ be an E-symmetric name on a poset P. Call τ *E-σ-trim* if it is E-ϕ-trim for the perpendicularity relation $\phi(V[G_i] \colon i \in \omega) =$"$\bigcap_i V[G_i] = V$". Call the equivalence relation E *P-σ-trim* if all E-σ-trim names on P are trivial, and *σ-trim* if it is P-σ-trim for all posets P.

Example 2.2.9. Suppose that $\mathfrak{F}$ is a class of pinned Borel equivalence relations. $\phi(V[G_i] \colon i \in \omega) =$"$\bigcap_i V[G_i] = V$, and for every $F \in \mathfrak{F}$, if an F-equivalence class has a representative in every model $V[G_i]$ for $i \in \omega$, then it has a representative in the ground model" is a perpendicularity relation.

Proof. It is only necessary to verify (4) of Definition 2.2.1. Suppose that, in some generic extension $V[G]$, $V[G_i]$ for $i \in n$ are generic extensions of V satisfyng ϕ, and $Q_i \in V[G_i]$ are posets. I will show that the finite support product $\prod_i Q_i$, adding generic filters $\dot{H}_i \subset Q_i$, forces that $\phi(V[G_i][\dot{H}_i] \colon i \in \omega)$.

The equality $\bigcap_i V[G_i][H_i] = V$ has been verified in the proof of Claim 2.2.6. Now suppose that $F \in \mathfrak{F}$ is an equivalence relation on a Polish space Y and $q \in \prod_i Q_i$ is a condition, $j \in \omega$ and $\tau \in V[G_j]$ is a Q_j-name for an element of Y and $q \Vdash \tau$ has a representative in every model $V[G_i][\dot{H}_i]$ for $i \in \omega$. I claim that below some stronger condition, $V[G_j] \models \tau$ is F-pinned. Once this is established, the argument is completed as follows. Since F is a pinned equivalence relation in V by the assumptions, it is also pinned in $V[G_j]$ by Theorem 5.2.1, and therefore $\prod_i Q_i$ forces that the F-class of τ is represented in $V[G_j]$. Varying the number $j \in \omega$, it becomes clear that $\prod_i Q_i$ forces that every F-equivalence class represented in each model $V[G_i][\dot{H}_i]$ for $i \in \omega$ is represented also in each model $V[G_i]$ for $i \in \omega$. Since $\phi(V[G_i] \colon i \in \omega)$ is assumed, it follows that such an F-class must be represented in V as desired.

To show that the Q_j-name τ is F-pinned under some stronger condition, pick a number $k \neq j$, and strengthen the condition q in the product if necessary to find a Q_k-name $\sigma \in V[G_j]$ such that the condition q forces $\tau/\dot{H}_j \; F \; \sigma/\dot{H}_k$. Now, argue that the Q_j-name τ is pinned below the condition $q(j)$. Indeed, whenever $H_j, H'_j \subset Q$ are mutually generic filters containing the condition $q(j)$, in some further forcing extension find a filter $H_k \subset Q_k$ containing the condition $q(k)$ and generic over $V[G][H_j][H'_j]$. By the product forcing theorem and the choice of the name σ, it must be the case that $\tau/H_j \; F \; \sigma/H_k \; F \; \tau/H'_j$. An appeal to the transitivity of the equivalence relation F concludes the argument. □

Definition 2.2.10. Suppose that E is an analytic equivalence relation and $\mathfrak{F}$ is a class of pinned Borel equivalence relations. The equivalence relation E is *σ-$\mathfrak{F}$-trim* if it is ϕ-trim for the perpendicularity formula $\phi(V[G_i] \colon i \in \omega) =$"$\bigcap_i V[G_i] = V$, and for every $F \in \mathfrak{F}$, if an F-equivalence class has a representative in every model $V[G_i]$ for $i \in \omega$, then it has a representative in the ground model".

The above examples all serve to prove ergodicity theorems in this book; there always are natural equivalence relations which exhibit one of the properties but not the other. There are other sensible options, such as ϕ-trim where $\phi(V[G_0], V[G_1], V[G_2])$ is the statement $\bigcap_{i\in 3} V[G_i] = V$; however, I do not see how to employ them–in the present case, I do not see an equivalence relation which would be trim but not ϕ-trim.

It is useful to observe that there are provably strongest variations of trimness for equivalence relations, and they are exactly the ones introduced above. This is the contents of the following theorem. The theorem also shows that below orbit equivalence relations, trimness and σ-trimness coincide. There are many ergodicity results for other equivalence relations that do rely on the distinction between trimness and σ-trimness though–see Section 3.3.

Theorem 2.2.11. *Let E be an equivalence relation on a Polish space X and let P be a poset and ϕ a perpendicularity relation.*

1. *If E is P-σ-trim then E is P-ϕ-trim for every perpendicularity relation ϕ;*
2. *if E is almost reducible to an orbit equivalence relation and E is P-trim then E is P-ϕ-trim for every perpendicularity relation ϕ.*

Proof. (1) follows immediately from the definition of σ-trimness. For (2), it is enough to show that P-trimness implies P-σ-trimness below orbit equivalence relations. Suppose that E is not P-σ-trim. Then, there must be, in some generic extension $V[G]$, filters $G_i \subset P$ for $i \in n$, separately generic over V and such that $\phi(V[G_i] \colon i \in n)$ holds, and points $x_i \in X$ in the respective models $V[G_i]$ which are pairwise E-related but not related to any point in the ground model. I must conclude that E is not P-trim. To this end, I will find, in some further generic extension, a filter $H \subset P$ generic over V which contains a point E-related to x_0 and such that $V[G_0] \cap V[H] = V$.

In the ground model V, find a Polish group Γ, a continuous action of the group Γ on a Polish space Y inducing an orbit equivalence relation F, and a Borel function $f \colon X \to Y$ which is an almost reduction from E to F. Consider the poset P_Γ of nonempty open subsets of the group Γ, introducing a generic point of Γ. Let $g \in \Gamma$ be a point which is P_Γ-generic over the model $V[G]$ and consider the point $y = g \cdot f(x_0)$.

Claim 2.2.12. $V[y] \cap V[G_0] = V$.

Proof. For every $j \in n$ let $g_j \in V[G] \cap \Gamma$ be some element of the acting group such that $g_j \cdot f(x_j) = f(x_0)$ and let $h_j = g g_j^{-1}$. Observe that for each $j \in n$,

- the point $h_j \in \Gamma$ is P_Γ-generic over $V[G]$ since the multiplication on g_j^{-1} on the right induces an automorphism of P_Γ;
- $y \in V[G_j][h_j]$ since $y = h_j \cdot f(x_j)$;

- $V[G_j][h_j] \cap V[G_0] = V[G_j] \cap V[G_0]$ by the genericity of h_j over $V[G_0, G_j]$.

Thus, $V[y] \cap V[G_0] \subseteq \bigcap_j V[G_j][h_j] \cap V[G_0] = \bigcap_j V[G_j] \cap V[G_0] = V$ as desired. □

Now, $V[G_i : i \in n][y]$ is a generic extension of $V[y]$, and so in the model $V[y]$, the following holds: there is a poset R adding a filter $H \subset P$ generic over V such that in $V[H]$ there is a point $x \in X$ such that $f(x)\ F\ y$. Pass to an R-extension generic over the model $V[G_0][y]$, and use a genericity argument together with the claim to argue that for the filter $H \subset P$ thus obtained, $V[G_0] \cap V[H] = 0$. Pick a point $x \in V[H]$ such that $f(x)\ F\ y$ and conclude that $x_0 \in V[G_0]$ and $x \in V[H]$ are the desired E-connected points. □

As the last remark in this section, all the trimness classes of equivalence relations are preserved under many natural operations:

Theorem 2.2.13. *Suppose that ϕ is a perpendicularity relation on tuples of generic extensions. Suppose that P is a partial ordering. The class of P-ϕ-trim analytic equivalence relations is closed under*

1. *countable unions, if they result in an equivalence relation and the arity of ϕ is 2;*
2. *increasing countable unions, if the arity of ϕ is finite;*
3. *product modulo J if J is an analytic ideal on ω such that $=_J^{2^\omega}$ is P-ϕ-trim and the arity of ϕ is finite;*
4. *Friedman–Stanley jump if P preserves $\aleph_1$;*
5. *countable product;*
6. *almost Borel reducibility.*

Proof. Let n be the arity of ϕ. The key point is the following claim, whose proof depends on demand (5) in the definition of perpendicularity.

Claim 2.2.14. *Let E be an analytic equivalence relation on a Polish space X. Suppose that $\langle V[G_i] : i \in n\rangle$ is a tuple of generic extensions such that $\phi(V[G_i] : i \in n)$ holds, and suppose that $x_i \in X$ for $i \in n$ are pairwise E-related points in the respective extensions. Let $P, \tau \in V$ be a poset and a name such that $G_0 \subset P$ is generic over V and $x_0 = \tau/G_0$. Then there is a condition $p \in G_0$ such that in V, the name τ on $P \restriction p$ is E-symmetric and ϕ-trim.*

Proof. For every $i \in n$, in the model $V[G_i]$ form the set $A_i = \{q \in P$: in some generic extension there is a filter $H \subset P$ generic over V containing q and such that $\tau/H\ E\ x_i\} \subset P$. Since the models $V[G_i]$ for $i \in n$ all belong to the same common generic extension and the points x_i for $i \in n$ come from the same equivalence class, the sets A_i for $i \in n$ are all equal to the same common value A. Since $\bigcap_i V[G_i] = V$, it follows that $A \in V$. By the definitions, $G_0 \subset A$ and by genericity there is a condition $p \in G_0$ such that below p the set $A \subset P$ is dense. I claim that the condition p works as required.

It is immediate from the definitions that the name τ on $P \restriction p$ is E-symmetric. To show that it is ϕ-trim, for every $i \in n$ in the model $V[G_i]$ find a poset Q_i forcing the existence of a filter $K_i \subset P$ generic over V such that $\tau/K_i \mathrel{E} x_i$. Use (4) of Definition 2.2.1 to find, in some further generic extension, filters $H_i \subset Q_i$ respectively generic over $V[G_i]$ such that $\phi(V[G_i][H_i] \colon i \in n)$ holds. Let $K_i \subset P$ for $i \in n$ be filters generic over V in the respective models $V[G_i][H_i]$ such that $\tau/K_i \mathrel{E} x_i$. Use (3) of Definition 2.2.1 to conclude that $\phi(V[K_i] \colon i \in n)$ holds and so the name τ on $P \restriction p$ is ϕ-trim. □

Now, for (1) suppose that $E = \bigcup_m E_m$ is a union of P-ϕ-trim analytic equivalence relations, $V[G_0], V[G_1]$ are P-generic extensions of V such that $\phi(V[G_0], V[G_1])$ holds and $x_0 \in V[G_0], x_1 \in V[G_1]$ are E-related points. It will be enough to show that these two points are E-related to some point in V. To this end, observe that there must be a number $n \in \omega$ such that $x_0 \mathrel{E_m} x_1$. By the claim, there must be a condition $p \in P$ and an E_m-symmetric ϕ-trim name τ on $P \restriction p$ such that $x_0 = \tau/G_0$. By the trimness assumption on the equivalence relation E_m, it follows that τ is trivial and therefore x_0 is E_m-related, and therefore E-related, to a point in V as desired.

The proof of (2) is similar. Suppose that $E = \bigcup_m E_m$ is an increasing union of P-ϕ-trim analytic equivalence relations, $V[G_i]$ for $i \in n$ are P-generic extensions of V such that $\phi(V[G_i] \colon i \in n)$ holds and $x_i \in V[G_i]$ are pairwise E-related points in the space X. It will be enough to show that these points are E-related to some point in V. To this end, observe that there must be a number $m \in \omega$ such that the points x_i for $i \in n$ are pairwise E_m-related. By the claim, there must be a condition $p \in P$ and an E_m-symmetric ϕ-trim name τ on $P \restriction p$ such that $x_0 = \tau/G_0$. By the trimness assumption on the equivalence relation E_m, it follows that τ is trivial and therefore x_0 is E_m-related, and therefore E-related, to a point in V as desired.

For (3), suppose that J is an analytic ideal on ω such that the equivalence relation $=_J^{2^\omega}$ is P-ϕ-trim, and suppose that for each $m \in \omega$ there is an analytic P-ϕ-equivalence relation E_m on a Polish space X_m. I must show that the product equivalence relation $E = \prod_J E_m$ on the space $Y = \prod_m X_m$ is P-ϕ-trim. Suppose that $n \in \omega$ is the arity of ϕ and, in some generic extension, $V[G_i]$ for $i \in n$ are P-generic extensions of the ground model such that $\phi(V[G_i] \colon i \in n)$ holds, and $y_i \in Y$ are pairwise E-related points in each respective model $V[G_i]$. It will be enough to show that the points are E-related to some point in the ground model. To this end, in V, for every $m \in \omega$ fix a bijection f_m between the E_m-quotient space and $2^\omega \setminus \{0\}$. For every $i \in n$, let $z_i \in (2^\omega)^\omega$ in the model $V[G_i]$ be defined by $z_i(m) = f_m([y_i(m)]_{E_m})$ if the latter equivalence class is represented in the ground model, and $z_i(m) = 0$ otherwise.

Claim 2.2.15. *The points z_i for $i \in n$ are pairwise $=_J$-related, and the set $\{m \in \omega \colon z_i(m) = 0\}$ belongs to J for every $i \in n$.*

Proof. For every number $m \in \omega$, if all points $y_i(m)$ for $i \in n$ are pairwise E_m-related, then they are E_m-related to an element of the ground model. This follows from the trimness assumption on E_m and Claim 2.2.14. Now observe that the set of all $m \in \omega$ such that the points $y_i(m)$ for $i \in n$ are not pairwise related belongs to J by the finite additivity of the ideal J. The claim immediately follows. $\square$

By the trimness assumption on $=_J$, there is $z \in (2^\omega)^\omega$ in V which is $=_J$-related to all points z_i for $i \in n$. Let $y \in Y$ be a point in V such that $f_m([y(m)]_{E_m}) = z(m)$ whenever $z(m) \neq 0$. Unraveling the definitions it becomes clear that the point $y \in Y$ is E-related to each y_i for $i \in n$ as desired.

For (4), suppose that $E = F^+$ is a Friedman–Stanley jump of a P-ϕ-trim equivalence relation F on a Polish space X. Let n be the arity of ϕ and let $V[G_i]$ for $i \in n$ be P-generic extensions such that $\phi(V[G_i] \colon i \in n)$ holds. Suppose that $y_i \in X^\omega$ for $i \in n$ are pairwise E-related points in the respective models $V[G_i]$; it will be enough to show that these points are E-related to a point in V. To this end, observe that every F-equivalence class represented in $\operatorname{rng}(y_i)$ is represented already in V by the claim and the trimness assumption on F. The set A of F-equivalence classes represented in $\operatorname{rng}(y_i)$ for any $i \in n$ belongs to all models $V[G_i]$ and therefore belongs to V. The set A is countable in every model $V[G_i]$, and by the assumption on the poset P must be countable in V. Thus, in V there is a point $y \in X^\omega$ visiting exactly all the equivalence classes in A, then clearly $y\ E\ y_i$ for all $i \in n$ as desired.

For (5), suppose that $\langle F_m \colon m \in \omega\rangle$ are analytic P-ϕ-trim equivalence relations on the respective Polish spaces X_m. To prove that their product F on the Polish space $Y = \prod_m X_m$ is also P-ϕ-trim, suppose that in some generic extension, $\langle V[G_i] \colon i \in n\rangle$ is a sequence of extensions satisfying ϕ and $y_i \in Y$ are pairwise F-related points in the respective models $V[G_i]$ for $i \in n$. I must show that there is a point $y \in Y \cap V$ F-related to all y_i for $i \in n$. The set $A = \{\langle m, x\rangle : m \in \omega, x \in X_m \cap V, \forall i \in n\ x\ F_m\ y_i(m)\}$ belongs to all models $V[G_i]$ for $i \in n$, and therefore to V. By the trimness assumption and Claim 2.2.14 applied to the relations F_m, for each $m \in \omega$ the m-th section of the set A is nonempty, and so by the countable choice in V there is a function $y \in V$ with domain ω uniformizing the set A. It is immediate that $y \in Y$ works as desired.

For (6), suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y, let $a \subset X$ be a countable set and $f \colon X \to Y$ be a Borel function which on $X \setminus [a]_E$ is a reduction of E to F. It will be enough to show that if τ is a nontrivial E-symmetric ϕ-trim name then $f\tau$ is a nontrivial F-symmetric ϕ-trim name. This, however, follows directly from the definitions. $\square$

The principal corollary of the previous two theorems says that the equivalence relations classifiable by countable structures are very simple from the point of view of variations of trimness.

Corollary 2.2.16. *If E is an equivalence relation classifiable by countable structures, then E is P-ϕ-trim for every $\aleph_1$-preserving poset P and every perpendicularity relation ϕ.*

Proof. First note that E is reducible to an orbit equivalence relation, and so by Theorem 2.2.11 it is enough to show that E is P-trim. To this end, note that E is Borel-reducible to isomorphism of two-place relations on ω, so by Theorem 2.2.13 it is enough to deal with the equivalence relation F of isomorphism of two-place relations on ω; the underlying space of F is $X = \mathcal{P}(\omega^2)$. In some generic extension, let $G_0, G_1 \subset P$ be filters separately generic over V and $V[G_0] \cap V[G_1] = V$. Let $x_0 \in V[G_0]$ and $x_1 \in V[G_1]$ be F-related points. I must show that x_0, x_1 are F-related to a point in V.

Let α be an ordinal such that x_0 has Scott rank α. Since $V[G_0]$ is a model of ZFC, the ordinal α is countable there, and by the assumption on the poset P α is countable in V. Consider the equivalence relation F restricted to the Borel set B of relations of rank α. This relation belongs to the ground model, it is Borel, classifiable by countable structures. Thus, by [12, Theorem 12.5.2] it is reducible to an equivalence relation obtained by a countable transfinite repetition of the operation of Friedman-Stanley jump and increasing union. In view of Theorem 2.2.13, $F \restriction B$ is P-trim, and so x_0, x_1 are $F \restriction B$-related to a point $x_2 \in B$ in the ground model. Clearly, $x_2 \mathrel{F} x_0$ as desired. □

2.3 Ergodicity theorems

Finally, I will state three general ergodicity theorems. Suppose that E is an analytic equivalence relation on a Polish space X, P is a poset and τ is an E-symmetric name. Let I_τ be the set of all analytic subsets $A \subset X$ such that $P \Vdash \tau \notin \dot{A}$. Since the name τ is E-symmetric, the statement $\tau \in \dot{A}$ is decided by the largest condition for every E-invariant analytic set $A \subset X$, and so I_τ is a maximal ideal on the collection of analytic E-invariant sets.

Theorem 2.3.1. *Suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y and let $h\colon X \to Y$ be a homomorphism from E to F. Suppose that ϕ is a perpendicularity relation, P a poset and τ an E-symmetric ϕ-trim P-name. Suppose that F is P-ϕ-trim. Then there is an element $y \in Y$ such that $X \setminus h^{-1}[y]_F \in I_\tau$.*

Proof. Consider the P-name $h\tau$ for an element of the space Y obtained by applying the (interpretation of the) function h to τ in the P-extension. Since h is a homomorphism of E to F in V, the Shoenfield absoluteness shows that h remains a homomorphism in the P-extension and so $h\tau$ is an F-symmetric ϕ-trim P-name. Since the equivalence relation F is P-ϕ-trim, the name $h\tau$ is F-trivial and so there is $y \in Y$ such that $P \Vdash \check{y} \mathrel{F} h\tau$. Thus, $P \Vdash \tau \in h^{-1}[y]_F$ and the theorem follows. □

Other ergodicity theorems do not deal with homomorphisms to other equivalence relations, but to the nonempty intersection of ranges graph:

Theorem 2.3.2. *Let E be an analytic equivalence relation on a Polish space X. Suppose that P is c.c.c., and τ is an E-symmetric trim P-name. Suppose that $h\colon X \to (2^\omega)^\omega$ be a Borel function. One of the following occurs:*

1. *either, there is a countable set $a \subset 2^\omega$ such that the set $\{x \in X\colon a \cap \operatorname{rng}(h(x)) = 0\}$ belongs to I_τ.*
2. *or, there is a nonempty perfect set $C \subset X$ of pairwise E-related points such that for all pairs of distinct points $x_0, x_1 \in C$, $\operatorname{rng}(h(x_0)) \cap \operatorname{rng}(h(x_1)) = 0$ holds.*

Proof. Consider the set $a = \{y \in 2^\omega\colon \{x \in X\colon y \in \operatorname{rng}(h(x))\} \notin I_\tau\}$ and note that $a \subset 2^\omega$ is countable. Otherwise, there would be pairwise distinct points $y_\alpha \in 2^\omega$ and conditions $p_\alpha \in P$ for $\alpha \in \omega_1$ such that $p_\alpha \Vdash \check{y}_\alpha \in \operatorname{rng}(h(\tau))$. By the c.c.c. of the poset P, there is a condition $p \in P$ which forces uncountably many of these conditions into the generic filter, in other words $\operatorname{rng}(h(\tau))$ to be uncountable. This is impossible.

Now, if the $\{x \in X\colon a \cap \operatorname{rng}(h(x)) = 0\}$ belongs to I_τ then case (1) occurs. If the set $\{x \in X\colon a \cap \operatorname{rng}(h(x)) = 0\}$ does not belong to I_τ, then I will show that (2) occurs. There is a condition $p \in P$ which forces $a \cap \operatorname{rng}(h(\tau)) = 0$, which by the choice of the set a means that $\operatorname{rng}(h(\tau)) \cap V = 0$. Let M be a countable elementary submodel of a large structure containing the condition p. Let $G \subset P$ be a filter generic over the model M. Since the name τ is trim, in the model $M[G]$ there is a poset Q and a Q-name $\dot{H}$ such that $M[G] \models Q \Vdash \dot{H} \subset \check{P}$ is a generic filter containing P, $M[G] \cap M[\dot{H}] = M$, and $\tau/\dot{H}\ E\ \tau/G$.

Use Lemma 6.2.8 to find a perfect collection $\{K_z\colon z \in 2^\omega\}$ of filters on Q pairwise mutually generic over the model $M[G]$. For each $z \in 2^\omega$ let $H_z = \dot{H}/K_z$ and $x_z = \tau/H_z$. All the points $x_z \in X$ come from the equivalence class $[\tau/G]_E$. Now, whenever $z_0, z_1 \in 2^\omega$ are distinct points then $M[H_{z_0}] \cap M[H_{z_1}] \subset M[G]$ by the mutual genericity of the filters $K_{z_0}, K_{z_1} \subset Q$, and so $M[H_{z_0}] \cap M[H_{z_1}] = M$ by the choice of the name $\dot{H}$. The set $\operatorname{rng}(x_{z_0}) \cap \operatorname{rng}(x_{z_1})$ is a subset of the intersection $M[H_{z_0}] \cap M[H_{z_1}] = M$; however, the sets $\operatorname{rng}(x_{z_0}), \operatorname{rng}(x_{z_1})$ contain no elements of M by the choice of the condition p, and so they must be disjoint. Thus, the set $\{x_z\colon z \in 2^\omega\}$ satisfies the last demand in (2). This set is analytic and uncountable, and so it has a nonempty perfect subset $C \subset X$. (2) follows. $\square$

Corollary 2.3.3. *Let E be an analytic equivalence relation on a Polish space X. Suppose that P is c.c.c., and τ is an E-symmetric trim P-name. If $h\colon X \to (2^\omega)^\omega$ is a Borel homomorphism of E to the nonempty intersection of ranges graph, then there is a countable set $a \subset 2^\omega$ such that the set $\{x \in X\colon a \cap \operatorname{rng}(h(x)) = 0\}$ belongs to I_τ.*

In case of equivalence relations reducible to orbit relations, Theorem 2.3.2 has a generalization to higher dimensions of sorts.

Theorem 2.3.4. *Let E be an equivalence relation on a Polish space X, Borel reducible to an orbit equivalence relation. Suppose that P is c.c.c., and τ is an*

E-symmetric trim P-name. Let $n \in \omega$. Suppose that $h\colon [X]^n \to (2^\omega)^\omega$ is a Borel function. One of the following occurs:

1. *either, there is a countable set $a \subset 2^\omega$ such that the set $\{x \in X\colon \forall b \in [X]^n\ x \in b \to \mathrm{rng}(h(b)) \cap a = 0\}$ belongs to I_τ;*
2. *or, there is a nonempty perfect set $C \subset X$ of pairwise E-related points such that for any disjoint sets $b_0, b_1 \in [C]^n$, $\mathrm{rng}(h(b_0)) \cap \mathrm{rng}(h(b_1)) = 0$ holds.*

Proof. Consider the set $D \subset P$, $D = \{p \in P\colon \exists y \in 2^\omega\ p \Vdash \exists b \in [X]^n\ \tau \in b \wedge y \in \mathrm{rng}(h(\tau))\}$. Suppose first that the set $D \subset P$ is dense and argue that case (1) has to occur. Indeed, use the c.c.c. of the poset P to find a countable maximal antichain $A \subset D$, for each $p \in A$ select $y_p \in 2^\omega$ witnessing the membership of p in A, let $a = \{y_p\colon p \in A\}$, and argue that (1) holds with the set $a \subset 2^\omega$. To see this, let M be a countable elementary submodel containing X, E, h and the set a. Let $B = \{x \in X\colon \exists G \subset P\colon G$ is generic over the model M and $x = \tau/G\}$. By the c.c.c. of the poset P, the complement of the set B is in the ideal I_τ. For every $x \in B$, by the forcing theorem $M[x] \models \exists b \in [X]^n\ \tau \in b$ and $a_p \cap \mathrm{rng}(h(\tau) \neq 0\}$. Thus, (1) holds.

Suppose now that the set $D \subset P$ is not dense and argue that (2) has to occur. Since the set $D \subset P$ is open, there must be a condition $p \in P$ such that p is not compatible with any element of D, in other words $p \Vdash \forall b \in [X]^n\ \tau \in b \to \mathrm{rng}(h(b)) \cap V = 0$. Let Γ be a Polish group continuously acting on a Polish space Y, inducing the orbit equivalence relation F, and let $g\colon X \to Y$ be a Borel reduction of E to F. Let M be a countable elementary submodel of a large enough structure containing all objects named so far. Let $G \subset P$ be a filter generic over the model M, containing the condition p.

Let $\gamma \in \Gamma$ be a point generic over the model $M[G]$ for the usual Cohen poset P_Γ on Γ. In the model $M[\gamma \cdot g(\tau/G)]$ there must be points $x \in X, \delta \in \Gamma$ such that $\delta \cdot g(x) = \gamma \cdot g(\tau/G)$–note that $\tau/G, \gamma$ are such points and then an applicaton of Mostowski absoluteness finds such points in the model $M[\gamma \cdot g(\tau/G)]$. Let $\dot{x}$ be a P_Γ-name for this point $x \in X$. Below, I will write $x = x(\gamma)$ and $\delta = \delta(\gamma)$.

Claim 2.3.5. *In the model $M[G]$, P_Γ forces that $M[G] \cap M[\dot{x}] = M$.*

Proof. This is just a repetition of Claim 2.2.12. □

Claim 2.3.6. *If $c \subset \Gamma$ is a finite set of points mutually P_Γ-generic over the model $M[G]$ and $c = b_0 \cup b_1$ is a partition, then $M[x(\gamma)\colon \gamma \in b_0] \cap M[x(\gamma)\colon \gamma \in b_1] = M$.*

Proof. Select points $\gamma_0 \in b_0$ and $\gamma_1 \in b_1$. Note that $M[x(\gamma_0)] \cap M[x(\gamma_1)]$ is a subset of $M[G]$ by the mutual genericity and Proposition 6.1.8. By Claim 2.3.5, the intersection is in fact equal to M.

Now, for every $\gamma \in b_0 \setminus \{\gamma_0\}$, let $\varepsilon(\gamma) = \gamma \cdot \gamma_0^{-1} \cdot \delta(\gamma_0)$. Similarly for $\gamma \in b_1 \setminus \{\gamma_1\}$, let $\varepsilon(\gamma) = \gamma \cdot \gamma_1^{-1} \cdot \delta(\gamma_0)$. Since multiplication on the right induces an automorphism of the Cohen poset on the group Γ, the collection

$\{\varepsilon(\gamma) \colon \gamma \in b_0 \setminus \{z_0\} \cup b_1 \setminus \{\gamma_1\}\}$ is mutually generic for the Cohen poset over the model $M[G][\gamma_0, \gamma_1]$. By Proposition 6.1.8, $M[x(\gamma_0)][\varepsilon(\gamma) \colon \gamma \in b_0 \setminus \{\gamma_0\}] \cap M[x(\gamma_1)][\varepsilon(\gamma) \colon \gamma \in b_1 \setminus \{\gamma_1\}] = M[x(\gamma_0)] \cap M[x(\gamma_1)]$, and this last intersection is by the first paragraph equal to M. Now note that if $\gamma \in b_0 \setminus \{\gamma_0\}$ then $x(\gamma) \in M[\gamma \cdot g(\tau/G)] = M[\varepsilon(\gamma) \cdot g(x(\gamma_0))] \subset M[x(\gamma_0)][\varepsilon(\gamma)]$, and similarly on the b_1 side. The Claim follows. □

Now, working in the model $M[G]$, find a P_Γ-name $\dot{Q}$ for a poset and a $\dot{Q}$-name $\dot{H}$ for a filter on P in $M[\dot{x}]$ such that $M[\dot{x}] \models \dot{Q} \Vdash \dot{H}$ is a filter on P generic over M, containing the condition p and $\tau/\dot{H}\ E\ \dot{x}$. Such a poset must exist since such a filter does exist in a generic extension of $M[x]$, namely in the model $M[G][x]$. Use Lemma 6.2.8 to find a perfect set $\{K_z \colon z \in 2^\omega\}$ of filters on $P_\Gamma * \dot{Q}$ mutually generic over $M[G]$. For each $z \in 2^\omega$ write $H_z = \dot{H}/K_z$ and $x_z = \tau/H_z$.

By Claim 2.3.6 and Proposition 6.1.8, whenever $b_0, b_1 \in [2^\omega]^n$ are disjoint sets then $M[x_z \colon z \in b_0] \cap M[x_z \colon z \in b_1] = M$. By the forcing theorem, for every $z \in b_0$ the model $M[x_z]$ satisfies that for every $y \in M \cap 2^\omega$ and every $d \in [X]^n$, if $x_z \in d$ then $y \notin \operatorname{rng}(h(d))$. By the Mostowski absoluteness, $\operatorname{rng}(h(x_z \colon z \in b_0)) \cap M = 0$. Thus, $\operatorname{rng}(h(x_z \colon z \in b_0)) \cap \operatorname{rng}(h(x_z \colon z \in b_1)) = 0$ and the option (2) of the theorem will be satisfied with any perfect subset C of the uncountable analytic set $\{x_z \colon z \in 2^\omega\}$. □

As one corollary, homomorphisms to treeable equivalence relations stabilize on a large set. Here, an equivalence relation F on a Polish space Y is *treeable* if there is a Borel acyclic graph G on Y such that F is exactly the connectedness relation in the graph G.

Corollary 2.3.7. *Let E be an equivalence relation on a Polish space X, Borel reducible to an orbit equivalence relation. Suppose that P is c.c.c., and τ is an E-symmetric trim P-name. Let F be a treeable equivalence relation on a Polish space Y and $h \colon X \to Y$ a Borel homomorphism of E to F. Then there is a point $y \in Y$ such that the set $\{x \in X \colon \neg h(x)\ F\ y\}$ belongs to the ideal I_τ.*

Proof. Let $g \colon [X]^2 \to Y^\omega$ be a Borel map such that $g(x_0, x_1)$ enumerates with repetitions the shortest path between $h(x_0)$ and $h(x_1)$ in the treeing of F if $x_0\ E\ x_1$ holds and $g(x_0, x_1) = 0$ if $x_0\ E\ x_1$ fails. The alternative (2) of Theorem 2.3.4 is excluded, since in a connected acyclic graph, among any four points there are two disjoint pairs such that the shortest paths beween the elements of the pairs intersect. Thus, option (1) must prevail. There must be an element $y \in Y$ such that the set $\{x \in X \colon \exists x'\ y \in \operatorname{rng}(h(x, x'))\}$ is I_τ-positive. The definitions imply that the h-preimage of $[y]_F$ must have I_τ-small complement. □

Question 2.3.8. Does the conclusion of Theorem 2.3.4 hold without the orbit assumption?

Chapter 3

The trim concept

3.1 Group actions and turbulence

The motivating result for the development of trimness and its variations is a restatement of Hjorth's turbulence. Suppose that Γ is a Polish group continuously acting on the space X. Recall that given a point $x \in X$, an open set $O \subset X$ containing x, and a set $U \subset \Gamma$ containing 1, the O, U-orbit of x is the set of all points $y \in O$ for which there exists a finite sequence $x = x_0, x_1, \dots x_n = y$ of points in O and elements $\gamma_i \in U$ for $i \in n$ so that for every $i \in n$, $x_{i+1} = \gamma_i \cdot x_i$. The group action is *turbulent* at x if for every open set $O \subset X$ containing x and every open set $U \subset G$ containing 1, the O, U-orbit of X is somewhere dense. Finally, the action is *generically turbulent* if the set of turbulent points is comeager in X. The following characterization of turbulence greatly simplifies all proofs of known consequences of turbulence:

Theorem 3.1.1. *Suppose that Γ is a Polish group acting continuously on a Polish space X with dense meager orbits. Let E be the orbit equivalence relation on X. The following are equivalent:*

1. *the action is generically turbulent;*
2. *the name for a generic element of X is E-trim.*

Proof. To begin, observe that the name $\dot{x}_{gen}$ is E-symmetric by Proposition 2.1.3. The following forcing- and equivalence-free claim is central:

Claim 3.1.2. *Suppose that the action $\Gamma \curvearrowright X$ is generically turbulent. Let $p \subset X$ be a Borel nonmeager set, $f \colon p \to 2^\omega$ a Borel map, and $U \subset \Gamma$ an open neighborhood of the unit. One of the following cases occurs:*

1. *either, there is $\gamma \in U$ and a Borel nonmeager set $q \subset p$ such that $\gamma \cdot q \subset p$ and $f''q \cap f''(\gamma \cdot q) = 0$;*
2. *or, there is a Borel nonmeager set $q \subset p$ such that $f \restriction q$ is constant.*

Proof. Thinning out the set p if necessary, I may assume that the function f is continuous on p. Let $U' \subset U$ be a dense countable set. The treatment divides into two cases:

Case 1. There is an element $\gamma \in U'$ such that the Borel set $\{x \in p \colon \gamma \cdot x \in p, f(x) \neq f(\gamma \cdot x)\}$ is nonmeager. In this case, use the σ-additivity of the meager ideal to find a number n and a bit $b \in 2$ such that the Borel set $q = \{x \in p \colon \gamma \cdot x \in p, f(x)(n) = b, f(\gamma \cdot x)(n) = b\}$ is nonmeager. Note that the set q and element $\gamma \in U$ are as required in (1) of the claim.

Case 2. The Borel set $\{x \in p \colon \gamma \cdot x \in p, f(x) \neq f(\gamma \cdot x)\}$ is meager for every $\gamma \in U'$. In this case, let $O \subset X$ be an open set in which p is comeager, let $x \in X$ be an arbitrary point, let $O' \subset O$ be a nonempty open set in which the O, U-orbit of x is dense, and write $q = p \cap O'$. I will show that $f \restriction q$ is constant, in other words (2) of the claim occurs and the claim follows.

Suppose that $f \restriction q$ is not constant. Since $f \restriction q$ is a continuous function, there would have to be open sets $O'_0, O'_1 \subset O'$ such that $f''(q \cap O'_0) \cap f''(q \cap O'_1) = 0$. There are finite O, U-walks from x to both O'_0 and O'_1. Adjusting the two walks, it is possible to get a finite walk from O'_0 to O'_1 such that all points on the walk are in p and all steps are in the set U'. However, such a walk is impossible since the value of f cannot change along its steps by the case assumption, and the values of f at the endpoints are distinct. $\square$

For the implication (1)→(2), consider the poset P_Γ of nonempty open subsets of G ordered by inclusion, adding a Cohen generic element $\dot\gamma_{\mathrm{gen}}$. Note that $P_X \times P_\Gamma \Vdash \dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}$ is a P_X-generic element of X, since in the P_Γ-extension, the application of $\dot\gamma_{\mathrm{gen}}$ on the right generates an automorphism of the poset P_X, and so the genericity of $\dot x_{gen}$ is inherited by the genericity of $\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}$. Now suppose that the action is generically turbulent; I will show that $P_\Gamma \times P_X \Vdash V[\dot x_{gen}] \cap V[\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}] = V$; this will prove the (1)→(2) implication.

First of all, note that in the Cohen forcing extension, whenever a is a set of ordinals which is not in the ground model, then there is a ground model countable set b such that $a \cap b$ is not in the ground model. As a result, it is enough to show that $V[\dot x_{gen}] \cap V[\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}] \cap 2^\omega = V \cap 2^\omega$. To this end, suppose that $\langle U, O\rangle$ is a condition in the product which forces some τ to be an element in $V[\dot x_{gen}] \cap V[\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}] \cap 2^\omega$; I must find a stronger condition forcing τ to the ground model. Strengthening the condition if necessary, it is possible to find a Borel function f on a comeager subset of O and some formula ϕ such that $\langle U, O\rangle \Vdash \tau = \dot f(\dot x_{gen})$, and τ is definable from $\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}$ by ϕ. Find nonempty open sets $U_0, U_1 \subset \Gamma$ such that $U_0 \subset U$ and $1 \in U_1$ and $U_0 \cdot U_1 \subset U$. The claim gives two options. Either, there is $\gamma \in U_1$ and a Borel nonmeager set $q \subset p$ such that $\gamma \cdot q \subset p$ and $f''q \cap f''(\gamma \cdot q) = 0$. In such a case, let δ, x be mutually P_Γ, P_X generic elements of Γ and X meeting the condition $\langle U_0, q\rangle \leq \langle U, O\rangle$. Then, $\delta \cdot \gamma^{-1}, \gamma \cdot x$ is also a pair of mutually generic elements meeting the condition $\langle U, O\rangle$. Note that $\delta \cdot x = \delta \cdot \gamma^{-1} \cdot \gamma \cdot x$ and by the forcing theorem, both $f(x)$ and $f(\gamma \cdot x)$ should be definable from this point using the formula ϕ. This is impossible though since $f(x) \neq f(\gamma \cdot x)$ by the case assumption. The other option yields a condition $q \subset p$ on which the function f is constant. Here, the condition $\langle U, q\rangle$ forces τ to be equal to this constant value and therefore to belong to the ground model as desired.

For the implication (2)→(1), assume that (2) holds and (1) fails, and work towards a contradiction. Since (1) fails, the Borel set of non-turbulent point is non-meager and so there is a condition $p \in P_X$ forcing the point $\dot{x}_{gen}$ to be non-turbulent. Since (2) holds, in some generic extension there are points $x, y \in 2^\omega$ which are E-related, separately P_X-generic over V, and such that $V[x] \cap V[y] = V$. Since x is non-turbulent, there are basic open neighborhoods $O \subset X$ containing x and $U \subset G$ containing 1 such that the local O, U-orbit of x is nowhere dense. Let $U' \subset \Gamma$ be a basic open neighborhood such that $U' \subset U, (U')^{-1} \subset U$, and $U' \cdot x \subset O$. Let $\gamma \in \Gamma$ be a group element such that $\gamma \cdot y = x$, and find a ground model element $\delta \in \Gamma$ such that $\delta \in U' \cdot \gamma$. Now consider the point $\delta \cdot y \in V[y]$. The point $h \cdot y$ belongs to the O, U-orbit of x, and the point x belongs to the O, U-orbit of $h \cdot y$, and therefore the two orbits are equal. Therefore, the set A of all basic open neighborhoods with empty intersection with the O, U-orbit of x belongs to both $V[x]$ and $V[y]$, and so to V. Now, the set $X \setminus \bigcup A$ is nowhere dense and contains the point x. This is impossible, since x is a P_X-generic point and therefore cannot belong to any ground model coded nowhere dense set. □

Corollary 3.1.3. *Every Borel homomorphism from an orbit equivalence relation of a generically turbulent action to a Cohen-trim equivalence relation or a treeable equivalence relation stabilizes on a comeager set.*

Note that the class of Cohen-trim equivalence relations includes those classifiable by countable structures by Corollary 2.2.16, and so this result strengthens orginial Hjorth's ??? considerably.

Proof. Apply Theorem 2.3.1 for trim equivalence relations or Corollary 2.3.7 for treeable relations. □

A measure theoretic counterpart of turbulence is much more difficult to find. In the terminology of this book, one would like to find a general answer to the following question.

Question 3.1.4. Suppose that $\Gamma \curvearrowright X$ is a continuous action of a Polish group, inducing an orbit equivalence relation E. Let μ be a Borel probability measure on X invariant under the action. Find conditions under which the name for the μ-random element of X is E-trim.

I can find a suitable characterization only for the usual actions of analytic P-ideals, see Theorem 3.2.10. I have no information in the case of non-abelian groups.

3.2 Trimness in ideals

Apart from the orbit equivalence relations, an important type of equivalence relations to understand are those of the form $=_J$ or $=_J^{2^\omega}$ for an analytic ideal J

on ω. Their importance is given by the result of Rosendal [21] which shows that equivalence relations of this form are cofinal in the Borel reducibility order. From the point of view of trimness technology, their importance is also documented by Theorem 2.2.13(3).

3.2.1 The generic element

I will first address the simplest question in this context:

Question 3.2.1. Characterize the class of analytic ideals J on ω such that the name for a generic element of 2^ω is $=_J$-trim.

It turns out that there is a complete resolution at hand, using the following standard concept:

Definition 3.2.2. An ideal J on ω is *ω-hitting* if for every collection $\{a_n : n \in \omega\}$ of infinite subsets of ω there is a set $b \in J$ such that $b \cap a_n \neq 0$ for all $n \in \omega$.

Example 3.2.3. Let p be a partition of ω into finite sets and let J_p be the ideal on ω generated by p-selectors. It is easy to see that J_p is ω-hitting. In fact, a determinacy argument [10] shows that every Borel ω-hitting ideal contains one of the ideals J_p as a subset.

Theorem 3.2.4. *Let J be an analytic ideal on ω. The following are equivalent:*

1. *J is ω-hitting;*
2. *the name for a generic element of 2^ω is $=_J$-trim;*
3. *for every Borel homomorphism $h\colon 2^\omega \to (2^\omega)^\omega$ from $=_J$ to the nonempty intersection of ranges graph, there is a countable set $c \subset 2^\omega$ such that the set $\{x \in 2^\omega : \operatorname{rng}(h(x)) \cap c \neq 0\}$ is comeager.*

Proof. (2) implies (3) by Corollary 2.3.3. To see why (3) implies (1), suppose that a collection $\{a_n : n \in \omega\}$ of infinite subsets of ω witnesses the failure of the ω-hitting property of J and consider any Borel function $h\colon 2^\omega \to (2^\omega)^\omega$ such that $\operatorname{rng}(x)$ includes exactly the sequences $z \in 2^\omega$ such that for some $n \in \omega$, $z(m) = 0$ for all $m \notin a_n$ and $z \restriction a_n = x \restriction a_n$ up to finitely many elements. I will argue that this is a Borel homomorphism violating (3). If $x_0, x_1 \in 2^\omega$ are $=_J$-related elements, then the set $b = \{m \in \omega : x_0(m) \neq x_1(m)\}$ belongs to the ideal J, there is $n \in \omega$ such that $a_n \cap b$ is finite and so $\operatorname{rng}(h(x_0)) \cap \operatorname{rng}(h(x_1)) \neq 0$, and h is a homomorphism. Also, whenever $c \subset 2^\omega$ is a countable set, then the set $B = \{x \in 2^\omega : \forall n\ x \restriction a_n$ is not modulo finite equal to $z \restriction a_n$ for any $z \in c\}$ is co-meager, for all $x \in B$ $\operatorname{rng}(h(x)) \cap c = 0$ holds, and (3) is violated.

Thus, the only thing remaining to be show is the (1)$\to$(2) implication. Let $x_0 \in 2^\omega$ be a generic point over V. In some further forcing extension I will produce a point $x_1 \in 2^\omega$ generic over V such that $x_0 =_J x_1$ and $V[x_0] \cap V[x_1] = V$; this will complete the proof of (2). Let $V[x_0][G]$ be a generic extension such that the set $\mathcal{P}(\omega) \cap V[x_0]$ is countable in it. By the Shoenfield absoluteness, the ideal J is still ω-hitting in $V[x_0][G]$ and so in this model there is a set $a \subset \omega$ in J meeting all infinite subsets of ω in $V[x_0]$. Let P be the poset of finite functions from a to 2, let $y \in 2^a$ be a point P-generic over $V[x_0][G]$ and in the model $V[x_0][G][y]$ define $x_1 = x_0$ rew y. I claim that $x_1 \in 2^\omega$ works as required.

It is essentially immediate that $x_1 \in 2^\omega$ is a point generic over V, and since the set $a \subset \omega$ was selected to belong to the ideal J, $x_0 =_J x_1$ must hold as well. It remains to check that $V[x_0] \cap V[x_1] = V$. Work in the model $V[x_0][G]$, suppose that $\tau \in V$ is a Cohen name for a set of ordinals, $z \in V[x_0] \setminus V$ is a set of ordinals, and $p \in P$; it will be enough to derive a contradiction from the assumption that $p \Vdash \tau/x_1 = \check{z}$. Consider the point $x_2 = x_0$ rew $p \in 2^\omega$ generic over V. There are two cases.

Case 1. If $\tau/x_2 \in V$, then find an initial segment $q \subset x_2$ which forces in the Cohen forcing that $\tau \in V$, and let $p' = q \restriction a$. Then the condition $p' \leq p$ forces in P that q is an initial segment of x_1 and therefore $\tau/x_1 \in V$ and $\tau/x_1 \neq z$. This is a contradiction.

Case 2. Suppose now that $\tau/x_2 \notin V$. Consider the graphing of the Cohen forcing connecting two binary strings of the same length and differing in exactly one entry. Corollary 6.1.13 applied to the point $x_2 \in 2^\omega$ shows that the set $b = \{m \in \omega \colon \tau/x_2 \neq \tau/x_2 * m\}$ is infinite, where $x_2 * m$ is the sequence x_2 with the m-th bit flipped. Now, the set $b \subset \omega$ belongs to the model $V[x_0]$ and so the set a has an infinite intersection with b. Thus, there is $m \in b \cap a$ such that $m \notin \mathrm{dom}(p)$, an ordinal α, and initial segments q_0, q_1 of x_2 and $x_2 * m$ respectively such that in the Cohen forcing, $q_0 \Vdash \check{\alpha} \notin \tau$ and $q_1 \Vdash \check{\alpha} \in \tau$. Now, consider the question of membership of the ordinal α in the set z and assume that $\alpha \in z$; the case $\alpha \notin z$ is symmetric. Let $p' = q_0 \restriction a$. The condition $p' \leq p$ forces in P that q_0 is an initial segment of x_1 and therefore $\check{\alpha} \notin \tau/x_1$ and $\tau/x_1 \neq z$. This is a contradiction again, completing the proof of the theorem. $\square$

Corollary 3.2.5. *Suppose that J is an analytic ideal on ω which is ω-hitting, contains all singletons, and does not contain ω. Suppose that E is a trim equivalence relation. If h is a Borel homomorphism from $=_J$ to E, then h stabilizes on a comeager set.*

Example 3.2.6. Let J be the Rado graph ideal; it is not ω-hitting. Let K be the ideal generated by convergent sequences in $\mathbb{Q} \cap [0,1]$; the equivalence relation $=_K$ is trim by Theorem 3.2.30. There is a Borel homomorphism h of $=_J$ to $=_K$ such that preimages of $=_K$-classes are meager. To construct h, let $\prec$ be an ordering of $\mathbb{Q} \cap [0,1]$ in ordertype ω and consider the partition π of pairs of rationals defined by $\pi(p,q) = 0$ if the rational ordering and $\prec$-ordering of p, q agree, and $\pi(p,q) = 1$ otherwise. Note that π-homogeneous sets are either decreasing or increasing sequences and therefore belong to K. Use the universality of the Rado graph to find an injection $\chi \colon \mathbb{Q} \to \omega$ such that $\pi(p,q) = 0$ if and only if $\chi(p)$ and $\chi(q)$ are connected with an edge in the Rado graph. Now define the function $h \colon 2^\omega \to 2^{\mathbb{Q} \cap [0,1]}$ by $h(x)(p) = x(\chi(p))$ and observe that h has the requested properties.

I do not know the answer to the following:

Question 3.2.7. Is there a trim equivalence relation E such that for an analytic ideal J on ω, the following are equivalent:

1. I is ω-hitting;
2. every Borel homomorphism from $=_J$ to E stabilizes on a comeager set.

Note that in some related cases such a testing equivalence relation exists– Theorem 3.2.10 or 3.3.3.

3.2.2 The random element

The dual question to Question 3.2.1 is

Question 3.2.8. Characterize the class of analytic ideals J on ω such that the name for a random element of 2^ω is $=_J$-trim.

The complete answer seems to be much more involved and out of reach at this point. However, a characterization is available in the important special case of analytic P-ideals.

Definition 3.2.9. A lower semicontinuous submeasure ϕ on ω *exhibits concentration of measure* if for ever positive real $\varepsilon > 0$ there is a positive real $\delta > 0$ such that for all but finitely many n, for all but finitely many m, for all sets $C_0, C_1 \subset 2^{m \setminus n}$ such that $|C_0|, |C_1| > (1-\delta)2^{m-n-1}$ there are $x_0 \in C_0$ and $x_1 \in C_1$ such that the set $\{k \in m \setminus n : x_0(k) \neq x_1(k)\}$ has ϕ-mass at most ε.

While the definition may seem convoluted at the first reading, it is simple in the sense that all its quantifiers are numerical. The statement will sound familiar to all attentive readers of [20].

Theorem 3.2.10. *Suppose that J is an analytic P-ideal on ω containing all finite sets, not containing ω. The following are equivalent:*

1. *the name for a random element of 2^ω is $=_J$-trim;*
2. *every Borel homomorphism from $=_J$ to E_0 stabilizes on a set of full measure;*
3. *for some (every) lower semicontinuous submeasure ϕ on ω such that $J = \{a \subset \omega : \lim_n \phi(a \setminus n) = 0\}$, ϕ exhibits concentration of measure.*

Note that the concentration of measure seems to be a combinatorial property of the submeasure, while the properties of $=_J$ do not depend on the choice of the submeasure. Thus, while there may be many choices of a submeasure generating a given analytic P-ideal, they all agree on the status of concentration of measure.

Proof. Let μ denote the usual Borel probability measure on 2^ω.

(1) certainly implies (2) by Theorem 2.3.1. To see that (2) implies (3), assume that (3) fails and work for the failure of (2). The failure of (3) yields a

positive real $\varepsilon > 0$, and for all $i \in \omega$ natural numbers $n_i < m_i < n_{i+1}$ and sets $C_0^i, C_1^i \subset 2^{m_i \setminus n_i}$ such that $|C_0^i|, |C_1^i| > (1-2^{-i})2^{m_i - n_i - 1}$ and for every $x_0 \in C_0^i$ and every $x_1 \in C_1^i$, the set $\{k \in m_i \setminus n_i \colon x_0(k) \neq x_1(k)\}$ has ϕ-mass at least ε. Let $B = \{x \in 2^\omega \colon$ for all but finitely many $i \in \omega$, $x \restriction (m_i \setminus n_i) \in C_0^i \cup C_1^i$. The set $B \subset 2^\omega$ is of full μ-mass as the sizes of the sets C_0^i, C_1^i keep growing. Let $h \colon B \to 2^\omega$ be defined by $h(x)(i) = 0 \leftrightarrow x \restriction (m_i \setminus n_i) \in C_0^i$. The definitions immediately imply that h is a homomorphism of $=_J$ to E_0. At the same time, h-preimages of singletons are μ-null. By Corollary 6.2.6, the homomorphism can be extended to all of 2^ω while preserving the fact that preimages of singletons are null. This completes the proof of the implication $\neg(2) \to \neg(1)$.

Thus, the only thing remaining is the proof of the implication (3)$\to$(1). The following abstract measure-theoretic claims will come handy:

Claim 3.2.11. *For every Borel set $B \subset 2^\omega$, every positive real $\delta > 0$, and every Borel function $h \colon B \to 2^\omega$, one of the following occurs:*

1. *either there are Borel sets $B_0, B_1 \subset B$ such that $\mu(B_0), \mu(B_1) > (1-\delta)\frac{\mu(B)}{2}$ and $h''B_0 \cap h''B_1 = 0$;*
2. *or there is $y \in 2^\omega$ such that $\mu(h^{-1}\{y\}) > 0$.*

Proof. There are two cases.

Case 1. There is a number $n \in \omega$ such that for every binary string $s \in 2^n$, $\mu(h^{-1}[s]) < \delta\frac{\mu(B)}{2}$. In such a case, it is easy to find a partition $2^n = A_0 \cup A_1$ so that setting $B_0 = h^{-1}(\bigcup_{s \in A_0}[s]$ and $B_1 = h^{-1}(\bigcup_{s \in A_1}[s]$, it is the case that $\mu(B_0), \mu(B_1) > (1-\delta)\frac{\mu(B)}{2}$. The set B_0, B_1 work as required in (1).

Case 2. Case 1 does not occur. Then, for each $n \in \omega$ there is a string $s_n \in 2^n$ such that writing $B_n = h^{-1}[s_n]$ the inequality $\mu(B_n) > \delta\frac{\mu(B)}{2}$ holds. By a compactness argument, there is an infinite set $a \subset \omega$ such that the strings s_n for $n \in a$ have a single limit point $y \in 2^{<\omega}$. The set $C = \limsup_{n \in a} B_n$ has μ-mass at least $\delta\frac{\mu(B)}{2}$, and for every $x \in C$ it must be the case that $h(x) = y$, proving (2). $\square$

Claim 3.2.12. *For every Borel set $B \subset 2^\omega$ and every positive real $\delta > 0$, for all but finitely many $n \in \omega$, $\mu(\{x \in B \colon \mu(B \cap [x \restriction n]) > \frac{1}{2}\mu([x \restriction n])\}) > (1-\delta)\mu(B)$.*

Proof. If this fails, then there is an infinite set $a \subset \omega$ such that, writing $C_n = \{x \in B \colon \mu(B \cap [x \restriction n]) > \frac{1}{2}\mu([x \restriction n])\}$, for every $n \in a$ the inequality $\mu(C_n) \leq (1-\delta)\mu(B)$ holds. By usual measure theoretic considerations, there are pairwise disjoint finite sets $a_i \subset a$ for $i \in \omega$ such that the set $D = \bigcap_i \bigcup_{n \in a_i} B \setminus C_n$ is μ-positive. Use the Lebesgue density theorem to find a binary string $u \in 2^{<\omega}$ such that the relative μ-mass of D in $[u]$ is greater than $1-\delta$. Find an index $i \in \omega$ such that $a_i \cap \mathrm{dom}(u) = 0$, and with an inductive construction find a binary string v extending u such that $\forall n \in a_i\ n \in \mathrm{dom}(v)$ and moreover for every $m \in \mathrm{dom}(v \setminus u)$, the relative μ-mass of D in $[v \restriction m]$ is at least $1-\delta$. Note that $D \subset B$ and so for every $m \in \mathrm{dom}(v \setminus u)$, the relative μ-mass of B in $[v \restriction m]$ is at least $1-\delta$. Let $x \in D$ be some point such that $v \subset x$. By the definition of the set D, there must be $n \in a_i$ such that $x \in B \setminus C_n$. By the definition of the set C_n though, $x \in C_n$. This is a contradiction. $\square$

Now suppose that the submeasure ϕ exhibits the concentration of measure. The following is a key consequence:

Claim 3.2.13. *Suppose that $B \subset 2^\omega$ is a μ-positive Borel set, $\varepsilon > 0$ is a real number, and $f \colon B \to 2^\omega$ is a Borel function. One of the following holds:*

1. *either there is a μ-positive Borel set $C \subset B$ such that $f \restriction C$ is constant;*
2. *or there is a binary string $s \in 2^{<\omega}$ with $\{k \in \mathrm{dom}(s) \colon s(k) = 1\} < \varepsilon$ and a μ-positive Borel set $C \subset B$ such that $s \cdot C \subset B$ and $f''C \cap f''(s \cdot C) = 0$.*

Proof. From the concentration of measure assumption, find a real number $\delta > 0$ which works for ε. Thinning out the set B if necessary, I may assume that $f \restriction B$ is continuous, B is compact, and there is a number $n \in \omega$ which works for ε, δ as in Definition 3.2.9, and there is a binary string $u \in 2^n$ of length some such that $B \subset [u]$ and B has relative mass $> 1 - \delta/4$ in $[u]$. Following Claim 3.2.11, the treatment now splits into cases.

Either, there is a point $y \in 2^\omega$ such that $f^{-1}\{y\}$ has positive μ-mass. In this case, (1) occurs and the claim is proved. Or, there is no such point. By Claim 3.2.11, there must be sets $B_0, B_1 \subset B$ such that their relative masses in $[u]$ are at least $\frac{1}{2}(1-\delta/2)$ and $f''B_0 \cap f''B_1 = 0$ holds. By Claim 3.2.12, there is a number $m \in \omega$ which works for ε, δ, n as in Definition 3.2.9 and moreover, the sets $C_0 = \{x \in B_0 \colon \mu(B_0 \cap [x \restriction m]) > \frac{1}{2}\mu[x \restriction m]\}$ and $C_1 = \{x \in B_0 \colon \mu(B_1 \cap [x \restriction m]) > \frac{1}{2}\mu[x \restriction m]\}$ have both relative μ-mass in $[u]$ greater than $\frac{1}{2}(1-\delta)$. Let $C_0' = \{x' \in 2^{m \setminus n} \colon \exists x \in C_0 \colon u^\frown x' \subset x\}$ and $C_1' = \{x' \in 2^{m \setminus n} \colon \exists x \in C_1 \colon u^\frown x' \subset x\}$. These sets must both have sizes $> (1-\delta)2^{m-n-1}$. By the concentration of measure assumption, there must be $x_0' \in C_0'$ and $x_1' \in C_1'$ such that the ϕ-mass of the set $\{k \in m \setminus n \colon x_0'(k) \neq x_1'(k)\}$ is less than ε.

Now, let $s \in 2^m$ be a string such that $s(k) = 1 \leftrightarrow x_0'(k) \neq x_1'(k)$. Let $v = u^\frown x_0'$, and let $C = \{x \in B_0 \colon v \subset x \text{ and } s \cdot x \in B_1\}$. It is not difficult to see that the set C and the string s work as required in (2). $\square$

Now, let Γ be the ideal J understood as a Polish group with the symmetric difference operation and the usual metric $d(x, y) = \phi\{k \in \omega \colon x(k) \neq y(k)\}$. Let P_Γ be the Cohen forcing on Γ, with the usual name $\dot\gamma_{\mathrm{gen}}$ for the generic element, and let P be the usual random forcing with the measure μ, with the usual name $\dot x_{gen}$ for a random element of 2^ω.

Claim 3.2.14. $P \times P_\Gamma \Vdash V[\dot x_{gen}] \cap V[\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}] = V$.

Proof. As the first remark, note that in the random extension, if α is any ordinal and $h \colon \alpha \to 2$ is any function which does not belong to the ground model, then there is a ground model countable set $b \subset \alpha$ such that $h \restriction b$ is not in the ground model. Thus, it is enough to show that $P \times P_\Gamma \Vdash V[\dot x_{gen}] \cap V[\dot\gamma_{\mathrm{gen}} \cdot \dot x_{gen}] \cap 2^\omega = V \cap 2^\omega$.

Thus, it is enough to show that in the model $V[\dot{x}_{gen}, \dot{\gamma}_{\mathrm{gen}}]$, the only elements definable from $\dot{\gamma}_{\mathrm{gen}} \cdot \dot{x}_{gen}$ in $2^\omega \cap V[\dot{x}_{gen}]$ from ground model parameters are already in the ground model. To this end, suppose that $\langle p, r\rangle \in P \times P_\Gamma$ is a condition, τ is a P-name for an element of 2^ω, ϕ is a formula with ground model parameters, and $\langle p, q\rangle \Vdash \tau$ is defined by $\phi(\dot{\gamma}_{\mathrm{gen}} \cdot \dot{x}_{gen})$. I will find a condition $q \leq p$ forcing τ into the ground model, and this will complete the proof.

Find a condition $r' \leq r$ and a real number $\varepsilon > 0$ such that ε-neighborhood of r' is a subset of r. Strengthen p if necessary to find a Borel function $f \colon 2^\omega \to 2^\omega$ such that $p \Vdash \tau = \dot{f}(\dot{x}_{gen})$. Now, consider Claim 3.2.13 with p, ε, f. If the first item of Claim 3.2.13 occurs, then this yields a condition $p' \leq p$ which forces τ to be an element of the ground model, completing the proof. Thus, it will be enough to show that the second item cannot occur.

If the second item of Claim 3.2.13 occurs as witnessed by a binary string s and a condition $q \leq p$, then $x \in X, \gamma \in \Gamma$ be mutually P-and P_Γ-generic points with $x \in q$ and $\gamma \in r'$. Then, $s \cdot x \in p$ and $s \cdot \gamma$ are mutually generic points as well, $s \cdot x \in p$ and $s \cdot \gamma \in r$, and $V[x, \gamma] = V[s \cdot x, s \cdot \gamma]$. Now, observe that $(s \cdot \gamma) \cdot (s \cdot x) = \gamma \cdot x$ and $f(x) \neq f(s \cdot x)$ both hold. Thus, the formula ϕ with parameter $\gamma \cdot x$ cannot in the model $V[x, \gamma]$ define both distinct points $f(x)$ and $f(s \cdot x)$. □

Finally, everything is ready to produce two random points $x_0, x_1 \in 2^\omega$ such that $x_0 =_J x_1$ and $V[x_0] \cap V[x_1] = V$, confirming (1). Just let $x_0 \in 2^\omega$ and $g \in G$ be points mutually generic for P and P_Γ respectively. In the model $V[\gamma \cdot x_0]$, note that there is a poset Q which adds a point $x \in 2^\omega$ which is $=_J$-related to $\gamma \cdot x$, its is random over V and such that $V[x] \cap V[\gamma \cdot x_0] = V$. This happens since the point x_0 which has the required properties belongs to a forcing extension of $V[\gamma \cdot x_0]$. Pass to a Q-generic extension of the model $V[\gamma, x_0]$ and let $x_1 \in 2^\omega$ be the point added by the generic filter. It is immediate that the two random points $x_0, x_1 \in 2^\omega$ have the required properties. □

The verification of the concentration of measure condition is quite complicated already in the simplest cases, and its status in most cases is open. I will provide several examples illustrating the possibilities.

Example 3.2.15. Let ϕ be the submeasure on ω defined by $\phi(a) = \sum_{n \in a} \frac{1}{n+1}$. The submeasure exhibits concentration of measure.

Proof. Let $\varepsilon > 0$ be arbitrary and consider $\delta = 1/4$. To see that δ witnesses the concentration of measure condition, find n large enough such that $1/4 \leq 2\exp(-\varepsilon^2/32 \sum_{i>n} \frac{1}{i+1}$. Let $m > n$ be any number larger than n. [20, Theorem 4.3.19] shows that whenever $A, B \subset 2^{m \setminus n}$ are sets of relative size $\geq 1/4$, their $\varepsilon/2$-neighborhoods have size $> 1/2$ and so they have nonempty intersection. It follows that there are $a \in A$ and $b \in B$ such that $\sum\{\frac{1}{i+1} \colon a(i) \neq b(i)\} \leq \varepsilon$ as desired. □

Corollary 3.2.16. *Let μ be the usual Borel probability measure on 2^ω. Every homomorphism from the summable equivalence relation on 2^ω to a trim equivalence relation or to a treeable equivalence relation stabilizes on a set of μ-mass 1.*

Proof. Apply Theorem 2.3.1 for trim equivalence relations or Corollary 2.3.7 for treeable relations. □

Example 3.2.17. There is a system of weights $w\colon \omega \to \mathbb{R}^+$ tending to zero such that the submeasure ϕ on ω defined by $\phi(a) = \sum_{n\in a} w(n)$ fails the concentration of measure.

Proof. The key tool is the following:

Claim 3.2.18. *For every $i \in \omega$ and every $\varepsilon > 0$ there is a number $n \in \omega$ and sets $a, b \subset 2^n$ of the same relative size $> \frac{1-\varepsilon}{2}$ each, such that for every $x \in a$ and $y \in b$ the set $\{m \in n\colon x(m) \neq y(m)\}$ contains at least i many elements.*

Proof. Fix i and ε. Elementary computation shows that there is $n \in \omega$ such that the size of the set $\{a \subset n : ||a| - \frac{n}{2}| < i + 1\}$ is less than $\varepsilon 2^n$. Let $a = \{x \in 2^n$: the set $\{m \in n : x(m) = 1\}$ contains at most $\frac{n}{2} - i$ many elements$\}$ and $b = \{x \in 2^n$: the set $\{m \in n\colon x(m) = 1\}$ contains at least $\frac{n}{2} + 1$ many elements$\}$. This works. □

Towards the proof of the example, find a partition $\omega = \bigcup_n I_n$ into finite sets such that for every $n \in \omega$, the set 2^{I_n} contains subsets A_n, B_n of the same size such that their relative size in 2^{I_n} is greater than $1/2 - 2^{-n}$, and if $x \in A_n$ and $y \in B_n$ are arbitrary elements, then the set $\{i \in I_n\colon x(n) \neq y(n)\}$ has size at least n. Now, define $w(m) = 1/n + 1$ if $m \in I_n$. □

Example 3.2.19. There is a Tsirelson submeasure on ω which fails the concentration of measure.

Proof. I will deal with a certain special kind of Tsirelson submeasures. Let $\alpha > 0$ be a real number and $f\colon \omega \to \mathbb{R}^+$ be a function. In a typical case, the function f will converge to 0 and never increase. By induction on $n \in \omega$ define submeasures μ_n on ω by setting $\mu_0(a) = \sup_{i\in a} f(i)$, and $\mu_{n+1}(a) = \sup\{\mu_n(a), \alpha \sum_{b\in\vec{b}} \mu_n(b)\}$ where the variable $\vec{b}$ ranges over all finite sequences $\langle b_0, b_1, \dots b_j\rangle$ of finite subsets of a such that $j < \min(b_0) \leq \max(b_0) < \min(b_1) \leq \max(b_1) < \dots$. In the end, let the submeasure μ be the supremum of μ_n for $n \in \omega$. Some computations are necessary to verify that μ is really a lower semicontinuous submeasure on ω. The ideal $J = \{a \subset \omega\colon \lim_m \mu(a \setminus m) = 0\}$ turns out to be an F_σ P-ideal [2].

By induction on $i \in \omega$ choose intervals $I_i \subset \omega$ such that $\max(I_i) > \min(I_{i+1})$ and such that $\min(I_i) > i/\alpha$ there are sets $a_i, b_i \subset 2^{I_i}$ of the same relative size $\geq \frac{1-2^{-i}}{2}$ such that for any elements $x \in a_i, y \in b_i$ the set $\{m \in I_i\colon x(m) \neq y(m)\}$ has size at least i/α. This is easily possible by Claim 3.2.18. Now, consider the function f defined by $f(m) = 1/i$ for $m \in (\max(I_{i-1}), \max(I_i)]$ and let μ be the derived submeasure. Observe that with this choice of the function f, for any $i \in \omega$ and elements $x \in a_i, y \in b_i$ the set $\{m \in I_i : x(m) \neq y(m)\}$ has μ-mass at least 1, since it has μ_1-mass at least 1. $\square$

Question 3.2.20. Is there a Tsirelson submeasure which exhibits concentration of measure?

In the class of F_σ-ideals, Question 3.2.8 does not seem to have a clear-cut combinatorial resolution. Even here, the concentration of measure appears to be the central concern:

Example 3.2.21. Let J be the Rado graph ideal. Then the name for the random element of 2^ω is not $=_J$-σ-trim. In fact, there is a Borel homomorphism h from $=_J$ to E_0 such that preimages of E_0-classes have zero mass.

Proof. To construct h, by induction on $n \in \omega$ find pairwise disjoint finite sets $C_n \subset \omega$ and sets $A_n, B_n \subset 2^{C_n}$ such that

- each C_n is an anticlique of the Rado graph;
- if $n \neq m$ then the graph connects every element of C_n with every element of C_m;
- for every $x \in A_n$ and every $y \in B_n$ the set $\{t \in C_n : x(t) \neq y(t)\}$ has size at least n;
- the sets $A_n, B_n \subset 2^{C_n}$ have the same relative size larger than $1/2 - 2^{-n}$.

This is easy to do using the universality properties of the Rado graph and Claim 3.2.13 repeatedly. Let $Y \subset 2^\omega$ be the set of all $x \in X$ such that for all but finitely many $n \in \omega$, $x \restriction C_n$ belongs to $A_n \cup B_n$. The last item implies that the set Y has unit mass. Let $h : Y \to 2^\omega$ be the function defined by $h(x)(n) = 0$ if $x \restriction C_n \in A_n$. To see that the function h is a homomorphism of $=_J$ to E_0, use the first two items to see that if $x, y \in Y$ are $=_J$-related points, then there is a number $m \in \omega$ such that for all but finitely many $n \in \omega$, the set $\{t \in C_n : x(t) \neq y(t)\}$ has size $< m$. The third item implies then that $h(x)\ E_0\ h(y)$. Finally, the last item implies immediately that h-preimages of singletons have zero mass, and so preimages of E_0-classes have zero mass as well. $\square$

3.2.3 The general situation

Finally, I will tackle the question of trimness in ideals in its full generality:

Question 3.2.22. Characterize the class of analytic ideals J on ω such that $=_J$ is trim.

While I do not have a full answer to this question, in the classes of F_σ ideals and analytic P-ideals there are useful combinatorial criteria that can be used to prove or disprove trimness.

Definition 3.2.23. An ideal J on ω is *tall* if every infinite subset of ω contains an infinite subset in J.

Theorem 3.2.24. *If J is an F_σ tall ideal on ω, then $=_J$ is not trim.*

Proof. I will simply find a J-positive set $b \subset \omega$ such that the ideal J on b is ω-hitting. Then Theorem 3.2.4 below shows that the name for a generic element of 2^b is nontrivial $=_J$-trim, proving the theorem.

By a theorem of ???, if J is an F_σ tall ideal, there is a J-positive set $a \subset \omega$ and a partition $\pi\colon [a]^2 \to 2$ such that all π-homogeneous subsets of a belong to J. By induction on $n \in a$ find bits $f(n) \in 2$ such that for every $m \in \omega$ the set $a_n = \{k \in a\colon \forall n \in a \cap m\ \pi(n,k) = f(n)\}$ is J-positive. Express the ideal J as a countable union of closed sets $J = \bigcup_i J_i$ and for each $n \in \omega$ pick a finite set $b_n \subset a_n$ such that no superset of b_n belongs to $\bigcup_{i\in n} J_i$. I claim that the set $b = \bigcup_n b_n$ works.

Plainly, the set b is J-positive. To prove the ω-hitting property, suppose that $\{c_i\colon i \in \omega\}$ are inifnite subsets of the set a. By induction on $i \in \omega$ choose numbers $k_i \in c_i$ so that for all $j < i$, $\pi(k_i, k_j) = f(k_i)$. In the end, the set $d = \{k_i\colon i \in \omega\}$ has nonempty intersection with all sets c_i and it can be split into two π-homogeneous sets $d \cap f^{-1}\{0\}$ and $d \cap f^{-1}\{1\}$. By the choice of the partition π, $d \in J$ as desired. □

Theorem 3.2.25. *Let J be an analytic P-ideal on ω containing all singletons. Exactly one of the following occurs:*

1. *there is an infinite set $a \subset \omega$ such that $J = \{b \subset \omega\colon b \cap a$ is finite$\}$;*
2. *there are pairwise disjoint infinite sets $a_i \subset \omega$ for $i \in \omega$ such that $J = \{b \subset \omega\colon b \cap a_i$ is finite for all $i \in \omega\}$;*
3. *$=_J$ is not Cohen-trim.*

Observe that in item (1) the equivalence relation $=_J$ is bireducible with E_0, and in item (2), $=_J$ is bireducible with E_0^ω; all of these equivalence relations are trim.

Proof. First observe that in item (1) the equivalence relation $=_J$ is bireducible with E_0 and in item (2) $=_J$ is bireducible with E_0^ω. As E_0^ω is not reducible to E_0, and both E_0, E_0^ω are trim relations, the three items are indeed mutually exclusive. I must prove that one of the items does occur. Using a theorem of Solecki [22], find a lower semicontinuous submeasure ϕ on ω such that $J = \{a \subset \omega\colon \lim_n \phi(a \setminus n) = 0\}$.

Case 1. There is $\varepsilon > 0$ such that the set $a_\varepsilon = \{n \in \omega\colon \phi(n) < \varepsilon\}$ is in J. This immediately implies that (1) occurs with $a = \omega \setminus a_\varepsilon$.

Case 2. Case 1 fails and for every $i \in \omega$ greater than 0 there is $\varepsilon_i > 0$ such that $\limsup_n \phi(a_{\varepsilon_i} \setminus n) < 2^{-i}$. Write also $a_{\varepsilon_0} = \omega$. Consider the sets $a_i = a_{\varepsilon_i} \setminus a_{\varepsilon_{i+1}}$. For every set $b \subset \omega$, if b has infinite intersection with some a_i, then $b \notin J$, since b contains infinitely many singletons of ϕ-mass $\geq \varepsilon_{i+1}$. On the other hand, if b has finite intersection with every a_i then $\limsup_n \phi(b \setminus n) = 0$ and so $b \in J$. Thus, item (2) occurs as witnessed by the sets $a_i \colon i \in \omega$ (possibly removing the finite ones among them).

Case 3. Both Case 1 and Case 2 fail. Then, there is a set $a \subset \omega$ which is not in J, and such that $\lim_{n\in a} \phi(\{n\})$. The ideal $J \restriction a$ is easily seen to be ω-hitting and therefore, by Theorem 3.2.4, the name for a generic element of 2^a is nontrivial $=_J$-trim. Item (3) immediately follows. □

For ideals which are not F_σ and not P-ideals, I can prove trimness only in certain somewhat ad-hoc cases. Still, the patterns are interesting enough to include them here.

Definition 3.2.26. [16] An ideal J on ω is *countably separated* if there is a collection $\{a_n \colon n \in \omega\}$ of subsets of ω such that for every $b \subset \omega$ in J and every $c \subset \omega$ not in J, there is a number $n \in \omega$ such that $a_n \cap b = 0$ and $a_n \cap c \notin J$.

Example 3.2.27. Suppose that X is a compact metrizable space, I an analytic σ-ideal of compact sets on X, and $D \subset X$ is a countable dense subset. Let $J = \{a \subset D\colon$ the closure of a in X belongs to $I\}$. Then J is countably separated.

In fact, Kwela and Sabok [16] showed that every countably separated tall analytic ideal can be derived in this way from an $\mathbf{\Pi}^0_2$ σ-ideal of compact sets, and it must be $\mathbf{\Pi}^0_3$-complete.

Proof. Let $\{O_n \colon n \in \omega\}$ be a basis of the topology of the space X closed under finite unions and intersections and let $a_n = O_n \cap D$. The collection $\{a_n \colon n \in \omega\}$ works as desired. If $b \in J$ and $c \notin J$ are subsets of D, then (as I is a σ-ideal of compact sets), there must be a basic open set O_n for some $n \in \omega$ such that $O_n \cap \bar{b} = 0$ and the closure of $O_n \cap c$ is I-positive. Then $a_n \cap b = 0$ and $a_n \cap c \notin J$ as required. □

Thus, the class of countably separated ideals includes such popular characters as the ideal of nowhere dense subsets of $\mathbb{Q} \cap [0,1]$ or the ideal of subsets of $\mathbb{Q} \cap [0,1]$ with Lebesgue null closure.

Theorem 3.2.28. *If J is countably separated analytic ideal on ω then $=^{2^\omega}_J$ is a trim equivalence relation.*

Proof. Let $\{a_n \colon n \in \omega\}$ be a collection witnessing the countable separation of the ideal J. Suppose that in some forcing extension $V[G]$, $x_0, x_1 \in (2^\omega)^\omega$ are $=^{2^\omega}_J$-related points and $V[x_0] \cap V[x_1] = V$; I must find a ground model element equivalent to both x_0, x_1.

Consider the set $d = \{n \in \omega\colon x_0 \restriction a_n$ is $=^{2^\omega}_J$ equivalent to some element in the ground model$\}$. The set d is defined from the $=^{2^\omega}_J$-class of x_0 and so it belongs to both $V[x_0]$ and $V[x_1]$ and therefore belongs to V. By a similar argument, there is in V a function f with domain d such that for every $n \in d$ the value $f(n)$ is some function in $(2^\omega)^{a_n}$ which is $=^{2^\omega}_J$-equivalent to $x_0 \restriction a_n$ and $x_1 \restriction a_n$.

Now, in the ground model there is a point $y \in (2^\omega)^\omega$ such that for every number $n \in d$, $f(n)$ is $=^{2^\omega}_J$-related to $y \restriction a_n$. This is true by the Mostowski absoluteness, since there is such a point in the model $V[G]$, namely x_0. Moreover, any two such points $y_0, y_1 \in (2^\omega)^\omega$ in the ground model must be $=^{2^\omega}_J$-related. To see this, suppose they are not related, and let $c = \{m \in \omega\colon y_0(m) \neq y_1(m)\} \notin J$ and $b = \{m \in \omega\colon x_0(m) \neq x_1(m) \in J$. By the Mostowski absoluteness, the collection $\{a_n\colon n \in \omega\}$ witnesses the strong trimness of J even in $V[G]$ and so there is $n \in \omega$ such that $a_n \cap b = 0$ and $a_n \cap c \notin J$. But then, $n \in d$ since $x_0 \restriction a_n = x_1 \restriction a_n$ and by the initial assumptions this point must be in the ground model. Thus, the value $f(n)$ is defined and both $y_0 \restriction a$ and $y_1 \restriction a$ must be $=^{2^\omega}_J$-related to it, contradicting the fact that $a_n \cap c \notin J$.

Pick a point $y \in (2^\omega)^\omega$ in the ground model such that for all $n \in d$, $y \restriction a_n =^{2^\omega}_J f(n)$. Thus, the ground model satisfies the statement "for every $z \in (2^\omega)^\omega$, if $\forall n\ z \restriction a_n =^{2^\omega}_J f(n)$ then $z =^{2^\omega}_J y$". By the Shoenfield absoluteness, the same statement must hold in the model $V[G]$ and therefore $y =^{2^\omega}_J x_0, x_1$ as required. $\square$

Further examples of analytic ideals J such that the equivalence relation $=^{2^\omega}_J$ is trim seem to be somewhat ad hoc.

Theorem 3.2.29. *Let J be the ideal on $2^{<\omega}$ generated by branches. The equivalence relation $=^{2^\omega}_J$ is trim.*

Note that the branch ideal is F_σ, but not tall: its restriction to any infinite antichain is the ideal of finite sets.

Proof. Let $X = (2^\omega)^{2^{<\omega}} = \mathrm{dom}(=^{2^\omega}_J)$. For every node $t \in \omega^{<\omega}$ write $[t]$ for the set of all nodes in $2^{<\omega}$ extending t. Let $V[G_0]$, $V[G_1]$ be two generic extensions containing respective $=_J$-related points $x_0, x_1 \in X$. Assume that $V[G_0] \cap V[G_1] = V$ and work to find a ground model point $x \in X$ which is $=_J$-related to both x_0, x_1.

Let $T = \{t \in 2^{<\omega}\colon x_0 \restriction [t]$ is not $=_J$-equivalent to any point in the ground model$\}$. This is a subtree of $\omega^{<\omega}$. If $0 \notin T$ then the proof is complete; thus, it is only necessary to derive a contradiction from the assumption $0 \in T$. First, observe that the tree T cannot have any terminal nodes: if t was a terminal node of T then one could combine the ground model witnesses for $t^\frown 0 \notin T$ and $t^\frown 1 \notin T$ to find a witness for $t \notin T$. Second, observe that the definition of the tree T depends only on the $=_J$-class of x_0 and so $T \in V[G_0] \cap V[G_1] = V$. Since T is a nonempty ground model tree without endnodes, it must have an infinite branch $y \in 2^\omega$ in the ground model. Since $x_0 =_J x_1$, there is a number $n \in \omega$ such that for every $t \in 2^{<\omega}$ such that $x_0(t) \neq x_1(t)$, either t is incompatible with $y \restriction n$ or else t is an initial segment of y. Let $e = [y \restriction n] \setminus y \restriction m\colon m \geq n\}$ and observe that $e \in V$ and x_0 and x_1 coincide on e, therefore $x_0 \restriction e \in V$. If $z \in V$ is any function in $2^{[y \restriction n]}$ extending $x_0 \restriction e$, then $z =_J x_0 \restriction [t]$, contradicting the assumption that $y \restriction n \in T$. $\square$

As one more interesting special case, let Y be a compact metrizable space without isolated points, let $D \subset Y$ be a countable dense set, let $\alpha \in \omega_1$ be a nonzero ordinal, and let J_α be the ideal of those subsets of $a \subset D$ such that the closure of a is countable and of Cantor–Bendixson rank $< \alpha$. It is not difficult to see that J_α is a Borel ideal. Up to isomorphism, the ideal J_α does not depend on the choice of the dense set D; it does depend on the characteristics of the underlying space Y. The dependence on Y will not be noted below

Theorem 3.2.30. *Let Y be a compact metrizable space, $D \subset Y$ a countable dense set, and let J_α be the derived ideals for nonzero countable ordinal α. The equivalence relation $=^{2^\omega}_{J_\alpha(Y)}$ is trim for every countable ordinal $\alpha > 0$.*

Note that $=^{2^\omega}_{J_1}$ is just another representation of the familiar equivalence relation E_1.

Proof. Write F_α for the equivalence relation $=^{2^\omega}_{J_\alpha(Y)}$ and $X = (2^\omega)^D = \mathrm{dom}(F_\alpha)$. The argument proceeds by induction on the ordinal α. For the basis step $\alpha = 1$, to argue that F_1 is trim, let $V[G_0], V[G_1]$ be generic extensions containing the respective F_1-related points $x_0, x_1 \in X$. By the definition of the ideal J_1, there is a finite set $a \subset D$ such that $x_0 \restriction (D \setminus a) = x_1 \restriction (D \setminus a)$. Now, if this common value belongs to V, then the F_1-class of x_0, x_1 has a representative in V; if the common value does not belong to V then $V[G_0] \cap V[G_1] \neq V$. This proves trimness of F_1.

The limit stage of the transfinite induction follows from Theorem 2.2.13, as for limit α it is the case that $F_\alpha = \bigcup_{\beta \in \alpha} F_\beta$. For the successor stage, say $\alpha = \beta + 1$ and the theorem has been proved for β. Let $V[G_0], V[G_1]$ be generic extensions such that $V[G_0] \cap V[G_1] = V$, let $x_0, x_1 \in X$ be two F_α-related points in these respective models, and work to find a representative of their F_α-class in the ground model. Choose a countable basis B for the space Y and let $A = \{O \in B \colon x_0 \restriction O \text{ is } F_\alpha\text{-related to some element of the ground model}\}$. Since the set A is defined from the F_α-class of x_0, it belongs to both $V[G_0]$ and $V[G_1]$ and therefore to V.

Now, if $Y = \bigcup A$, then by a compactness argument the space Y is covered by finitely many elements of A and x_0 is F_α-related to an element of the ground model. Thus, it is enough to derive a contradiction from the assumption that $Y \neq \bigcup A$.

Suppose then that $Y \neq \bigcup A$ and pick a point $y \in Y \setminus \bigcup A$. Use the compactness of the space Y to pick a sequence $\langle O_n \colon n \in \omega\rangle$ of neighborhoods in the basis such that $\bigcup_n O_n = Y \setminus \{y\}$, each O_n is bounded away from y, and the neighborhoods converge to the point y. Consider the space $\bar{X} = \prod_n (2^\omega)^{D \cap O_n}$ and the equivalence relation F on it obtained as modulo finite product of F_β-equivalence on each coordinate. The equivalence relation F is trim by the induction hypothesis and Theorem 2.2.13 (3). It is clear from the definition of Cantor–Bendixson rank that the points $\bar{x}_0 = \langle x_0 \restriction O_n \colon n \in \omega\rangle \in \bar{X}$ and $\bar{x}_1 = \langle x_1 \restriction O_n \colon n \in \omega\rangle \in \bar{X}$ are F-related. Thus, there is a point $\bar{x}$ in the ground model which is F-related to $\bar{x}$. Pick a number $m \in \omega$ such that for all $n > m$, $\bar{x}(n)\ F_\beta\ \bar{x}_0(n)$. For every $n \leq m$, use the fact that $y \notin O_n$ to conclude that $\bar{x}_0(n)$ has an F_α-equivalent in the ground model, and alter $\bar{x}$ on the first $m+1$ coordinates if necessary to ensure that $\bar{x}_0(n)\ F_\alpha\ \bar{x}(n)$ for all $n \leq m$. Now, in the ground model define the point $x \in X$ by setting $x(d) = \bar{x}(n)(d)$ where $n \in \omega$ is the smallest number such that $d \in O_n$. A review of definitions shows that $x \; F_\alpha \; x_0$. This means that in fact $A = B$ holds, and so $Y = \bigcup A$, contradicting the assumption that $Y \neq \bigcup A$. □

3.3 σ-trimness in ideals

In this section, I will attempt the same feat as in the previous one, except for σ-trimness.

3.3.1 The generic point

Question 3.3.1. Characterize the class of analytic ideals J on ω for which the name for the generic element of 2^ω is $=_J$-σ-trim.

It turns out that there is a complete resolution of this natural question.

Definition 3.3.2. An ideal J on ω is *tall* if every infinite subset of ω has an infinite subset in J.

Theorem 3.3.3. *Let J be an analytic ideal on ω. The following are equivalent:*

1. *J is tall;*
2. *the name for a generic element of 2^ω is $=_J$-σ-trim;*
3. *every Borel homomorphism from $=_J$ to E_0 stabilizes on a comeager set.*

Proof. (2) implies (3) by Theorem **??**, since E_0 is clearly σ-trim. To see that (3) implies (1), suppose that (1) fails, and work to prove that (3) must fail too. Let $a \subset \omega$ be an infinite set witnessing the failure of tallness of the ideal J, i.e. every subset of a which is in J is in fact finite. The function $h \colon 2^\omega \to 2^\omega$, defined by $h(x)(m) = 0$ if $m \notin a$ and $h(x)(m) = x(m)$ if $m \in a$, is a continuous homomorphism of $=_J$ to E_0 which does not stabilize on a comeager set. Thus, (3) fails as desired.

This leaves us with the (1)→(2) implication. Let $x \in 2^\omega$ be a V-generic point and $y \in V[x] \setminus V$ be an element of 2^ω; it will be enough, in some further forcing extension, to find a V-generic point $x' \in 2^\omega$ $=_J$-related to x and such that $y \notin V[x']$.

Work in the model $V[x]$. Let $\sigma \in V$ be a name such that $\sigma/x = y$. Using Corollary 6.1.13, find pairwise distinct numbers n_i for $i \in \omega$ such that $\sigma/x_i \neq y$ where $x_i \in 2^\omega$ is the binary sequence equal to x everywhere except in its n_i-th entry. The analytic ideal J is tall in V, by the Shoenfield absoluteness it is tall in $V[x]$, and therefore there must be an infinite set $a \subset \{n_i \colon i \in \omega\}$ which belongs to J. Let $z \in 2^a$ be a $V[x]$-generic point, and let $x' = x$ rew z. I claim that $x' \in 2^\omega$ works as required.

To show that $y \notin V[x']$, let $\tau \in V$ be a name for an element of 2^ω and p be a finite partial function from a to 2. I will find a partial function q extending

p, a number m such that t rew $q \vdash \tau(\check{m}) \neq y(m)$. Then, the condition q forces $(\tau/x')(m) \neq y(m)$ and so $\tau/x' \neq \check{y}$ as required.

Find a condition $s \in P$ such that $s \subset x$ and s decides the statement $\tau/\dot{x}_{gen}$ rew $p = \sigma/\dot{x}_{gen}$. There are two cases.

Case 1. The decision is negative. In such a case, find an extension t of s such that $s \subset t$ and $t \subset x$, a number $m \in \omega$, and distinct bits $b_0 \neq b_1 \in 2$ such that $t \Vdash (\tau/\dot{x}_{gen}$ rew $p)(m) = \check{b}_0$ and $\sigma/\dot{x}_{gen})(m) = \check{b}_1$. Let $q = (t$ rew $p) \restriction a$ and observe that t, q, m work as desired.

Case 2. The decision is affirmative. In this case, find a number $k \in a$ such that $k \notin \mathrm{dom}(s)$, a binary string t extending s, a number $m \in \omega$ and distinct bits $b_0, b_1 \in 2$ such that $t \Vdash \sigma(\check{m}) = \check{b}_0$ and $t' \Vdash \sigma(\check{m}) = \check{b}_1$, where t' is the string equal to t except at its k-th entry. Note that by the case assumption, $t' \Vdash (\tau/\dot{x}_{gen}$ rew $p)(\check{m}) = \check{b}_1$. Now let $q = (t'$ rew $p) \restriction a$ and observe that t, q, m work as desired. □

Corollary 3.3.4. *If J is a tall analytic ideal and h is a Borel homomorphism from $=_J$ to an equivalence relation classifiable by countable structures, then h stabilizes on a comeager set.*

Note that the homomorphism may not stabilize on a co-null set by Example 3.2.21.

3.3.2 The random element

Question 3.3.5. Characterize the class of ideals J on ω such that the name for the random element of 2^ω is $=_J$-σ-trim.

As in the case of trimness, this question appears to be much harder than its category counterpart. The special case of analytic P-ideals is fully resolved by Theorem 3.2.10: if the P-ideal J exhibits concentration of measure, then the name for the random point is in fact $=_J$-trim and so $=_J$-σ-trim. On the other hand, if the P-ideal J fails to have the concentration of measure, there is a Borel homomorphism from $=_J$ to E_0 which does not stabilize on any set of full measure, which proves the failure of σ-trimness of the name for the random point in view of Theorem 2.3.1 and the fact that E_0 is σ-trim. The case of F_σ-ideals seems to be very difficult and out of reach. Another class of ideals in which the name for the random point is σ-trim is isolated in the following definition:

Definition 3.3.6. An ideal J on ω is *of convergence type* if there is a countable collection $\{a_n : n \in \omega\}$ of subsets of ω such that every set $b \subset \omega$ such that for every $n \in \omega$, b is modulo finite disjoint from or modulo finite included in a_n, $b \in J$ holds.

Example 3.3.7. The ideal J on a countable dense set D of a compact metrizable space X generated by convergent sequences is of convergence type. To see this, let $\{a_n : n \in \omega\}$ consist of all intersections of basic open subsets of X with D.

The ideal J appears prominently in Hrušák's work on the Katětov order of Borel ideals on ω. It is the smallest ideal which is not below any F_σ-ideal in this order [10]. I will prove a great strengthening of this result, which shows in particular that any Borel homomorphism for $=_J$ to $=_K$ for any F_σ-ideal K stabilizes on a set of full measure and on a co-meager set.

Theorem 3.3.8. *Suppose that J is an analytic ideal of convergence type on ω. Then the names for a generic or random elements of 2^ω are $=_J$-E_{K_σ}-σ-trim.*

The proof uses an approach that will be useful in Section 3.4, and so I will first fix certain technical definitions and facts.

Definition 3.3.9. Suppose that $V[G]$ is a generic extension, $x_0 \in 2^\omega \cap V[G]$ is a point generic over V, and $A = \{a_n : n \in \omega\}$ is a collection of ground model subsets of ω. The poset $P(x_0, A) \in V[G]$ consists of all pairs $p = \langle b_p, t_p\rangle$ where $b_p \subset \omega$ is a finite set and $t_p \in 2^{<\omega}$. The ordering is defined by $q \leq p$ if $b_p \subset b_q$, $t_p \subset t_q$, and $\{m \in \mathrm{dom}(t_q \setminus t_p) : t_q(m) \neq x_0(m)\} \cap \bigcup_{n\in b_p} a_n = 0$. Let $\dot{x}_1$ be a $P(x_0, A)$-name for the union of the second coordinates of conditions in the generic filter.

Proposition 3.3.10. *Suppose that J is an analytic ideal on ω. Suppose that $V[G]$ is a generic extension, $x_0 \in 2^\omega \cap V[G]$ is a point generic over V, and $A = \{a_n : n \in \omega\}$ is a collection of ground model subsets of ω such that every J-positive set has infinite intersection with one of them.*

1. *$P(x_0, A) \Vdash \dot{x}_1 \in 2^\omega$ is a point generic over V;*
2. *the point $\dot{x}_1$ is forced to be $=_J$-equivalent to x_0;*
3. *for every Polish space Z, K_σ-equivalence relation E on it, and points $z_0 \in Z \cap V[x_0]$ and $z_1 \in Z \cap V[x_1]$, the points z_0, z_1 are E-related if and only if there is a finite set $b \subset \omega$ such that the points z_0, z_1 are both E-related to the same point in $V[x_0 \restriction \bigcup_{n\in b} a_n]$.*

Proof. Let Q be the poset of all finite partial functions from ω to 2 ordered by reverse inclusion. Thus,the point $x_0 \in 2^D$ is generic over V for the poset Q.

For (1), suppose that $C \subset Q$ is a dense open set in the ground model and $p \in P = P(x_0, A)$ is a condition. I have to find a condition $q \leq p$ such that $t_q \in C$. To do this, note that x_0 rew t_p, as a finite modification of a point generic over V, is also generic over V. Therefore, there is a finite fragment $t_q \subset x_0$ rew t_p such that $t_p \subset t_q$ and $t_q \in C$. The pair $q = \langle b_p, t_q\rangle$ belongs to the poset P, it is stronger than p, and it has the requested properties.

For (2), use the definition of the poset P to argue that the set $\{m \in \omega : x_0(m) \neq x_1(m)\}$ has a finite intersection with every element of the set A and therefore belongs to J. Note that the critical property of the set A is assumed to hold in $V[G]$, but by the Shoenfield absoluteness it also holds in the $P(x_0, A)$-extension of $V[G]$.

For (3), the right-to-left direction is clear. For the left-to-right direction, suppose that $z \in Z \cap V[x_0]$ is a point, $\tau \in V$ is a Q-name for an element of the space Z, and $p \in P$ is a condition; write $a = \bigcup_{n\in b_p} a_n$. By a density argument, it is enough to show that there is a condition $q \leq p$ which either forces τ/x_1 to be E-equivalent to some element of $V[x_0 \restriction a]$ or else forces $\langle \tau, z\rangle$ into some basic open set with an empty intersection with E. In the model $V[x_0 \restriction a]$ consider the poset Q_p of all finite partial functions $t\colon D \to 2$ such that $t_p \subset t$ and $t \restriction a \subset x_0$ rew t_p. The poset Q_p is ordered by reverse extension. The union of conditions in the generic filter is a point $x_2 \in 2^\omega$ which is forced to be Q-generic over the ground model.

Case 1. Suppose first that there is a condition $t \in Q_p$ which forces that τ/x_2 is E-related to some point in the model $V[x_0 \restriction a] = V[x_2 \restriction a]$. By the forcing theorem applied to the poset Q, strengthening the function t if necessary it must be the case that $V \models t \Vdash_Q \tau$ has an E-equivalent in the model $V[\dot{x}_{gen} \restriction a]$. Then, the condition $q \leq p$ defined by $q = \langle b_p, t\rangle$ forces in P that τ/x_1 has some E-equivalent in the model $V[x_1 \restriction a] = V[x_0 \restriction a]$, as desired.

Case 2. Q_p forces $\tau/\dot{x}_2$ not to be equivalent to any point in the model $V[x_0 \restriction a]$. Let $x_2, x_3 \in 2^D$ be points mutually generic over the model $V[x_0 \restriction a]$ for the poset Q_p. Since the equivalence relation E is K_σ, it is pinned and so it must be the case that $\tau/x_2 \; E \; \tau/x_3$ fails. Thus, the point z is not E-related to one of the points τ/x_2 or τ/x_3; say it is not equivalent to the former. Then $\tau/x_2 \; E_n \; z$ must fail, and since the equivalence relation E is F_σ, there must be basic open sets $U, W \subset Z$ such that $(U \times W) \cap E = 0$ and $\tau/x_2 \in U$ and $z \in W$. By the forcing theorem applied to the poset Q in the ground model, there must be a finite fragment $t \subset x_2$ extending t_p such that $V \models t \Vdash_Q \tau \in U$. Then the condition $q = \langle b_p, t\rangle \leq p$ forces $\langle \tau, z\rangle \in U \times W$, and the proof is complete. $\square$

All the ergodicity results below also have a measure counterpart. This counterpart relies on measure variations of the above proposition.

Definition 3.3.11. Suppose that $V[G]$ is a generic extension, $x_0 \in 2^\omega \cap V[G]$ is a point random over V, and $A = \{a_n \colon n \in \omega\}$ is a collection of ground model subsets of ω. The poset $R(x_0, A) \in V[G]$ consists of all pairs $p = \langle b_p, t_p.B_p\rangle$ where $b_p \subset \omega$ is a finite set, $t_p \in 2^{<\omega}$ and $B_p \subset 2^\omega$ is a Borel set of positive measure coded in V such that x_0 rew $t_p \in B_p$. The ordering is defined by $q \leq p$ if $b_p \subset b_q$, $t_p \subset t_q$, $B_q \subset B_p$, and $\{m \in \mathrm{dom}(t_q \setminus t_p) \colon t_q(m) \neq x_0(m)\} \cap \bigcup_{n\in b_p} a_n = 0$. Let $\dot{x}_1$ be a $R(x_0, A)$-name for the union of the second coordinates of conditions in the generic filter.

Proposition 3.3.12. *Suppose that J is an analytic ideal on ω. Suppose that $V[G]$ is a generic extension, $x_0 \in 2^\omega \cap V[G]$ is a point random over V, and $A = \{a_n \colon n \in \omega\}$ is a collection of ground model subsets of ω such that every J-positive set has infinite intersection with one of them.*

1. *$R(x_0, A) \Vdash \dot{x}_1 \in 2^\omega$ is a point random over V;*
2. *the point $\dot{x}_1$ is forced to be $=_J$-equivalent to x_0;*
3. *for every Polish space Z, K_σ-equivalence relation E on it, and points $z_0 \in Z \cap V[x_0]$ and $z_1 \in Z \cap V[x_1]$, the points z_0, z_1 are E-related if*

and only if there is a finite set $b \subset \omega$ such that the points z_0, z_1 are both E-related to the same point in $V[x_0 \restriction \bigcup_{n \in b} a_n]$.

Proof of Theorem 3.3.8. I will treat the case of the random element. Let $\{a_n : n \in \omega\}$ be a collection of subsets of ω witnessing the convergence type of J. Let Y be a Polish space with a K_σ equivalence relation E on it. Let $x_0 \in 2^\omega$ be a point random over V, and in $V[x_0]$, let $y \in Y$ be a point which has no E-equivalent in V. It will be enough to find, in some further generic extension, a point $x_1 \in 2^\omega$ which is random over V, $=_J$-related to x_0, and such that y has no E-equivalent in the model $V[x_1]$.

The set $\{a \subset \omega : a \in V$ and y has an E-equivalent in $V[x_0 \restriction a]$ is closed under intersections by Claim 6.1.14. Extend it to an ultrafilter F on ω, let $A = \{a_n : n \in \omega, a_n \in F\}$ and consider the poset $P(x_0, A)$. By the assumptions on the collection $\{a_n : n \in \omega\}$, every set which has finite intersection with all elements of the set A belongs to the ideal J. Proposition 3.3.12 and the choice of the ultrafilter F now show that the poset $P(x_0, A)$ forces $\dot{x}_1$ to be a point in 2^ω generic over V, $=_J$-related to x_0, and such that y has no E-equivalent in $V[\dot{x}_1]$. This completes the proof. □

Corollary 3.3.13. *Suppose that J is an analytic ideal of convergence type on ω. Every Borel homomorphism of $=_J$ to an equivalence classifiable by countable structures or a K_σ-equivalence relation stabilizes on a comeager set and also on a set of full measure in 2^ω.*

Proof. Use the general ergodic Theorem 2.3.1 together with Theorem 3.3.8 and Corollary 2.2.16. □

3.3.3 The general situation

Question 3.3.14. Characterize the class of analytic ideals on ω such that the equivalence relation $=_J$ is σ-trim.

In view of the characterization of trimness in analytic P-ideals in Theorem 3.2.25 and Solecki's dichotomy about the equivalence relations of the form $=_J$ [22, Corollary 4.1(i)], the main problematic case is that of the relation E_1. Note that E_1 can be viewed as $=_J$ where J is the ideal on $\omega \times \omega$ consisting of sets with only finitely many vertical sections nonempty. Alas, the case of E_1 is quite tricky. I can prove that E_1 is *consistently* not σ-trim. At the same time I have no result to the effect that σ-trimness of analytic equivalence relations is absolute–this is in contradistinction of trimness or pinnedness, as Corollary 5.2.7 and Theorem 5.2.1 show.

Theorem 3.3.15. *The equivalence relation E_1 is trim. It is consistent with ZFC that E_1 is not σ-trim.*

Proof. Write $X = (2^\omega)^\omega$. The first sentence is a special case of Theorem 3.2.30 for $\alpha = 1$. For the second sentence, consider the countable support product P of countably many copies of the Sacks forcing. For every number $n \in \omega$ let

P_n be the countable support product of the copies of Sacks forcing indexed by numbers $\geq n$. Let $\dot{G}_n$ be the P_n-name for a generic filter and $\dot{x}_n$ be the P_n name for an ω-sequence of elements of 2^ω defined by $\dot{x}_n(m) = 0$ if $m < n$ and $\dot{x}_n(m)$ is the m-th Sacks real added by the product.

The poset P forces $\dot{x}_n \in V[\dot{G}_n]$ for $n \in \omega$ to be pairwise E_1-related points. I will show that P forces that $\bigcap_n V[\dot{G}_n] = V[\dot{H}]$ for some filter H generic over V such that $X \cap V[H] = X \cap V$. Then, Claim 2.2.14 shows that in $V[\dot{H}]$, there is a poset R and a nontrivial E_1-σ-trim name on the poset R, proving the second sentence of the theorem.

Consider the poset Q consisting of Borel subsets of X which happen to be E_1-saturations of products $\prod_n C_n$ where each $C_n \subset 2^\omega$ is a nonempty perfect set; the order is that of inclusion. It was proved in [13, Theorem 9.3.4] and not difficult to see that the poset Q is σ-closed and if $G \subset P$ is generic then the set $H \subset Q$ of all E_1-saturations of conditions in G is in fact a filter generic over V.

It is immediate that the filter H belongs to the model $V[G_n]$ for each $n \in \omega$, in other words $V[H] \subseteq \bigcap_n V[G_n]$ holds. Proving the opposite inclusion is similarly easy. Suppose that τ is a P-name and $p \in P$ is a condition forcing τ to be a set of ordinals in the intersection $\bigcap_n V[\dot{G}_n]$. A simple fusion argument shows that strengthening the condition p if necessary, it is possible to find P_n-names τ_n for $n \in \omega$ such that $p \Vdash \tau = \tau_n/\dot{G}_n$ for every $n \in \omega$. Then, there cannot be a number $n \in \omega$, an ordinal α and conditions $p_0, p_1 \leq p$ such that $\forall m \geq n\ p_0(m) = p_1(m)$ and $p_0 \Vdash \check{\alpha} \notin \tau$ and $p_1 \Vdash \check{\alpha} \in \tau$–the projections of such two conditions to the poset P_n coincide and therefore cannot force contradictory information about τ_n. This, however, means that τ can be recovered in the model $V[\dot{H}]$ as the set of those ordinals α for which there is a condition $q \leq p$ in the poset P such that the E_1-saturation of q is in H and $q \Vdash \check{\alpha} \in \tau$. □

Corollary 3.3.16. *It is consistent that for an analytic ideal J on ω, exactly on of the following occurs:*

1. *there is an infinite set $a \subset \omega$ such that $J = \{b \subset \omega \colon a \cap b$ is finite$\}$;*
2. *there are pairwise disjoint infinite subsets $\{a_n \colon n \in \omega\}$ of ω such that $J = \{b \subset \omega \colon a_n \cap b$ is finite for all $n \in \omega$;*
3. *$=_J$ is not σ-trim.*

Proof. Use Theorem 3.3.15 to pass to a generic extension in which E_1 is not σ-trim, and argue that the trichotomy holds there.

Let J be an analytic ideal on ω. By Solecki's dichotomy [22, Corollary 4.1(i)], either E_1 is continuously reducible to $=_J$ or J is an analytic P-ideal. In the former case, $=_J$ is not σ-trim by Theorem 2.2.13 (6). In the latter case, use the trichotomy in Theorem 3.2.25 to show that either one of the first two items holds or else the equivalence relation $=_J$ is not even Cohen-trim, and so certainly not σ-trim. □

Question 3.3.17. Is the conclusion of Corollary 3.3.16 true in ZFC?

As a final remark in this subsection, the E_1-σ-trim name exhibited in Theorem 3.3.15 lives on a poset that is very far from definable–it is a remainder forcing of some sort. Of course, this would make the theorem difficult to use as a vehicle for ergodicity results. The following theorem shows that even for an equivalence relation strictly more complicated than E_1, there are no σ-trim names on simple posets:

Theorem 3.3.18. *Let J be the branch ideal on $2^{<\omega}$. Let I be a c.c.c. $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal of analytic subsets of some Polish space X and let P denote the collection of all Borel I-positive sets ordered by inclusion. The equivalence relation $=_J$ is P-σ-trim.*

The posets covered by the theorem as the Cohen poset (associated with I =meager ideal), the random poset (associated with I =the Lebesgue null ideal), the Maharam algebras, the eventually different real forcing [25, Proposition 3.8.12] and some other posets. The posets do not add dominating reals by [25, Proposition 3.8.15], and so the theorem does not resolve the case of P =Hechler forcing.

The proof uses a lemma of independent pure forcing interest.

Lemma 3.3.19. *Suppose that $p \in P$ is a condition and $p \Vdash \langle \dot{y}_n : n \in \omega \rangle$ is a sequence of points in 2^ω. Then one of the following holds:*

1. *there is $q \leq p$ and $n \in \omega$ such that $q \Vdash \dot{y}_n$ is in the ground model;*
2. *there is $q \leq p$ and a function $g \in \omega^\omega$ such that $q \Vdash \langle \dot{y}_n \restriction g(n) : n \in \omega \rangle$ is not in the ground model.*

Proof. Let I be a σ-ideal on a Polish space X such that $P = P_I$. Strengthening the condition $p \in P$ if necessary, I may assume that there are Borel functions $f_n : p \to 2^\omega$ such that $p \Vdash \forall n\ \dot{y}_n = \dot{f}_n(\dot{x}_{gen})$. Let $A \subset \omega^\omega \times 2^{\omega \times \omega}$ be the set $A = \{\langle z, w \rangle : \forall n \forall m > z(n)\ w(n,m) = 0$ and the set $\{x \in p : \forall n \forall m \leq z(n)\ f_n(x)(m) = w(n,m)\} \subset X$ is I-positive$\}$. As the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the set A is analytic. As the poset P_I is c.c.c., the set A has countable vertical sections. By the first reflection theorem [14, Theorem 35.10], it is covered by a Borel set with countable vertical sections, which in turn is covered by graphs of countably many Borel functions $\{h_k : k \in \omega\}$ from ω^ω to $2^{\omega \times \omega}$ by the Lusin-Novikov theorem [14, Theorem 18.10]. The usual Miller forcing type fusion arguments then yield a superperfect tree $T \subset \omega^{<\omega}$ such that all functions $h_k \restriction [T]$ are continuous, and in fact whenever $k \in \omega$, $t \in T$ is a splitnode of length $> k$, $j \in \omega$ is such that $t^\frown j \in T$, and $m \in j$, then the value of $h_k(z)(|t|, m)$ is the same for all $z \in [T \restriction t^\frown j]$. Further thinning out the infinite branchings of the tree T if necessary, I can also assume that whenever $k \in \omega$, $t \in T$ is a splitnode of length $> k$ and $m \in \omega$, there is a bit $y_{k,t}(m) \in 2$ such that for all but finitely many j such that $t^\frown j \in T$, $h_k(z)(|t|, m) = y_{k,t}(m)$ for all $z \in [T \restriction t^\frown j]$. Thus, $y_{k,t} \in 2^\omega$ for every such k, t.

For every $k \in \omega$ and a splitnode $t \in T$ longer than k, consider the set $q_{k,t} = \{x \in p : \exists^\infty j\ \exists z \in [T \restriction t^\frown j]\ \forall n\ \forall m \leq n\ f_n(x)(m) = h_k(z)(n,m)\}$. This is a Borel subset of p, since the existential quantification over z can be replaced with universal by the choice of the tree T. The treatment splits into two cases.

Case 1. Suppose first that there is k, t such that the set $q = q_{k,t}$ is I-positive. Then write $n = |t|$, and observe that for every point $x \in q_{k,t}$, $f_n(x) = y_{k,t}$. Thus, $q \Vdash \dot{y}_{|t|}$ is in the ground model, equal to $\check{y}_{k,t}$.

Case 2. Suppose that all sets $q_{k,t}$ are in I. In such a case, consider the set $C = \{\langle x, z\rangle \colon x \in p \setminus \bigcup_{k,t} q_{k,t},\ z \in [T]$, and there is $k \in \omega$ such that $\forall n\ \forall m \leq z(n)\ f_n(x)(m) = h_k(z)(m)\}$.

Claim 3.3.20. *C is a Borel set with σ-bounded vertical sections.*

Proof. If $x \in p$ was such that C_x is not σ-bounded, there would have to be $k \in \omega$ such that the set $\{z \in [T] \colon \forall n\ \forall m \leq z(n)\ f_n(x)(m) = h_k(z)(m)\}$ is not σ-bounded, and so there would have to be $t \in \omega^{<\omega}$ longer than k such that for infinitely many $j \in \omega$ there is $y \in [T \restriction t^\frown j]$ such that $\forall m \leq z(n)\ f_n(x)(m) = h_k(z)(m)$. This would, however, put x into the set $q_{k,t}$, an impossibility. $\square$

Now, since I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the poset P does not add dominating reals by [25, Proposition 3.8.15], and so it has the Fubini property with the σ-bounded ideal on ω^ω [25, Definition 3.2.1]. As the Borel set $C \subset p \times [T]$ has σ-bounded vertical sections, this means that the complement $p \times [T] \setminus C$ must have an I-positive horizontal section corresponding to some $z \in \omega^\omega$. Let $q = q' \setminus \bigcup_{k,t} q_{k,t}$, and review the definition of the set A to conclude that for every $w \in 2^{\omega\times\omega}$, it must be the case that the set $\{x \in q \colon \forall n \forall m \leq z(n)\ f_n(x)(m) = w(n,m)\} \subset X$ is in I. This is to say that $q \Vdash \langle \dot{y}_n \restriction g(n) \colon n \in \omega\rangle$ is not in the ground model. The lemma follows. $\square$

Proof of Theorem 3.3.18. Let P be a poset as in the assumptions of the theorem. Assume that $V[G_n]$ for $n \in \omega$ are generic extensions of the ground model via the poset P, containing the respective pairwise $=_J$ equivalent points $x_n \in X$. Assume that $\bigcap_n V[G_n] = V$ and work to find a ground model point $x \in X$ which is $=_J$-related to all x_n's.

Let $T = \{t \in 2^{<\omega} \colon x_0 \restriction [t]$ is not $=_J$-equivalent to any point in the ground model$\}$. Clearly, T is a tree. Its definition depends only on the $=_J$-class of x_0, therefore T belongs to all $V[G_n]$ for $n \in \omega$ and so to the ground model. Since the witnesses for the failure of membership of a binary sequence in the tree T can be combined, T has no terminal nodes. Assume for contradiction that $0 \in T$. Since T is a nonempty ground model tree without endnodes, it has a branch in the ground model. To simplify the notation, assume that this branch has only 0 entries along it. For every $m \in \omega$ write $t_m = (0^m)^\frown 1$.

Let $y_m = x_0 \restriction [t_m]$ for $m \in \omega$. Note that for every $n \in \omega$, for all but finitely many $m \in \omega$, $y_m = x_n \restriction [t_m]$ as $x_0 =_J x_n$. Thus, the function $z \in 2^\omega$ defined by $z(m) = 0$ if $y_m \in V$ is in all models $V[G_n]$ for $n \in \omega$, and therefore in V. There are two cases, both of which end in contradiction:

Case 1. For all but finitely many $m \in \omega$, $z(m) = 0$. Then, for some $m_0 \in \omega$, the sequence $\langle y_m : m > m_0\rangle$ consists of ground model points only. Since for every $n \in \omega$ the model $V[G_n]$ contains a finite modification of this sequence, the sequence must belong to the ground model. Then, working in the ground model, let $x\colon [0^{m_0}] \to 2$ be any function extending all y_m for $m > m_0$. The definitions show that the values of x and $x_0 \restriction [0^{m_0}]$ can differ only on the branch y and therefore $x =_J x_0 \restriction [0^{m_0}]$. This contradicts the definition of the tree T and the assumption that $0^{m_0} \in T$.

Case 2. The set $a = \{m \in \omega\colon z(m) = 1\}$ is infinite. As $z \in V$, $a \in V$ as well. Apply Lemma 3.3.19 to find a function $g : a \to \omega$ such that the sequence $y = \langle y_m \restriction 2^{\leq g(m)} \cap [t_m]\colon m \in a\rangle$ does not belong to the ground model. For every number $n \in \omega$, the model $V[G_n]$ contains a finite modification of y, namely the sequence $\langle x_n \restriction 2^{\leq g(m)} \cap [t_m]\colon m \in a\rangle$. Therefore, $y \in \bigcap_n V[G_n]$. This contradicts the assumption that $\bigcap_n V[G_n] = V$. □

Corollary 3.3.21. *Let J be an analytic tall ideal on ω and let K be the branch ideal. Every Borel homomorphism from $=_J$ to $=_K$ stabilizes on a comeager set.*

Proof. Use Theorems 3.3.3 and 2.3.1. □

3.4 Variations

In this section, I use other certain trimness variations to prove nonreducibility results between equivalence relations of the form $=_J$ where J is an analytic ideal on ω. Typically, I isolate a combinatorial property of an ideal J which is preserved under taking a larger ideal, and show that for a great number of equivalence relations E the combinatorial property implies that every Borel homomorphism from $=_J$ to E stabilizes on a comeager and on a conull set.

3.4.1 σ-ideal type ideals

The first class of ideals, beside being very natural, serves the purpose of showing that the class of trim equivalence relations is much wider than the class isolated by Kanovei and Reeken in their study of turbulence.

Definition 3.4.1. An ideal J on ω is *of the σ-ideal type* if there is a collection $\{A_n\colon n \in \omega\}$ such that each C_n is a countable subset of $\mathcal{P}(\omega)$ which cannot be covered by finitely many centered sets, and for every J-positive set $b \subset \omega$ there is $n \in \omega$ such that b has infinite intersection with every element of A_n.

Note that every σ-ideal type ideal J on ω is also of convergence type. If a collection $\{A_n\colon n \in \omega\}$ witnesses the σ-ideal type of J, then $\bigcup_n A_n$ witnesses the convergence type of J: whenever $b \subset \omega$ is a J-positive set, then there must be $n \in \omega$ such that b has infinite intersection with every element of A_n. Then, there must be $a \in A_n$ such that both sets $b \cap a$ and $b \setminus a$ are infinite–otherwise A_n would be covered by the filter of sets which contain a cofinite piece of b.

Example 3.4.2. Suppose that I is an analytic σ-ideal of compact sets on a compact metrizable space X containing all singletons, $D \subset X$ is a countable dense set, and let $J = \{a \subset D\colon$ the closure of a belongs to $I\}$. Then the ideal J is analytic and of the σ-ideal type.

Proof. The ideal I is in fact G_δ subset of the hyperspace $K(X)$ by ??? This means that there are collections B_n for $n \in \omega$ consisting of open subsets of X such that for every compact set $K \subset X$, $K \in I$ if and only if there are sets $O_n \in B_n$ for $n \in \omega$ such that $K \subset \bigcap_n O_n$. For every number $n \in \omega$ let $A_n = \{P \cap D \colon P \subset X \text{ is a basic open set and } O \cup P = X \text{ for some } O \in B_n\}$ and argue that the sets A_n work as desired.

First of all, suppose that $\{F_i \colon i \in j\}$ are centered subsets of A_n. For each $i \in j$, the collection $\{\bar{a} \colon a \in F_i\}$ is centered and consists of closed subsets of the compact space X, therefore it has a nonempty intersection containing some point $x_i \in X$. The set $\{x_i \colon i \in j\}$ belongs to the ideal I. Thus, there must be a set $O \in C_n$ such that $\{x_i \colon i \in j\} \subset O$ and a basic open set P such that $O \cup P = X$ and $\{x_i \colon i \in j\} \cap \bar{P} = 0$. This means that the set $P \cap D \in C_n$ does not belong to any of the centered sets F_i for any $i \in j$.

Second, suppose that $a_n \in A_n$ are sets for each $n \in \omega$ and $b \subset \omega$, and argue that b has to have an infinite intersection with one of the sets a_n. If not, pick basic open sets $O_n, P_n \subset X$ such that $O_n \in B_n$, $a_n = P_n \cap D$ and $O_n \cup P_n = X$ and note that all of the accumulation points of b must lie in the set $\bigcap_n O_n$. By the definitions, the set of accumulation points of b belongs to I, and since I is a σ-ideal of compact sets containing all singletons, it follows that the closure of b is in I and so $b \in J$; a contradiction. $\square$

Definition 3.4.3. $\mathfrak{L}$ is the smallest class of analytic equivalence relations containing the identity and closed under the operations of Borel reduction, countable union, Friedman–Stanley jump, and infinite product modulo any F_σ-ideal.

Theorem 3.4.4. *Suppose that J is a σ-ideal type analytic ideal on ω, containing all singletons and not ω. Suppose that $E \in \mathfrak{L}$ is an equivalence relation on a Polish space Y and $h \colon 2^\omega \to Y$ is a homomorphism from $=_J$ to E. The homomorphism stabilizes on a comeager set and also on a conull subset of 2^ω.*

Proof. Let $x_0 \in 2^\omega$ be an element generic over V, F an equivalence relation in the class $\mathfrak{L}$ on a Polish space Y and $y \in Y$ a point which has no F-equivalent in V. It will be enough to produce, in a further generic extension, an element $x_1 \in 2^\omega$ generic over V such that $x_0 =_J x_1$ and y has no F-equivalent in the model $V[x_1]$.

Fix a ground model guiding system $C_n \colon n \in \omega$ for the anti-Kanovei ideal J and work in the model $V[x_0]$. Let P be a partial order of all tuples $p = \langle s_p, t_p, A_p \rangle$ where $s_p \in 2^{<\omega}$, t_p is a finite function such that for every number $n \in \omega$ $t_p(n) \in C_n$ holds. The finite set A_p consists of pairs $\langle F, y \rangle$ where F is a ground model coded analytic equivalence relation on some Polish space Y, $y \in Y$, and y has no F-equivalent in the model $V[x_0 \restriction a_p]$. The ordering is by coordinatewise extension.

Claim 3.4.5. *For every condition $p \in P$ and every open dense set D of the Cohen forcing in the ground model there is a condition $q \leq p$ such that $t_q \in D$.*

It follows that if $G \subset P$ is a filter generic over $V[x_0]$ then the point $x_1 = \bigcup_{p \in G} t_p \in 2^\omega$ is a point generic over V.

Claim 3.4.6. *For every condition $p \in P$ and every number $n \in \omega$ there is a condition $q \leq p$ such that $n \subset \mathrm{dom}(t_q)$.*

It follows that the generic point $x_1 \in 2^\omega$ is $=_J$-related to x_1, since the set $\{m \in \omega \colon x_0(m) \neq x_1(m)\}$ is a subset of $\omega \setminus \bigcup\{\mathrm{rng}(t_p) \colon p \in G\}$. The latter set belongs to the ideal J by the claim. Note that the collection $\{C_n \colon n \in \omega\}$ remains guiding for the anti-Kanovei ideal J in the extension $V[x_0][G]$ by the Shoenfield absoluteness.

The following claim contains the central ingredient of the proof.

Claim 3.4.7. *Whenever F is a ground model coded equivalence relation on a Polish space Y and $y \in Y$ is a point in $V[x_0]$ which has no F-equivalent in the ground model, then the condition $\langle 0, 0, \{\langle F, y\rangle\}\rangle$ forces that y has no equivalent in the model $V[x_1]$.*

??? □

3.4.2 Dimension type ideals

Among the σ-ideal type ideals, there are many interesting distinctions. Some of these distinctions result in ergodicity theorems. In this subsection, I will show that the ideals derived from σ-ideals connected with category or dimension are much more complicated than the ideals associated with capacities.

Definition 3.4.8. An ideal J on a countable set ω is of *dimension type* if there is a collection $\{A_n \colon n \in \omega\}$ such that each A_n is a countable set of pairwise disjoint subsets of ω and for every J-positive set $b \subset \omega$ there is $n \in \omega$ such that for every $a \in A_n$, the intersection $a \cap b$ is infinite.

Observe that every dimension type ideal is also of the σ-ideal type.

Example 3.4.9. Let X be a compact metrizable space without isolated points, $D \subset X$ a dense set, and J the ideal of nowhere dense subsets of D. Then J is of dimension type. To see this, let $\{O_n \colon n \in \omega\}$ enumerate a basis for the space Y and let A_n be an collection of intersections with D of infinitely many pairwise disjoint open subsets of O_n.

Example 3.4.10. Let X be a compact metrizable space without isolated points, $D \subset X$ a dense set, and J the ideal of subsets of D whose closure is zero-dimensional. To se why this is a dimension type ideal, let $\langle O_n \colon n \in \omega\rangle$ enumerate all basic open balls in the space Y, each with center y_n and radius ε_n with infinitely many repetitions, and let A_n consist of intersections with D of pairwise disjoint open annuli around the central point y_n whose radii converge to the number ε_n.

Theorem 3.4.11. *Let J be a dimension type analytic ideal. Let Y be a compact metrizable space, $D \subset Y$ a countable dense set, ν a capacity on Y, and K an ideal on D consisting of sets whose closure has zero ν-mass. Then the names for generic and random points of 2^ω are both $=_J$-σ-$\{=_K\}$-trim.*

Proof. The argument depends on a small observation. Given $\varepsilon > 0$, consider the function f_ε which assigns to every $z \in 2^D$ the set $\{z \restriction U : U$ is a basic open set and $\nu(Y \setminus U) < \varepsilon\}$. Observe that if $z_0, z_1 \in 2^D$ are $=_K$-related, then $f_\varepsilon(z_0) \cap f_\varepsilon(z_1) \neq 0$: the closure C of the set $\{d \in D : z_0(d) \neq z_1(d)\}$ has ν-mass zero, and since the capacity ν is continuous in decreasing intersections of compact sets, there must be a basic open set $U \subset Y_1$ such that $\nu(Y_1 \setminus U) < \varepsilon$ and $U \cap C = 0$, and therefore $z_0 \restriction U = z_1 \restriction U$. This means that each function f_ε is a homomorphism from $=_K$ to the graph of nonempty intersection on countable sets.

Now, suppose that $x_0 \in 2^\omega$ is a point generic (or random) over V and $z \in 2^D$ is a point in $V[x_0]$. I must show that either z has an $=_K$-equivalent in V, or else in some generic extension there is a point $x_1 \in 2^\omega$ generic over V and $=_J$-related to z such that $V[x_1]$ contains no $=_K$-equivalent of z. The argument neatly splits into several cases.

Case 1. There is a real number $\varepsilon > 0$ such that no element of $f_\varepsilon(z)$ belongs to V. In this case, reach to the ground model for a collection $\{A_n : n \in \omega\}$ witnessing the fact that J is a dimension-type ideal, and find sets $a_n \in A_n$ for $n \in \omega$ such that for no finite set $b \subset \omega$ and no element of $f_{\varepsilon/2}(z)$ belongs to the model $V[x_0 \restriction \bigcup_{n \in b} a_n]$. Once this is done, let $A = \{a_n : n \in \omega\}$, force with the poset $P(x_0, A)$ to add a point $x_1 \in 2^\omega$. By Proposition 3.3.10, this point is generic over V, $=_J$-related to x_0, and the model $V[x_1]$ contains none of the elements of $f_{\varepsilon/2}(z)$. This means that the model $V[x_1]$ cannot contain any $=_K$-equivalent of z.

To build the sets a_n, by induction on $n \in \omega$ choose sets $a_n \in A_n$ and find numbers $\varepsilon_n > \varepsilon/2$ so that no element of $f_{\varepsilon_n}(z)$ belongs to the model $V[x_0 \restriction \bigcup_{m \in n} a_m]$. The basis step consists of setting $\varepsilon_0 = \varepsilon$. Now suppose that the sets a_m for $m \in n$ and the number ε_n have been found. Let ε_{n+1} be any number strictly between ε_n and $\varepsilon/2$, and suppose for contradiction that the set $a_n \in A_n$ cannot be selected to fit the induction hypothesis. This means that for every $a \in A_n$ there is a basic open set $U_a \subset Y$ and a point $z_n \in 2^{D \cap U_a}$ in the model $V[x_0 \restriction (a \cup \bigcup_{m \in n} a_m)]$ such that $z \restriction U = z_n$ and $\nu(Y_1 \setminus U_a) \leq \varepsilon_{n+1}$.

Consider the open set $U \subset Y$ of all elements of Y which belong to some sets U_a, U_b for $a \neq b \in A_n$.

Claim 3.4.12. $\nu(Y \setminus U) \geq \varepsilon_n$.

Proof. Write A_n as an increasing union of finite sets B_k for $k \in \omega$. Let $U_k \subset Y$ be an open set of all points which belong to some sets U_a, U_b for some $a \neq b \in B_k$. Clearly, U is an increasing union of the sets U_k for $k \in \omega$. Since the capacity ν is continuous in decreasing intersections of compact sets, it is eanough to show that for every $k \in \omega$, $\nu(Y \setminus U_k) \geq \varepsilon_n$.

For this, observe that if $a \neq b \in B_k$ are distinct sets, then $z \restriction (U_a \cap U_b)$ belongs to both models $V[x_0 \restriction a \cup \bigcup_{m \in n} a_n]$ and $V[x_0 \restriction b \cup \bigcup_{m \in n} a_n]$. By Claim 6.1.14 and the assumption that $a \cap b = 0$, $z \restriction (U_a \cap U_b)$ belongs to the model $V[x_0 \restriction b \cup \bigcup_{m \in n} a_n]$. It follows that $z \restriction U_k \in V[x_0 \restriction b \cup \bigcup_{m \in n} a_n]$ and by the induction assumption, $\nu(Y \setminus U_k) \geq \varepsilon_n$ must hold. ☐

Now, for each $a \in A_n$ let $W_a = U_a \setminus \bigcup_{b \neq a} U_a$. The sets $W_a \subset Y$ for $a \in A_n$ are pairwise disjoint. Since the capacity ν is continuous in increasing unions, the claim implies that there must be a set $a \in A_n$ such that $\nu((Y \setminus U) \setminus W_a) > \varepsilon_{n+1}$. Now, $((Y \setminus U) \setminus W_a) \cap U_a = 0$ by the definition of the sets U and W_a. This contradicts the assumption that $\nu(Y \setminus U_a) \leq \varepsilon_{n+1}$. This contradiction shows that the induction step can be completed, and concludes the proof in this case.

Case 2. If Case 1 fails, consider the map g sending every basic open set $U \subset Y_1$ to the $=_K$-class of $z \restriction U$. By the c.c.c. of the Cohen forcing, there must be a countable set A in the ground model containing $V \cap \text{rng}(g)$.

Case 2a. If $\text{rng}(g) \cap V$ belongs to the ground model, then I will work to show that z has an $=_K$-equivalent in the ground model. Use the Mostowski absoluteness to find a point $z' \in 2^D \cap V$ such that for every basic open set $U \subset Y$, if $g(U) \in V$ then $g(U)$ is the $=_K$-class of $z' \restriction U$. Still work in V and note that any two such points z' and z'' must be $=_K$-related: if they were not, then the closure C of the set $\{d \in D\colon z'(d) \neq z''(d)\}$ has positive ν-mass, bigger than some positive real $\varepsilon > 0$. Use the case assumption to find a basic open set U such that $z \restriction U \in V$ and $\nu(Y \setminus U) < \varepsilon$. It follows immediately that $z' \restriction U$ and $z'' \restriction U$ are not $=_K$-related, while they should both be related to $z \restriction U$. Now, by the Shoenfield absoluteness between V and $V[x_0]$ argue that the point $z' \in V$ must be $=_K$-related to z and conclude the proof in this case.

Case 2b. If $\text{rng}(g) \cap V$ does not belong to the ground model, then use the fact that J is also a convergence type ideal to find a point $x_1 \in 2^\omega$ generic over V which is $=_J$-related to x_0 and such that $\text{rng}(g) \cap V$ does not belong to $V[x_1]$. This means that the point z does not have any $=_K$-equivalent in the model $V[x_1]$, since such an equivalent could be used to reconstruct the set $\text{rng}(g) \cap V$. This completes the proof in this last case. $\square$

Corollary 3.4.13. *If h is a Borel homomorphism from $=_J$ to $=_K$, where J, K are ideals on $\mathbb{Q} \cap [0,1]$ of nowhere dense sets and sets with Lebesgue null closure respectively, stabilizes on a comeager sets and also on a set of full measure.*

3.4.3 Measure type ideals

Within the ideals generated by capacities on compact spaces, it is possible to make further ergodicity type distinctions.

Definition 3.4.14. An ideal J on ω is of *measure type* if there is a collection $\{A_n \colon n \in \omega\}$ such that

1. each A_n is a countable set of subsets of ω such that for every $j \in \omega$ there is $k \in \omega$ such that if $B_i \colon i \in j$ are subsets of A_n, each of size k, then there are infinitely many elements of A_n with empty intersection with $\bigcup_i \bigcap B_i$;

2. whenever $b \subset \omega$ is a J-positive set, then there is $n \in \omega$ such that b has infinite intersection with each element of A_n.

Example 3.4.15. If Y is a compact metrizable space, μ is a Borel probability measure on Y, and $D \subset Y$ is a countable dense set, then the ideal J of subsets of D whose closure has zero μ-mass is of measure type.

Theorem 3.4.16. *Suppose that J is an analytic ideal of measure type. Suppose that Y is a compact metrizable space, ν a Ramsey capacity on it, $D \subset Y$ a countable dense set, and K is the ideal on D consisting of sets whose closure has zero ν-mass. Then the names for both the generic and the random elements of 2^ω are $=_J$-σ-$\{=_K\}$-trim.*

Proof. For every real number $\varepsilon > 0$ define f_ε to be the map on the space Y, assigning to each point $y \in Y$ the set $\{y \restriction U : U \subset Y$ is a basic open set and $\nu(Y \setminus U) < \varepsilon\}$. As in the proof of ???, it is just enough to show the following: whenever $x_0 \in 2^\omega$ is a generic (or random) element over V and $y \in Y$ is a point in $V[x_0]$ and $\varepsilon > 0$ is a real number such that $f_\varepsilon(y) \cap V \neq 0$, there is a collection of sets $\{a_n : n \in \omega\}$, each of them a subset of ω in the ground model, such that every J-positive set has infinite intersection with a_n for some n, and for every finite set $b \subset \omega$, $f_{\varepsilon/2}(y) \cap V[x_0 \restriction \bigcup_{n\in b} a_n] = 0$.

To construct the sets $a_n \subset \omega$ for each $n \in \omega$, step back to V and pick a collection $\{A_n : n \in \omega\}$ witnessing the fact that the ideal J is of measure type. By induction on $n \in \omega$ build sets $a_n \in A_n$ and numbers $\varepsilon_n > \varepsilon/2$ so that no element of $f_{\varepsilon_n}(z)$ belongs to the model $V[x_0 \restriction \bigcup_{m\in n} a_m]$. The basis step consists of setting $\varepsilon_0 = \varepsilon$. Now suppose that the sets a_m for $m \in n$ and the number ε_n have been found. Let ε_{n+1} be any number strictly between ε_n and $\varepsilon/2$, and suppose for contradiction that the set $a_n \in A_n$ cannot be selected to fit the induction hypothesis. Thus, for each set $a \in A_n$ there is an open set $U_a \subset Y$ such that $\nu(Y \setminus U) \leq \varepsilon_{n+1}$ and $y \restriction U_a \in V[x_0 \restriction a \cup \bigcup_{m\in n} a_m]$. I will show that this is impossible.

By induction on $k < l$ build sets $B_k \subset A_n$ such that

- each set B_k has size ???
- $\nu(Y \setminus \bigcap_{a\in B_k} U_a) < \delta$;
- the intersections $\bigcap B_k \subset \omega$ are pairwise disjoint.

To construct the set B_k on the assumption that $B_{k'}$ for $k' < k$ have been already constructed, just use the fact that the ideal J is of measure type; the set $C = \{a \in A_n : a \cap \bigcup_{k'<k} \bigcap B_{k'} = 0\}$ is infinite. By the Ramsey assumption on the capacity ν it has an infinite subset $C' \subset C$ such that $\nu(Y \setminus \bigcap_{a\in C'} U_a) < \delta$, and then any subset of C' of size ??? will work as B_k.

In the end, use the Ramsey assumption on the capacity ν to find indices $k_0, k_1 < l$ such that $\nu(Y \setminus \bigcap_{a\in B_{k_0}\cup B_{k_1}} U_a) < \varepsilon_n$ and use Claim 6.1.14 to observe that the point $y \restriction Y \setminus \bigcap_{a\in B_{k_0}\cup B_{k_1}} U_a$ belongs to the model $V[x_0 \restriction \bigcup_{m\in n} a_m \cup (\bigcap B_{k_0} \cap \bigcap B_{k_1})] = V[x_0 \restriction \bigcup_{m\in n} a_m]$, which contradicts the inductive assumption. $\square$

3.4.4 Cantor-Bendixson type ideals

This subsection deals with the equivalence relations F_α for a countable ordinal $\alpha > 0$ introduced in Theorem 3.2.30. For notational reasons let F_0 to be the identity. It is clear that these equivalence relations grow with the ordinal α, which yields nonstabilizing Borel homomorphism from F_{α_0} to F_{α_1} whenever $\alpha_0 < \alpha_1$ and both relations are derived from the same compact metrizable space. It turns out that in the opposite direction, if $\alpha_0 > \alpha_1$, every Borel homomorphism must stabilize on a large set, and this is the contents of the following theorem.

Theorem 3.4.17. *Let Y_0, Y_1 be compact metrizable spaces without isolated points. Let $D_0 \subset Y_0, D_1 \subset Y_1$ be dense countable sets. Let $\alpha_0 > \alpha_1$ be nonzero countable ordinals. The names for a generic and random element of 2^{D_0} are F_{α_1}-σ-$\{F_{\alpha_0}\}$-trim.*

Proof. I will deal with the case of a generic element, the case of a random element is similar. To streamline the notation, let Q be the poset of finite partial functions from D_0 to 2 ordered by inclusion, and write F_0 for the identity equivalence relation. Choose countable bases for the spaces Y_0, Y_1 closed under finite unions and intersections.

Let $x_0 \in 2^{D_0}$ be a generic point and $z \in V[x_0]$ be an element of 2^{D_1} which has no F_{α_1}-equivalent in the ground model. It will be enough to find, in some further forcing extension, a point $x_1 \in 2^{D_0}$ generic over V such that the model $V[x_1]$ does not contain any F_{α_1}-equivalent of the point z in the model $V[x_1]$.

Consider the poset P of all tuples $p = \langle t_p, a_p\rangle$ where t_p is a finite partial function from D_0 to 2 and a_p is a finite set of triples of the form $\langle y, O, \beta\rangle$ such that $O \subset Y_0$ is a basic open set, $y \in O$, and $\beta > 0$ is an ordinal; I also require that the point coordinates in the triples in a_p are pairwise distinct. The ordering is defined by $q \leq p$ if $t_p \subset t_q$ and $\{d \in \mathrm{dom}(t_q) \setminus t_p) \colon t_q(d) \neq x_0(d)\}$ is a subset of the union of second coordinates of triples in a_p, and for every triple $\langle y, O, \beta\rangle \in a_p$ there is a triple $\langle y, O', \beta\rangle \in a_q$ such that $O' \subset O$, and for every triple $\langle y', O', \beta'\rangle \in a_q$ there is a triple $\langle y, O, \beta\rangle$ such that $O' \subset O$ and either $\beta = \beta'$ and $y = y'$, or else $\beta' < \beta$.

It is not difficult to see that the set of conditions $p \in P$ such that the second coordinates of triples in a_p have pairwise disjoint closures of diameter smaller than any fixed $\varepsilon > 0$ is dense in the poset P. I consider only the conditions in P below a certain initial condition. The collection of basic closed subsets $C \subset Y_0$ such that z has an F_{α_0}-equivalent in the model $V[x_0 \restriction C]$ is a filter by Claim 6.1.14. By a compactness argument, its intersection is not empty, containing some point $y \in Y_0$. The initial condition is defined as $\langle 0, \{\langle y, Y_0, \alpha_1 + 1\rangle\}\rangle$.

Suppose that $G \subset P$ is a filter generic over $V[x_0]$, and let $x_1 = \bigcup_{p \in G} t_p$. I claim that the point $x_1 \in 2^{D_0}$ works as required. The genericity of the point $x_1 \in 2^{D_0}$ follows immediately from the following claim.

Claim 3.4.18. *If $p \in P$ is a condition and $C \subset Q$ is a dense open set in V, then there is a condition $p' \leq p$ in P such that $t_{p'} \in C$.*

Proof. Note that the point x_0 rew $t_p \in 2^{D_0}$ is a point generic over V. Thus, there is a condition $q \in C$ such that $t_p \subset q \subset x_0$ rew t_p. Set p' to be equal to p except for $t_{p'} = q$. It is immediate that the condition $p' \leq p$ works as required. □

In order to show that $x_0 \; F_{\alpha_0} \; x_1$, consider the set $C \subset Y_0$ consisting of all points of Y_0 mentioned in some triple in some condition in the filter G.

Claim 3.4.19. *The set $C \subset Y_0$ is closed. Whenever $p \in P$ is a condition and $\langle y, O, \beta\rangle \in a_p$ is a triple, then $p \Vdash$ the Cantor–Bendixson rank of $y \in C$ is $\leq -1 + \beta$.*

Proof. For the first sentence, suppose that $\langle y_n : n \in \omega\rangle$ is a converging sequence in the set C. By recursion on $m \in \omega$, pick a descending sequence $\langle p_m : m \in \omega\rangle$ of conditions in the filter G and triples $\langle v_m, O_m, \beta_m\rangle \in a_{p_m}$ such that for every number $m \in \omega$, $O_{m+1} \subset O_m$, $O_m \subset Y_0$ has diameter smaller than 2^{-m} in some fixed metric for the space Y_0, and there are infinitely many $n \in \omega$ such that $y_n \in \bar{O}_m$. The definition of the partial ordering P shows that the ordinals β_m must form a nonincreasing sequence, and by a wellfoundedness argument, they must stabilize at some $k \in \omega$. The definition of the ordering applied again shows that the points $v_m \in Y_0$ also stabilize at that k. The stable value must be the limit of the sequence, and so the limit belongs to the set C as desired.

The second sentence is proved by a straightforward transfinite induction argument on β. □

By the definition of the poset P, the set $\{d \in D_0 : x_0(d) \neq x_1(d)\}$ accumulates in C, and in view of the choice of the initial condition in P, its closure has rank $\leq \alpha_0 + 1 \leq \alpha_1$. This shows that $x_0 \; F_{\alpha_0} \; x_1$ holds.

Finally, I need to show that $V[x_1]$ contains no F_{α_1}-equivalent of z. For this, work in the model $V[x_0]$ and for a nonempty open set $U \subset Y_1$, an ordinal β and a point $y \in Y_0$, say that y is U, β-*good* if for every basic open set $O \subset Y_0$, if $y \in O$ then the point $z \restriction U$ has no F_β-equivalent in the model $V[x_0 \restriction (Y_0 \setminus O)]$. I will need two abstract properties of this notion:

1. If $z \restriction U$ has no F_β-equivalent in the model $V[x_0 \restriction Y_0 \setminus O]$ then there is a point $y \in \bar{O}$ such that y is U, β-good. This follows from the fact that the set $\mathfrak{F}$ of those closed subsets $C \subset Y_0$ for which $z \restriction U$ has an F_β-equivalent in $V[x_0 \restriction C]$ is a filter by Claim 6.1.14, the closure $\bar{O}$ is $\mathfrak{F}$-positive, and by a compactness argument the set $\bar{O} \cap \bigcap \mathfrak{F}$ is nonempty. Any element y of this nonempty intersection will have the requested properties.

2. If y_0 is U, β-good, then there is a point $y_1 \in \bar{U}$ such that for every neighborhood $W \subset Y_1$ of y_1, y_0 is $W \cap U, \beta$-good. If this failed, by a compactness argument the closure $\bar{U}$ would be covered by a finite set A consisting of open sets W such that y is not $U \cap W, \beta$-good. For each $W \in A$ there is an open neighborhood $O_W \subset Y_0$ of y_0 such that $z \restriction U \cap W$ has an F_β-equivalent in the model $V[x_0 \restriction (Y_0 \setminus O_W)$. Let $O = \bigcap_{W \in A} O_W$; this is an open neighborhood of the point y_0. The model $V[x_0 \restriction (Y_0 \setminus O)]$ contains

an F_β-equivalents for all $z \restriction U \cap W$ for $W \in A$, and therefore also for $z \restriction U$. This contradicts the assumption that y_0 was U, β-good.

Claim 3.4.20. *If $p \in P$ is a condition, $\langle y, O, \beta\rangle \in a_p$ is a triple, $U \subset Y_1$ is a basic open set, and y is U, β-good, then $p \Vdash \check{z} \restriction U$ has no $F_{-1+\beta}$-equivalent in the model $V[x_1]$.*

Proof. Let $O' \subset O$ be a basic open set containing y and such that $\bar{O}' \subset O$. Write M for the model $V[x_0 \restriction (Y_0 \setminus O')]$ and R for the poset of all finite functions $t\colon D_0 \to 2$ such that $t_p \subset t$ and $t_p \restriction (D_0 \setminus O') \subset (x_0 \text{ rew } t_p)$ ordered by reverse extension. Note that the poset R belongs to the model M; it adds a point $\dot{x}_{gen} \in 2^{D_0}$ generic over V, which agrees with x_0 rew t_p on the set $D_0 \setminus O'$.

The proof proceeds by induction on the ordinal β. For the base case $\beta = 1$, suppose for contradiction that $q \leq p$ is a condition, $\tau \in V$ is a Q-name for an element of $2^{D_1 \cap U}$, and $q \Vdash \tau = \check{z} \restriction U$. By the definition of the partial ordering P, the condition q contains a triple $\langle y, O', \beta\rangle$ such that $O'' \subset O'$. In the model $V[x_0 \restriction (Y_0 \setminus O'')]$, consider the set $z' = \{\langle d, b\rangle \colon d \in D_1 \cap U, b \in 2$, and there is a condition $t \in Q$ such that $t_q \subset t$, $t_q \restriction (Y_0 \setminus O') \subset (x_0 \restriction (Y_0 \setminus O''))$ rew t_q and $t \Vdash \tau(d) = b\}$. Since the point $y \in Y_0$ is U_1, β-good, $z \restriction U_1 \notin V[x_0 \restriction (Y_0 \setminus O'')]$ holds, and in particular $z \neq z'$. This means that there must be a condition $t \in Q$ and a point $d \in D_1 \cap U_1$ such that $t_q \subset t$, $t_q \restriction (Y_0 \setminus O') \subset (x_0 \restriction (Y_0 \setminus O''))$ rew t_q, and t decides the value of $\tau(d)$ to be the opposite of $z(d)$. The condition $r \in P$ which is the same as q except that t_q is replaced with t, then satisfies $r \leq q$ and $r \Vdash (\tau/x_1)(d) \neq z(d)$, a contradiction.

For the induction step, suppose that the claim has been proved for all ordinals below β. Suppose that $\tau \in V$ is a Q-name for an element of $2^{D_1 \cap U}$. By a genericity argument, it will be enough to find, for every ordinal $\alpha \in \beta$, a condition $q \leq p$ and basic open sets $U_1, U_2 \subset U$ with disjoint closures such that y is U_1, β-good and $r \Vdash \neg(\tau/x_1) \restriction U_2 \ F_\alpha \ z \restriction U_2$.

Use (2) above to find a point $y_1 \in Y_1$ in the closure of the set U such that for every neighborhood W of y_1, the point y is $W \cap U, \beta$-good. Now, the treatment divides into cases.

Case 1. The set $\mathfrak{F}$ does not have the finite intersection property, where $\mathfrak{F}$ consists of closures of basic open subsets $W \subset U$ such that either there is $t \in R$ such that $R \Vdash \tau/\dot{x}_{gen} \restriction W$ to have no F_α-equivalent in the model M, or there are two conditions $s, t \in R$ which force two F_α-distinct elements of M to be equivalent to $\tau\dot{x}_{gen} \restriction W$. In this case, there must be $U_2 \in \mathfrak{F}$ whose closure does not contain the point $y_1 \in \bar{U}$. Let U_1 be a basic open neighborhood of y_1 such that $W \cap U_1 = 0$. There are two subcases:

Case 1a. $z \restriction U_2$ has no F_α-equivalent in the model M. In this case, use (1) above to find a point $y' \in \bar{O}'$ which is U_2, α-good, let $q \leq p$ be any condition which contains a triple $\langle y', O'', \alpha\rangle$ for some open set $O'' \subset O'$ and use the induction hypothesis to see that q is as required.

Case 1b. $z \restriction U_2$ does have an F_α-equivalent in the model M. In this case, by the definition of the filter $\mathfrak{F}$, there must be a condition $t \in R$ which forces τ

not to be F_α-equivalent to $z \restriction U_2$. Let $q \leq p$ be the condition $q = \langle t, a_p \rangle$ and check that the condition q with the sets U_1, U_2 works.

Case 2. The set $\mathfrak{F}$ has the finite intersection property. By a compactness argument, the set $\bigcap \mathfrak{F}$ is nonempty, containing a point $y_2 \in \bar{U}$. It follows that for every basic open set $W \subset U$ not containing the point y_2, there is a point $v_W \in 2^{D_1 \cap W}$ such that $R \Vdash v_W \; F_\alpha \; \tau \restriction W$. In the R-extension of the model M, there is a point in $2^{D_1 \cap U}$ whose restriction to each W is F_α-related to v_W, namely the point $(\tau/\dot{x}_{gen}) \restriction U$. By analytic absoluteness between the model M and its R-extension, there must be such a point $v \in 2^{D_1 \cap U}$ already in the model M. This point is not F_β-equivalent to $z \restriction U$, and so there is a basic open set $U_2 \subset U$ not containing neither y_1 nor y_2 such that $z \restriction U_2$ is not F_α-related to $v \restriction U_2$. The choice of the point v shows that p forces $\tau \restriction U_2$ not to be F_α-related to $z \restriction U_2$. Let $U_1 \subset U$ be a basic open set containing the point y_1 and disjoint from U_2. The condition p and sets U_1, U_2 complete the treatment of this case. $\square$

In view of the choice of the initial condition in the poset P, it follows that the point z has no F_{α_1}-equivalent in the model $V[x_1]$, and this concludes the verification of the desired properties of the point $x_1 \in 2^{D_0}$. $\square$

Corollary 3.4.21. *Every Borel homomorphism of F_{α_1} to F_{α_0} stabilizes on a comeager set and also on a set of full measure.*

Chapter 4

The pinned concept

4.1 Classifying the pinned names

Let E be an analytic equivalence relation on a Polish space X. The narrowest natural $\bar{E}$-invariant class of E-symmetric names is that of E-pinned names. It is therefore natural to attempt some sort of classification of $\bar{E}$-classes. Even in this very restrictive case, the success is only partial. Still, the information obtained leads to interesting non-reducibility results. The starting point is an easy technical proposition.

Proposition 4.1.1. *Suppose that E is an analytic equivalence relation on a Polish space X, P, Q are posets, τ, σ are E-pinned names on the respective posets. The following are equivalent:*

1. $\langle P, \tau\rangle \; \bar{E} \; \langle Q, \sigma\rangle$;
2. $P \times Q \Vdash \tau \; E \; \sigma$;
3. *in every (forcing) extension, whenever $G \subset P$ and $H \subset Q$ are filters separately generic over V, then $\tau/G \; E \; \sigma/H$.*

Proof. The implication (1)→(2) needs to be first proved in the simpler case where $P = Q$ and $\tau = \sigma$. Towards a contradiction, suppose that (1) holds and (2) fails, and let $p_0, p_1 \in P$ be conditions forcing in the product that τ evaluated with the left generic filter is not E-related to τ evaluated with the right generic filter. Use the assumption that τ is a pinned name to find, in some generic extension, filters $G_0, G_1 \subset P$ mutually generic over V such that $\tau/G_0 \; E \; \tau/G_1$. Use Proposition 2.1.7 in the model $V[G_0]$ to find there a poset Q_0 which forces the existence of a filter $K_0 \subset P_0$ generic over V such that $p_0 \in K_0$ and $\tau/G_0 \; E \; \tau/K_0$. Similarly, find a poset $Q_1 \in V[G_1]$ forcing in $V[G_1]$ the existence of a filter $K_1 \subset P$ generic over V such that $p_1 \in K_1$ and $\tau/G_1 \; E \; \tau/K_1$. Let $H_0 \times H_1 \subset Q_0 \times Q_1$ be a filter generic over $V[G_0, G_1]$, and find the corresponding filters $K_0 \in V[G_0][H_0]$ and $K_1 \in V[G_1][H_1]$. The filters

K_0, K_1 are mutually generic by Example 2.2.4, they contain the conditions p_0, p_1 respectively, and $\tau/K_0\ E\ \tau/K_1$ by the transitivity of E. This contradicts the initial choice of the conditions p_0, p_1.

The proof of (1)→(2) in the case of general $\langle Q, \sigma\rangle$ is now easy. Suppose towards a contradiction that (1) holds and (2) fails, and the failure of (2) is witnessed by conditions $p \in P$ and $q \in Q$ which force in the product that $\tau\ E\ \sigma$ fails. Use (1) to find, in some generic extension, filters $G_0 \subset P$ and $H_0 \subset Q$ such that $\tau/G_0\ E\ \sigma/H_0$. Let $G_1 \times H_1 \subset P \times Q$ be a filter generic over $V[G_0][H_0]$. Note that $G_0 \times G_1 \subset P \times P$ and $H_0 \times H_1 \subset Q \times Q$ are filters generic over V by the product forcing theorem. The previous paragraph applied to P and Q shows that $\tau/G_0\ E\ \tau/G_1$ and $\sigma/H_0\ E\ \sigma/H_1$. The transitivity of the relation E then shows that $\tau/G_1\ E\ \tau/H_1$ holds, contradicting the initial choice of the conditions p, q.

The proof of (2)→(3) is similar. Suppose that (2) holds. Towards the proof of (3), suppose that $G \subset P$ and $H \subset Q$ are filters separately generic over V. Let $G' \times H' \subset P \times Q$ be a filter generic over any model containing V as a subset and G, H as elements. Note that $G \times G' \subset P \times P$ and $H \times H' \subset Q \times Q$ are filters generic over V by the product forcing theorem. A repeated application of (2) then shows that $\tau/G\ E\ \tau/G'\ E\ \sigma/H'\ E\ \sigma/H$. The transitivity of the relation E then shows that $\tau/G\ E\ \tau/H$ holds as desired.

Finally, the implication (3)→(1) follows directly from the definitions. □

To start the classification effort, there is a great number of analytic equivalence relations which are known to be pinned:

Fact 4.1.2. [12, Theorem17.1.3] *The following equivalence relations are pinned:*

1. *every F_σ-equivalence relation;*
2. *every Borel equivalence relations with $\mathbf{\Sigma}^0_3$ classes;*
3. *every orbit equivalence relation generated by a continuous action of a cli Polish group.*

In these cases, the $\bar{E}$-equivalence classes essentially coincide with E-equivalence classes and so the whole enterprise of reaching to pinned names on various posets yields no information. A standard way of obtaining unpinned equivalence relations is using the Friedman–Stanley jump operation. Here, the classification of pinned names is right at hand: a pinned name for the jump is essentially just a set of pinned names for the original equivalence relation. Let E be an analytic equivalence relation on a Polish space X. Recall that E^+ is an equivalence relation on X^ω connecting x, y if they enumerate the same set of E-classes. For a nonempty set $S = \{[\langle P_i, \tau_i\rangle]_{\bar{E}} : i \in I\}$ of $\bar{E}$-classes, let τ_S be the name on the poset $Q_S = \prod_i P_i \times \mathrm{Coll}(\omega, I)$ for an element of X^ω enumerating the set $\{\tau_i : i \in I\}$.

Theorem 4.1.3. *Let E be an analytic equivalence relation on a Polish space X.*

1. *If S is a set of E-pinned names then τ_S is an E^+-pinned name;*

2. *$\tau_S\ \bar{E}^+\ \tau_T$ iff the set of $\bar{E}$-classes represented by names in S is the same as the set of $\bar{E}$-classes represented by names in T;*

3. *every E^+-pinned name is $\bar{E}^+$-equivalent to τ_S for some nonempty set S of $\bar{E}$-classes.*

Proof. Items (1) and (2) are immediate. To prove (3), suppose that σ is an E^+-pinned name on a poset P. Consider the set $S = \{\langle P \restriction p, \sigma(\check{n})\rangle : p \in P, n \in \omega,$ and $\sigma(n)$ is a name E-pinned on $P \restriction p\}$ and argue that $\langle P, \sigma\rangle\ \bar{E}^+\ \langle Q_S, \tau_S\rangle$. To this end, suppose that $G \times H \subset P \times Q_S$ is a filter generic over V and argue that $\sigma/G\ E\ \tau_S/H$.

To see that every E-class enumerated by σ/G is also enumerated by τ_S/H, let $n \in \omega$ be a number, and select a filter $K \subset P$ generic over $V[G \times H]$. Since $\sigma/G\ E^+\ \sigma/K$, there must be $m \in \omega$ such that $\sigma(n)/G\ E\ \sigma(m)/K$, and by Claim 2.2.14 there must be $p \in G$ such that $\sigma(n)$ is an E-pinned name on the poset $P \restriction p$. Thus the pair $\langle P \restriction p, \sigma(n)\rangle$ is an element of S and so τ_S enumerates the E-class of $\sigma(n)$.

To see that every E-class enumerated by τ_S/H is enumerated by σ/G, note that by the definition of the poset Q_S all such classes come from various components of the product filter H which are mutually generic with G, and so they must be enumerated by σ/G. $\square$

Corollary 4.1.4. *If τ is a $=^+$-pinned name on a poset P, then there is a nonempty set $A \subset 2^\omega$ such that $\langle P, \tau\rangle$ is equivalent to $\langle \mathrm{Coll}(\omega, A), \sigma\rangle$, where σ is the $\mathrm{Coll}(\omega, A)$-name for a generic enumeration of A.*

Proof. Note that the $=$-pinned names are essentially elements of 2^ω, and use Theorem 4.1.3. $\square$

Theorem 4.1.3 also leads to a new and informative proof of a theorem of Friedman and Stanley [5, Theorem 8.3.6]:

Corollary 4.1.5. *Suppose that E is an analytic equivalence relation on a Polish space X with only set many $\bar{E}$-classes. Then E^+ is not reducible to E.*

Proof. Suppose for a contradiction that $h\colon x^\omega \to X$ is a Borel reduction of E^+ to E. Let $\{\langle P_i, \tau_i\rangle : i \in I\}$ be an enumeration of a maximal pairwise $\bar{E}$-inequivalent list of E-pinned names. For every set $J \subset I$, the name τ_I is E^+-pinned and the names $\{\tau_J : J \subset I\}$ are pairwise $\bar{E}^+$-unrelated, the names $h\tau_J$ for $J \subset I$ must be pairwise $\bar{E}$-unrelated, and each of them is represented by som τ_i. This contradicts the fact that $|\mathcal{P}(I)| > |I|$. $\square$

Corollary 4.1.6. *Whenever E is a Borel equivalence relation, E^+ is not Borel reducible to E.*

Proof. Borel equivalence relations have only set many $\bar{E}$-classes of pinned names by Theorem 4.2.3(1). $\square$

In the following two theorems, I will show how to classify the pinned names for isomorphism of models of certain type. Let X be the standard Borel space of structures with universe ω, let E be the equivalence relation of isomorphism, and let $A \subset X$ be a coanalytic E-invariant set. Write E_A for the analytic equivalence relation $E \restriction A$. I will attempt to classify E_A-pinned names at least in some specific cases. The natural pinned names come in the following shape. Let $\hat{A}$ be the class of all (possibly uncountable) models M such that $\mathrm{Coll}(\omega, M) \Vdash \check{M}$ is isomorphic to some structure in $\dot{A}$. For every model $M \in \hat{A}$, let τ_M be a $\mathrm{Coll}(\omega, M)$-name for some isomorph of $\check{M}$ in $\dot{A}$. Clearly, τ_M is an E_A-pinned name.

Theorem 4.1.7. *Let X be the standard Borel space of structures with universe ω, let E be the equivalence relation of isomorphism, and let $A \subset X$ be a coanalytic E-invariant set consisting of rigid structures.*

1. *Every E_A-pinned name is $\bar{E}_A$-equivalent to some τ_M for some $M \in \hat{A}$;*
2. *for $M, N \in \hat{A}$, $\tau_M \; \bar{E}_A \; \tau_N$ if and only if M is isomorphic to N.*

In other words, in the rigid case the pinned names with the $\bar{E}_A$-equivalence are classified as (perhaps uncountable) models in $\hat{A}$ with isomorphism.

Proof. Before I begin the argument, note that the statement that every structure in the set A is rigid is $\mathbf{\Pi}^1_2$ and so holds also in all generic extensions by the Shoenfield absoluteness.

For (1), suppose that σ is an E_A-pinned name on a poset P and let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over V. In the model $V[G_0, G_1]$, let $N_0 = \tau/G_0$ and $N_1 = \tau/G_1$. To define the model $M \in V$, let $x_0 = \{s \colon s$ is the Scott sentence of the model $\langle N_0, a\rangle$ for some $a \in N_0\} \in V[G_0]$ and $x_1 = \{s \colon s$ is the Scott sentence of the model $\langle N_1, a\rangle$ for some $a \in N_1\} \in V[G_1]$. Since N_0 is isomorphic to N_1, it follows from Karp's theorem ([5, Lemma 12.1.6]) that $x_0 = x_1$, so $x_0 = x_1 \in V[G_0] \cap V[G_1] \in V$. The set x_0 will be the universe of the model M. Note that since the model N_0 is rigid, the elements of N_0 are in one-to-one correspondence with x_0 by Karp's theorem again and the unique isomorphism between N_0 and N_1 factors through the identity on the set $x_0 = x_1$. To construct the realizations of relational and functional symbols of the model M, for every relational symbol R (say binary) of the language of the models and $s, t \in x_0$, let $s \; R^M \; t$ if for the unique $a, b \in N_0$ such that s is the Scott sentence of a and t is a Scott sentence of b, $N_0 \models s \; R \; t$. The same definition using the model N_1 yields the same relation, and so $R^M \in V[G_0] \cap V[G_1] = V$. Define the realizations of all functional and relational symbols of the model M in this way. As a result, M is a model in V and the map $a \mapsto$the Scott sentence of $\langle M, a\rangle$ is an isomorphism of N_0 and M in the model $V[G_0]$. Thus, $\tau_M \; \bar{E}_A \; \tau$ as desired.

For (2), if M is isomorphic to N then clearly $\tau_M \; \bar{E}_A \; \tau_N$ as E_A is the isomorphism relation. For the other direction, suppose that $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N) \Vdash \tau_M \; E_A \; \tau_N$. Therefore $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N) \Vdash$there is a unique isomorphism from M to N. Since $\mathrm{Coll}(\omega, M) \times \mathrm{Coll}(\omega, N)$ is a homogeneous notion of forcing, for each $m \in M$ the value of the image of m under the unique isomorphism between M and N is decided by the largest condition to be some $h(m) \in N$. The function $h : m \to N$ is an isomorphism of M to N present already in V. $\square$

In the case of non-rigid structures, the classification may become more complicated. I will treat the case of acyclic countable graphs, which has the virtue of being Borel-complete among the equivalence relations classifiable by countable structures. For a perhaps uncountable acyclic graph H, let τ_H be a $\mathrm{Coll}(\omega, H)$-name for a graph on ω isomorphic to H. Recall that two structures M, N are *Ehrenfeucht–Fraissé equivalent* if Player II has a winning strategy in the Ehrenfeucht–Fraissé game of length ω. In the i-th round of this game, Player I indicates an element $m_i \in M$ or $n_i \in N$, and Player II answers with an element n_i or m_i in the other model. Player II wins if the set $\{\langle m_i, n_i\rangle : i \in \omega\}$ is a partial isomorphism between the models M and N.

Theorem 4.1.8. *Let X be the Polish space of graphs on ω, let $A \subset X$ be the Borel set of acyclic graphs, and let E be the relations of isomorphism on A.*

1. *every E-pinned name is $\bar{E}$-equivalent to some τ_H for some acyclic graph H;*
2. *for acyclic graphs H, K, $\tau_H \ \bar{E}\ \tau_K$ if and only if H is Ehrenfeucht–Fraissé equivalent to K.*

Note that the right-hand side of (2) cannot be strengthened to isomorphism of H and K, as the example of empty graphs on $\aleph_1$ and $\aleph_2$ vertices shows.

Proof. The point of the proof is that an acyclic graph can be explicitly built from automorphism orbits of its elements. This procedure is captured in the following observation. Suppose x is a set, $f : x^2 \to \omega + 1$ is a function such that $f(s,t) > 0 \leftrightarrow f(t,s) > 0$, and $g : x^2 \to \omega + 1$ is a function such that $f(s,t) > 0$ implies $g(s) = g(t)$. Then there is, up to an isomorphism unique, acyclic graph $H(x, f, g)$ together with an onto map $h : y \to x$, where y is the set of vertices of $H(x, f, g)$, such that

- for every $s, t \in x$ and every vertex $v \in y$, if $h(v) = s$ then the set of all neighbors of v mapped to t has size $f(s,t)$;
- for every $s \in x$ there are $g(s)$ many connected components of the graph H containing a vertex v with $h(v) = s$.

The construction of the graph $H(x, f, g)$ is straightforward. Note that whenever $u, v \in y$ are two vertices such that $h(u) = h(v)$ then there is an automorphism of the graph $H(x, f, g)$ sending u to v.

For (1), suppose that σ is an E-pinned name on a poset P and let $G_0 \times G_1 \subset P \times P$ be mutually generic filters over V. In the model $V[G_0, G_1]$, let $H_0 = \sigma/G_0$ and $H_1 = \sigma/G_1$. To define the graph $H \in V$, let $x_0 = \{s : s$ is the Scott sentence of the model $\langle H_0, v\rangle$ for some vertex v of $H_0\} \in V[G_0]$ and $x_1 = \{s : s$ is the Scott sentence of the model $\langle H_1, v\rangle$ for some vertex v in $H_1\} \in V[G_1]$. Since H_0 is isomorphic to H_1, it follows from Karp's theorem ([5, Lemma 12.1.6]) that $x_0 = x_1$, so $x = x_0 = x_1 \in V[G_0] \cap V[G_1] \in V$. Let $f : x^2 \to \omega+1$ be the function defined by $f(s, t) = i$ if every vertex of H_0 of type s has i-many neighbors of type t when $i \in \omega$, and $f(s, t) = \omega$ if every vertex of H_0 of type s has infinitely many neighbors of type t. Let $g : x \to \omega + 1$ be the function defined by $f(s) = i$ if there are i-many connected components of H_0 containing a node of type s when $i \in \omega$, and $g(s) = \omega$ if there are infinitely many connected components of H_0 containing a node of type s. Note that these functions are well-defined and the graph H_0 is isomorphic to $H(x, f, g)$ in the model $V[G_0]$. Similar definitions using the grap H_1 yield the same functions, and so $f, g \in V[G_0] \cap V[G_1] = V$. Working in V, consider the graph $H = H(x, f, g)$. This graph is isomorphic to H_0 in $V[G_0]$, and so $\tau_H\ \bar{E}\ \sigma$.

(2) does not use the fact that H, K are acyclic graphs. If, on one hand, H, K are Ehrenfeucht–Fraissé equivalent in V, they remain so in the $\mathrm{Coll}(\omega, H) \times \mathrm{Coll}(\omega, K)$-extension. In that extension, they are in addition countable, and therefore they are isomorphic by the usual back-and-forth argument. Thus, $\tau_H\ \bar{E}\ \tau_K$ must hold. On the other hand, assume that $\tau_H\ \bar{E}\ \tau_K$. Then, one can use the existence of isomorphism between H and K in the $\mathrm{Coll}(\omega, H) \times \mathrm{Coll}(\omega, K)$ extension to produce a winning strategy in the E-F game for the isomorphism player. Therefore, the graphs H, K are Ehrenfeucht–Fraissé equivalent as desired. □

Question 4.1.9. Classify the pinned names for the measure equivalence.

Question 4.1.10. Let ϕ be an arbitrary $L_{\omega_1\omega}$ sentence. Classify the pinned names for the equivalence of isomorphism between countable models of ϕ.

4.2 Estimating the pinned cardinal

There is a cardinal invariant which carries a great amount of information regarding the classification of pinned names.

Definition 4.2.1. Let E be an analytic equivalence relation on a Polish space X.

1. If τ is an E-symmetric name on a poset P, let $\kappa(\tau)$ be the minimum cardinal κ such that τ has a $\bar{E}$-equivalent on a poset of size κ.
2. $\kappa(E)$, the *pinned cardinal* of E, equals to $\sup\{\aleph_1, \kappa(\tau)^+ : \tau$ is an E-pinned name$\}$ or to ∞ if the aforementioned supremum does not exist.

In the definition of $\kappa(E)$, it is necessary to restrict attention to a small class of names such as the E-pinned names, otherwise the value of the invariant would be ∞ even for very simple equivalence relations. I will show that for every infinite cardinal κ there is an E_0-symmetric name τ on some poset P such that

$\kappa(\tau) = \kappa$. Let X be the space of all linear orderings on ω, let Γ be the group of permutations of ω, and let E be the equivalence relation on X obtained by the natural action of Γ on X. It is not difficult to see that E is hyperfinite and therefore bireducible with E_0. Now, given an infinite cardinal κ let P be the poset of finite injections from ω to κ ordered by reverse extension, and let τ be the P-name for the linear ordering on ω which is the inverse image of the linear ordering on κ under the generic bijection between ω and κ. It is easy to see that τ is an E-symmetric name and each poset with an $\bar{E}$-equivalent name on it collapses κ to ω, and therefore $\kappa = \kappa(\tau)$ as desired.

The inclusion of $\aleph_1$ in the supremum in (2) above is purely for notational convenience. This way, $\kappa(E) = \aleph_1$ for all pinned equivalence relations E and $\kappa(E) \geq \aleph_2$ for all unpinned equivalence relations E holds.

The most important feature of the pinned cardinal is the fact that it respects the Borel reducibility and almost Borel reducibility of analytic equivalence relations:

Theorem 4.2.2. *Suppose that E, F are analytic equivalence relations on respective Polish spaces X, Y, and $h\colon X \to Y$ is a Borel function which is an almost reduction of E to F.*

1. *If τ is a nontrivial E-symmetric name on a poset P, then $\kappa(\tau) = \kappa(h\tau)$;*
2. *$\kappa(E) \leq \kappa(F)$.*

Proof. Let $a \subset X$ be a countable set such that h is a reduction of E to F on the set $X \setminus [a]_E$. By the Shoenfield absoluteness, h maintains this property in every forcing extension.

For (1), let P be a poset and τ a nontrivial E-symmetric name. By Proposition 2.1.9, (1) will follow if I show that whenever Q is a poset and in some generic extension $G \subset P$ and $H \subset Q$ are filters separately generic over V, then $V[H]$ contains an E-equivalent of τ/G if and only if $V[H]$ contains an F-equivalent of $h\tau/G$.

For the left-to-right direction, if $x \in V[H]$ is E-equivalent to τ/G. Since the name τ is not trivial, $\tau/G \notin [a]_E$ holds, and therefore $h(x)\ F\ h(\tau/G)$ holds as well. Thus, $V[H]$ contains an F-equivalent to $h\tau/G$. For the right-to-left direction, suppose that $y \in V[H]$ is F-equivalent to $h\tau/G$. By the Shoenfield absoluteness for the model $V[H]$, there is in $V[H]$ a point $x \in X \setminus [a]_E$ such that $h(x)\ E\ y$, since such a point (namely τ/G) exists in the model $V[G]$. The point $x \in X$ must be E-related to τ/G as desired.

(2) follows from (1) immediately using the fact that if τ is a nontrivial E-pinned name then $h\tau$ is a nontrivial F-pinned name. □

This section is devoted to finding upper bounds for the pinned cardinal based on the rough complexity features of the equivalence relation E.

Theorem 4.2.3. *Let E be an analytic equivalence relation on a Polish space X.*

1. *if E is Borel of rank $\mathbf{\Pi}^0_\alpha$, then $\kappa(E) \leq (\beth_\alpha)^+$;*

2. *if E is weakly reducible to an orbit equivalence relation and $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first ω_1-Erdős cardinal;*

3. *if E is arbitrary analytic and $\kappa(E) < \infty$ then $\kappa(E)$ is not greater than the first measurable cardinal.*

The bounds obtained in (1) and (2) are more or less optimal; in Theorems 4.3.3 and 4.3.4 for a given equivalence relation α I will find a Borel equivalence relation E such that $\kappa(E) \geq (\beth_\alpha)^+$ and an analytic equivalence relation E weakly reducible to an orbit equivalence relation such that $\kappa(E)$ is the successor of the first α-Erdős cardinal. I do not have a similar complementary result for the bound in (3).

Proof. For (1), suppose that E is Borel of rank $\alpha \in \omega_1$ and let τ be a pinned P-name; I must produce a $\mathrm{Coll}(\omega, \beth_\alpha)$-name σ which is $\bar{E}$-related to τ. Note that $[\tau]_E$ is a P-name for a Borel set of rank $\leq \alpha$. As is the case for every name for a Borel set, [23, Corollary 2.9] shows that in the $\mathrm{Coll}(\omega, \beth_\alpha)$ extension $V[G]$ there is a Borel code for a Borel set $B \subset X$ such that in every further forcing extension $V[G][H]$ and every $x \in X \cap V[G][H]$ in that extension, $x \in B$ if and only if $V[x] \models P \Vdash \check{x} \in [\tau]_E$. Note that if $H \subset P$ is generic over $V[G]$, then the set B is nonempty in $V[G][H]$, containing the point τ/H; this follows from the fact that τ is E-pinned. Thus, the set B is nonempty already in $V[G]$ by the Mostowski absoluteness between $V[G]$ and $V[G][H]$. Back in V, let σ be any $\mathrm{Coll}(\omega, \beth_\alpha)$-name for an element of the set B. This clearly works.

For (2), let E be an orbit equivalence relation on a Polish space X almost reducible to an orbit equivalence relation of a Polish group action. Let κ be the first ω_1-Erdős cardinal, and suppose that $\kappa(E) > \kappa$; I must show that $\kappa(E) = \infty$. Since the cardinal κ is the Hanff number for the class of wellfounded models of first order sentences, for every cardinal λ there is a wellfounded model M such that $M \models \kappa(E) > \lambda$. Now, since E is almost reducible to an orbit equivalence relation, Corollary 5.3.2 shows that the wellfounded model M is correct about $\kappa(E)$ to the extent that $|\kappa(E)^M| \leq \kappa(E)$. It follows that $\kappa(E) > \lambda$, and since λ was arbitrary, $\kappa(E) = \infty$.

For (3), let κ be a measurable cardinal and suppose that there is a poset P and an E-pinned name τ on P which is not $\bar{E}$-related to any name on a poset of size $< \kappa$. I will produce a proper class of pairwise non-$\bar{E}$-related E-pinned names.

First note that the poset P and the name τ can be selected so that $|P| = \kappa$. Simply take an elementary submodel M of size κ of large structure with $V_\kappa \subset M$ and consider $Q = P \cap M$ and $\sigma = \tau \cap M$; so $|Q| = \kappa$. As M is correct about pinned names and the equivalence $\bar{E}$ by Theorem 5.1.1, σ is an E-pinned name on Q and it is not $\bar{E}$-equivalent to any pinned names on posets of size $< \kappa$.

Thus, assume that the poset P has size κ. Let $j \colon V \to N$ be any elementary embedding into a transitive model with critical point equal to κ. Note that $H(\kappa) \subset N$ and so both P, τ are (isomorphic to) elements of N. Let $\langle N_\alpha, j_{\beta\alpha} \colon \beta \in$

$\alpha\rangle$ be the usual system of iteration of the elementary embedding j along the ordinal axis. Let $P_\alpha = j_{0\alpha}(P)$ and $\tau_\alpha = j_{0,\alpha}(\tau)$. It will be enough to show that the pairs $\langle P_\alpha, \tau_\alpha\rangle$ for $\alpha \in \mathrm{Ord}$ are pairwise $\bar{E}$-unrelated. To see this, pick ordinals $\alpha \in \beta$. As the original poset had size κ, it is the case that $P_\alpha, \tau_\alpha, P_\beta, \tau_\beta$ are in the model N_β. By the elementarity of the embedding $j_{0\beta}$, $N_\beta \models \neg\langle P_\alpha, \tau_\alpha\rangle \ \bar{E} \ \langle P_\beta, \tau_\beta\rangle$. The wellfounded model N_β is correct about $\bar{E}$ by Theorem 5.1.1 and so $\langle P_\alpha, \tau_\alpha\rangle \ \bar{E} \ \langle P_\beta, \tau_\beta\rangle$ fails also in V as required. □

One interesting corollary is that $\kappa(E) < \infty$ for every Borel equivalence relation E. Note that the definition of $\kappa(E)$ does not depend on the Polish topology chosen for the space X, and so we conclude that if the value of the cardinal $\kappa(E)$ is too large, then there is no topology on X inducing the same Borel structure that makes E into a $\mathbf{\Pi}^0_\alpha$ subset of $X \times X$. This connects the pinned cardinal with the subject of potential Borel classes studied by Lecomte [18].

4.3 Examples

The whole point of the definition of the pinned cardinal is that it attains interesting values for fairly common equivalence relations and therefore can be used for Borel nonreducibility results via Theorem 4.2.2. This section contains a long list of examples with computations of $\kappa(E)$.

4.3.1 Basics

I start with the ultimate uninteresting example.

Example 4.3.1. $\kappa(E_{\omega_1}) = \infty$.

Proof. It is enough to produce a proper class of pairwise nonequivalent pinned names. For every ordinal α, let τ_α be the $\mathrm{Coll}(\omega, \alpha)$-name for a relation on ω which is isomorphic to the order relation on α. It is clear that each τ_α is pinned, and if $\alpha \neq \beta$ then the names τ_α, τ_β are nonequivalent. □

Theorem 4.4.1 shows that in fact E_{ω_1} is a minimal example of an equivalence relation with infinite value of $\kappa(E)$.

Example 4.3.2. $\kappa(=^+) = \mathfrak{c}^+$.

Proof. Corollary 4.1.4 shows that for every $=^+$-pinned name τ there is a set $S \subset 2^\omega$ such that $\mathrm{rng}(\tau) = \check{S}$ is forced. Such a name τ is clearly equivalent to the $\mathrm{Coll}(\omega, S)$-name for the generic enumeration of the set S. Since $|\mathrm{Coll}(\omega, S)| \leq \mathfrak{c}$, this implies that $\kappa(=^+) \leq \mathfrak{c}^+$. On the other hand, the $\mathrm{Coll}(\omega, 2^\omega)$-name for the generic enumeration of 2^ω cannot be equivalent to any pinned name on a poset of size $< \mathfrak{c}$, since it entails a collapse of $\mathfrak{c}$ to $\aleph_0$. Thus, $\kappa(=^+) = \mathfrak{c}^+$ as desired. □

Theorem 4.3.3. *For every countable ordinal α there is a Borel equivalence relation F_α such that $\kappa(F_\alpha) = \beth^+_\alpha$.*

This shows that the estimate in Theorem 4.2.3 (1) is essentially the best possible; the values of $\kappa(E)$ for Borel equivalence relations E are cofinal in $\beth_{\omega_1}$.

Proof. Let ϕ_α be the $L_{\omega_1\omega}$ sentence saying that R is an extensional binary relation which is wellfounded of rank $\leq \omega + \alpha$. The equivalence relation F_α of isomorphism of models of ϕ_α is Borel since wellfounded extensional relations have no nontrivial automorphisms by Mostowski transitive collapse theorem.

By Theorem 4.1.7, for every F_α-pinned name τ there is a transitive set A of rank $\leq \omega + \alpha$ such that τ is forced to be isomorphic to $\langle A, \in\rangle$. The name τ is then equivalent to the $\mathrm{Coll}(\omega, A)$-name for a generic isomorph of $\langle A, \in\rangle$ with universe ω. Since $|\mathrm{Coll}(\omega, A)| = |A| \leq \beth_\alpha$, it follows that $\kappa(F_\alpha) \leq \beth_\alpha^+$.

On the other hand, $V_{\omega+\alpha}$ is a set of rank $\omega+\alpha$, it has size $\beth_\alpha$, the $\mathrm{Coll}(\omega, V_{\omega+\alpha})$-name for its generic isomorph is an E_α-pinned name, and it cannot be equivalent to a pinned name on a poset of size $< \beth_\alpha$ since it entails the collapse of $|V_{\omega+\alpha}|$ to $\aleph_0$. Thus, $\kappa(F_\alpha) = \beth_\alpha^+$ as required. □

Theorem 4.3.4. *For every countable ordinal α there is an analytic equivalence relation E_α almost classifiable by countable structures such that $\kappa(E_\alpha)$ =the first α-Erdős cardinal.*

This shows that the estimate in Theorem 4.2.3 (2) is essentially the best possible; the values of $\kappa(E)$ for equivalence relations almost reducible to orbit equivalence relations can come quite close to the first ω_1-Erdős cardinal.

Proof. Let X be the Polish space of binary relations on ω, with relation E of isomorphism. Let B_α be the set of all binary relations on ω which are extensional, wellfounded, and do not admit a sequence of indiscernibles of ordertype α. This is a coanalytic set of rigid structures invariant under E. Let $E_\alpha = E \restriction B_\alpha$. This is an analytic equivalence relation almost reducible to isomorphism of countable structures. I claim that $\kappa(E_\alpha)$ =the first α-Erdős cardinal κ_α.

On one hand, by Theorem 4.1.7 for every E_α-pinned name τ there is a transitive set A without indiscernibles of ordertype α such that τ is forced to be isomorphic to $\langle A, \in\rangle$. It must be the case that $|A| < \kappa_\alpha$ and therefore τ is equivalent to the $\mathrm{Coll}(\omega, A)$-name for the generic isomorph of $\langle A, \in\rangle$ with domain ω. This means that $\kappa(E_\alpha) \leq \kappa_\alpha$.

On the other hand, whenever $\lambda < \kappa_\alpha$ is an ordinal, then the structure $\langle V_\lambda, \in\rangle$ has no indiscernibles of ordertype α, and it remains such in every forcing extension by a wellfoundedness argument. Thus, the $\mathrm{Coll}(\omega, V_\lambda)$-name for the generic isomorph of this structure is an E_α-pinned name, and it is not equivalent to any pinned name on a poset of size $< |V_\lambda|$ since it entails the collapse of $|V_\lambda|$ to $\aleph_0$. Thus, $\kappa(E_\alpha) = \kappa_\alpha$ as desired. □

4.3.2 Alephs and the Singular Cardinal Hypothesis

Theorem 4.3.5. *For every countable ordinal $\alpha > 0$ there is a Borel equivalence relation E_α such that (provably) $\kappa(E_\alpha) = \aleph_\alpha$.*

Proof. First argue that for every countable ordinal $\alpha > 0$ there is an $L_{\omega_1\omega}$ sentence ϕ_α which has models of all infinite cardinalities $< \aleph_\alpha$ but no model of size $\aleph_\alpha$. The proof goes by induction on α.

For $\alpha = 1$ just let ϕ_α be any sentence which describes the natural ordering on ω. For a limit ordinal α let ϕ_α be the disjunction of ϕ_β for $\beta \in \alpha$. For a successor ordinal $\alpha = \beta + 1$ distinguish the case of β limit or β successor. If β is limit, then let the language of ϕ_α contain new unary predicates A_γ for $\gamma \in \beta$ and let ϕ_α say "the predicates A_γ for $\gamma \in \beta$ partition the universe and $A_\gamma \models \phi_\beta$". If $\beta = \gamma + 1$ is a successor ordinal, then let the language of ϕ_α contain new unary predicates A, B, a binary predicate $<$ and a binary functional symbol F and let ϕ_α say "the predicates A, B partition the universe, $A \models \phi_\beta$, $B \models <$ is a linear order, and $F\colon A \times B \to B$ is a function such that for every $x \in B$, the initial segment of $<$ up to x is a subset of the range of $F(\cdot, x)$". This clearly works. For example, in the latter case, in any model M of ϕ_α, the predicate A^M has size at most $\aleph_\gamma$, and the predicate B^M has a linear order on it whose proper initial segments have size $\leq |A^M|$, so $|B^M| \leq \aleph_{\gamma+1}$ as desired.

Now, for every ordinal $\alpha > 0$ let ψ_α be the sentence in the language of ϕ_α together with a new binary relational symbol R which says "ϕ_α holds and R is a relation satisfying the axiom of extensionality, which is wellfounded of rank $\leq \omega + \alpha$". Clearly, the models of ψ_α have no nontrivial automorphisms by the Mostowski collapse theorem. Moreover, every model of ϕ_α can be equipped with an additional relation R with which it becomes a model of ψ_α, since $|M| < \aleph_\alpha \leq \beth_\alpha = |V_{\omega+\alpha}|$ and the $\in$-relation on V_α is extensional and wellfounded of rank $\omega + \alpha$. Thus, the sentence ψ_α has models of all infinite cardinalities $< \aleph_\alpha$. Theorem 4.1.7 now says that $\kappa(E_{\psi_\alpha}) = \aleph_\alpha$. $\square$

Note that an equivalence relation E_α as above for $\alpha \geq 2$ cannot be reducible to $=^+$ and $=^+$ cannot be reducible to it. This answers a question of Kechris [12, Question 17.6.1] in the negative as well as some related questions of Simon Thomas. To see that $=^+$ cannot be Borel reducible to any E_α, suppose for contradiction that $h\colon \mathrm{dom}(E) \to \mathrm{dom}(F)$ is a Borel reduction. Pass to a generic extension in which $\mathfrak{c} > \aleph_{\omega_1}$. There, h is still a reduction of E to F, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 4.2.2. To see that E_α cannot be reducible to $=^+$ for any $\alpha > 2$, pass to a generic extension in which the Continuum Hypothesis holds instead.

The next example is motivated by the Singular Cardinal Hypothesis. This is the statement that for every singular cardinal κ, if $2^{\mathtt{cof}(\kappa)} < \kappa$ then $\kappa^{\mathtt{cof}(\kappa)} = \kappa^+$. The validity of this statement remained open long after the Generalized Continuum Hypothesis was shown to be independent of ZFC. In a major breakthrough, Magidor [19] proved that the singular cardinal hypothesis can fail at $\aleph_\omega$. The following corollary shows that Magidor's result can be injected into the Borel reducibility ordering of Borel equivalence relations classifiable by countable structures.

Example 4.3.6. There are Borel equivalence relations E, F such that (provably) $\kappa(E) = (\aleph_\omega^{\aleph_0})^+$ and $\kappa(F) = \max\{\mathfrak{c}, \aleph_{\omega+1}\}^+$.

Proof. For E, the first step is the construction of an $L_{\omega_1\omega}$ sentence which has models of size $\aleph_\omega^{\aleph_0}$ but no larger. First, let χ be an $L_{\omega_1\omega}$ sentence that has models of size $\aleph_\omega$ but not any larger, as obtained in the previous example. Let ϕ be a sentence in the language of χ with additional unary predicates A, B, C and a binary functionaly symbol f. ϕ will say: "the predicates A, B, C partition the universe, $A \models \chi$, B is ordered in type ω, and $f : B \times C \to A$ is a function such that for distinct $i \neq j \in C$ the sets $f''(\cdot, i)$ and $f''(\cdot, j)$ are distinct". The sentence ϕ clearly works as desired.

Let ψ be a sentence in the language of ϕ with an additional binary relational symbol R which says "ϕ holds and the relation R satisfies the axiom of extensionality and it is well-founded with rank $< \omega + \omega + 2$". The sentence ψ has only rigid models, and it has models of size $\aleph_\omega^{\aleph_0}$ but no larger. Theorem 4.1.7 now shows that $\kappa(E_\psi) = (\aleph_\omega^{\aleph_0})^+$ as desired.

For F, just use the previous two examples to find sentences ϕ_0, ϕ_1 with only rigid models such that ϕ_0 has models of size $\mathfrak{c}$ but no larger, and ϕ_1 has models of size $\aleph_{\omega+1}$ but no larger. It is not difficult to see that $F = E_{\phi_0 \vee \phi_1}$ works as required. $\square$

I can now show that under suitable large cardinal hypothesis, E cannot be Borel reducible to F. Suppose for contradiction that $h \colon \mathrm{dom}(E) \to \mathrm{dom}(F)$ is a Borel reduction. Use a classical result of Magidor [19] and pass to a generic extension in which the Singular Cardinals Hypothesis fails at $\aleph_\omega$: $\aleph_\omega^{\aleph_0} > \max(\mathfrak{c}, \aleph_{\omega+1})$. There, h is still a reduction of E to F, while $\kappa(E) > \kappa(F)$. This contradicts Theorem 4.2.2.

Thus, one can (oh horror!) encode the status of the Singular Cardinal Hypothesis at $\aleph_\omega$ into the value of cardinal invariants of Borel equivalence relations E, F which are even classifiable by countable structures, and turn the proof of independence of SCH into a proof of Borel nonreducibility of E to F.

4.3.3 Sierpinski's theorem

The previous examples were to some extent artificial in the sense that the values of the pinned cardinal were directly built into them. The following examples, all Borel reducible to $=^+$, work in the context of Martin's Axiom, and they are more natural.

Definition 4.3.7. Let X be a set, $n \in \omega$ a number, and $R \subset [X]^n \times X$ a relation. A set $a \in [X]^{n+1}$ is *R-free* if for every element $x \in a$, $(a \setminus \{x\}, x) \notin R$ holds.

Definition 4.3.8. Let $n \in \omega$ be a number and $R \subset [2^\omega]^n \times 2^\omega$ a Borel relation. E_R is the equivalence relation $=^+ \restriction B_R$ where $B_R = \{x \in (2^\omega)^\omega \colon \mathrm{rng}(x)$ contains no R-free $n+1$-tuple$\}$.

There is a natural directed quasiorder here. For Borel relations $R, S \subset [2^\omega]^n \times 2^\omega$ write $R \leq S$ if there is a Borel injection $\pi \colon 2^\omega \to 2^\omega$ which induces a homomorphism from R to S. It is clear that $R \leq S$ implies $E_R \leq E_S$ since the function π induces also a Borel reduction $x \mapsto \pi \circ x$ of E_R to E_S.

Theorem 4.3.9. *Let $n \in \omega$ be a number and let $R \subset [2^\omega]^n \times 2^\omega$ be a Borel relation with all vertical sections countable.*

1. $\kappa(E_R) \leq \aleph_{n+1}$;
2. *if R is sufficiently high in the $\leq$-quasiorder, then* $\mathrm{MA}_{\aleph_n} \vdash \kappa(E_R) = \aleph_{n+1}$.

An example for which (2) holds is the relation $R = \{(a, x) \in [2^\omega]^n \times 2^\omega \colon x$ is computable from $a\}$.

Proof. The argument reflects a classical partition theorem:

Fact 4.3.10. (Sierpiński [6]) *Let $n \in \omega$ be a number. $\aleph_{n+1}$ is the smallest cardinal κ such that every relation $T \subset [\kappa]^n \times \kappa$ with countable vertical sections has a T-free $n+1$-tuple.*

Now, let $n \in \omega$ be a number and $R \subset [2^\omega]^n \times 2^\omega$ a Borel relation with all vertical sections countable.

To prove (1), use Corollary 4.1.4 to see that for every E_R-pinned name τ on a poset P there is a set $A \subset 2^\omega$ such that A contains no free $n+1$-tuple and $\langle P, \tau\rangle \; \bar{E}_R \; \langle \mathrm{Coll}(\omega, A), \sigma\rangle$ where σ is the generic enumeration of the set A. Now, the set A must have size $< \aleph_{n+1}$ by Fact 4.3.10 and so $\kappa(E_R) \leq \aleph_{n+1}$.

To prove (2), note that $R \leq S$ implies $E_R \leq_{\mathrm{B}} E_S$ and so $\kappa(E_R) \leq \kappa(E_S)$. Therefore, it will be enough to prove that $\mathrm{MA}_{\aleph_n} \vdash \kappa(R) = \aleph_{n+1}$ for the computability relation R. To this end, I will prove a small coding result: for every cardinal κ, every $n \in \omega$ and every relation $T \subset [\kappa]^n \times \kappa$ with countable sections there is a c.c.c. poset adding an injection $\pi \colon \kappa \to \mathcal{P}(\omega)$ which induces a homomorphism from T to R. Once this is known, use Fact 4.3.10 to find a relation $T \subset [\omega_n]^n \times \omega_n$ with all vertical sections countable and no T-free $n+1$-tuple. Use the coding and $\mathrm{MA}_{\aleph_n}$, to find an injection $\pi \colon \omega_n \to 2^\omega$ which is a homomorphism of T to R. Then, the range $A = \mathrm{rng}(\pi)$ is a set of size $\aleph_n$ with no R-free $n+1$-tuple, the $\mathrm{Coll}(\omega, A)$-name for the generic enumeration of A is E_R-pinned, and it has no $\bar{E}_R$-equivalent on a poset of size $< \aleph_n$ since it entails collapsing $|A|$ to $\aleph_0$. Thus, $\kappa(E_R) \geq \aleph_{n+1}$ as required.

Towards the coding result, fix $n \in \omega$, a cardinal κ and a relation $T \subset [\kappa]^n \times \kappa$ with countable vertical sections. let $\langle k_m : m \in \omega\rangle$ be a recursive sequence of increasing functions in ω^ω with disjoint ranges. For a finite set $b \subset \mathcal{P}(\omega)$ let e_b be the increasing enumeration of the set $\bigcap b$, for every $m \in \omega$ let $h_m(b) \subset \mathcal{P}(\omega)$ be the set of all l such that $e_b \circ k_m(l)$ is an odd number. I will produce the map π such that for every set $a \in [\kappa]^n$ and every $\alpha \in T_a$, there is a number $m \in \omega$ such that $\pi(\alpha)$ is modulo finite equal to $h_m(\pi'' a)$.

Let P be the poset of all tuples $p = \langle n_p, \pi_p, \nu_p\rangle$ so that

- $n_p \in \omega$, π_p is a partial function from κ to $\mathcal{P}(n_p)$ with finite domain $\mathrm{dom}(p)$;
- ν_p is a finite partial function from $[\mathrm{rng}(p)]^n \times \omega$ to $\mathrm{rng}(p)$ such that $(a, \nu_p(a, m)) \in T$ whenever $\langle a, m\rangle \in \mathrm{dom}(\nu_p)$.

The ordering on P is defined by $q \leq p$ if $n_p \leq n_q$, $\mathrm{dom}(p) \subset \mathrm{dom}(q)$, $\forall\alpha \in \mathrm{dom}(p)\ \pi_p(\alpha) = \pi_q(\alpha) \cap n_p$, $\nu_p \subset \nu_q$, and for every $\langle a, m\rangle \in \mathrm{dom}(\nu_p)$, whenever l is a number in the domain of $(e_{\pi_q'' a} \setminus e_{\pi_q'' a}) \circ k_m$ then $e_{\pi_q'' a} \circ k_m(l)$ is odd if and only if $l \in \pi_q(\nu_p(a, m))$. It is not difficult to see that P is indeed an ordering. The following three claims set the stage for the application of Martin's Axiom.

Claim 4.3.11. *The poset P is c.c.c.*

Proof. Let $\langle p_\alpha \colon \alpha \in \omega_1\rangle$ be conditions in P. The usual Δ-system and counting arguments can be used to thin down the collection if necessary so that the sets $\mathrm{dom}(p_\alpha)$ for $\alpha \in \omega_1$ form a Δ-system with root b and for all $a \in [b]^n$ and all $\alpha \in \omega_1$, $T_a \cap \mathrm{dom}(p_\alpha) \subset b$. Moreover, I can require that the increasing bijection between $\mathrm{dom}(p_\alpha)$ and $\mathrm{dom}(p_\beta)$ extends to an isomorphism of p_α and p_β for every $\alpha, \beta \in \omega_1$.

I claim that any two conditions in such a collection are compatible. Indeed, whenever $\alpha, \beta \in \omega_1$, then the condition q defined by $n_q = n_{q_\alpha}$, $\pi_q = \pi_{p_\alpha} \cup \pi_{p_\beta}$ and $\nu_q = \nu_{p_\alpha} \cup \nu_{p_\beta}$ is easily checked to be a common lower bound of the conditions p_α, p_β. ☐

Claim 4.3.12. *Whenever $a \in [\kappa]^n$ and $\beta \in T_a$, the set $D_{a,\beta} = \{p \in P : a \cup \{\beta\} \subset \mathrm{dom}(p), \exists m\ \nu_p(a, m) = \beta\}$ is dense in P.*

Proof. Let $p \in P$; I must find a condition $q \leq p$ in the set $D_{a,\beta}$. For definiteness assume that $\beta \notin \mathrm{dom}(p)$. Choose $m \in \omega$ such that $\langle a, m\rangle \notin \mathrm{dom}(\nu_p)$. Consider the condition $q \leq p$ defined by $n_q = n_p$, $\pi_q = \pi_p \cup \{\langle \alpha, 0\rangle \colon \alpha \in a \setminus \mathrm{dom}(p), \langle \beta, 0\rangle\}$, $\nu_q = \nu_p \cup \{\langle a, m, \beta\rangle\}$. The condition $q \leq p$ is in the set $D_{a,\beta}$ as required. ☐

Claim 4.3.13. *For every $a \in [\kappa]^n$ and every $k \in \omega$, the set $D_{a,k} = \{p \in P : a \subset \mathrm{dom}(p)$ and the set $\bigcap \pi_p'' a$ has at least k elements$\}$ is dense in P.*

Proof. Fix a, k and let $p \in P$ be an arbitrary condition. I must find a condition $q \leq p$ in the set $D_{a,k}$. First of all, the previous claim shows that one can strengthen p to include all ordinals in a. Increasing n_p if necessary, I may also assume that $k < n_p$.

Consider the set $b = \pi_p'' a$ and the function e_b; write $k' = \mathrm{dom}(e_b)$. If $k \leq k'$ then $q = p$ will work. Otherwise, it is easy to find an increasing sequence $d = \langle m_i : k' \leq i < k\rangle\rangle$ of numbers larger than n_p such that, writing $e = e_b \cup d$, for every natural number m such that $\langle a, m\rangle \in \mathrm{dom}(\nu_p)$ and every l such that $k' \leq k_m(l) < k$, $m_{k_m(l)}$ is odd if and only if $l \in p(\nu_r(a, m))$. The condition $q \leq p$ defined by $n_q = m_{k-1} + 1$, $\mathrm{dom}(\pi_q) = \mathrm{dom}(\pi_p)$, $\forall\beta \in a\ \pi_q(\beta) = \pi_p(a) \cup \{m_i : k' \leq i < k\}$, $\forall\beta \in \mathrm{dom}(\pi_p) \setminus a\ \pi_q(\beta) = \pi_p(\beta)$, and $\nu_q = \nu_p$, is in the set $D_{a,k}$ as desired. ☐

Now, use Martin's Axiom to find a filter $G \subset P$ meeting all the dense sets obtained in the previous claims. Define the map $\pi \colon \kappa \to 2^\omega$ by setting $\pi(\alpha)$ =the characteristic function of $\bigcup_{r \in G} \pi_r(\alpha)$. It is clear that the function works as desired. ☐

One can ask whether the recursivity can be replaced by some other conditions more intimately tied to some preexisting mathematical structures.

Question 4.3.14. Let M be a Borel model on 2^ω and let $R \subset [2^\omega]^n \times 2^\omega$ be the M-algebraicity relation. Compute $\kappa(E_R)$.

4.3.4 Komjáth-Shelah theorem

The next example relies on a partition theorem discovered by Komjáth and Shelah.

Definition 4.3.15. Let $n \in \omega$ be a natural number and let R be a Borel equivalence relation on $[2^\omega]^{<\omega}$. E_{Rn} is the equivalence relation $=^+ \restriction B_{Rn}$ where $B_{Rn} = \{x \in (2^\omega)^\omega$: no finite set $a \subset \mathrm{rng}(x)$ can be written in more than $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $b\ R\ c\}$.

There is a natural directed quasiorder among such equivalence relations again. If R, S are two Borel equivalence relations on $[2^\omega]^{<\omega}$ write $R \leq S$ if there is a Borel injection $\pi\colon 2^\omega \to 2^\omega$ which induces a homomorphism of $\neg R$ to $\neg S$. If $R \leq S$ then E_{Rn} is Borel reducible to E_{Sn} as the function $x \mapsto \pi \circ x$ will be the desired Borel reduction.

Theorem 4.3.16. *Let $n \in \omega$ and let R be a Borel equivalence relation on $[2^\omega]^{<\omega}$ with countably many classes.*

1. $\kappa(E_{Rn}) \leq \aleph_{n+1}$;
2. *if R is sufficiently high in the quasiorder $\leq$ then* $\mathrm{MA}_{\aleph_n} \vdash \kappa(E_{Rn}) = \aleph_{n+1}$.

Proof. I will use the following partition theorem:

Fact 4.3.17. (Komjáth, Shelah [15]) *Let n be a nonzero natural number.*

1. *For every function $f : [\omega_n]^{<\aleph_0} \to \omega$ there is a finite set $a \subset \omega_n$ which can be written in at least $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $f(b) = f(c)\}$.*
2. *If $MA_{\aleph_n}$ holds then there is a function $f : [\omega_n]^{<\aleph_0} \to \omega$ such that every finite set $a \subset \omega_m$ can be written in at most $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $f(b) = f(c)\}$.*

For (1), use Corollary 4.1.4 to see that for every E_{Rn}-pinned name on a poset P there is a set $A \subset 2^\omega$ such that no finite set $a \subset A$ can be written in more than $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $b\ R\ c$, and $\langle P, \tau\rangle\ \bar{E}_{Rn}$ $\langle \mathrm{Coll}(\omega, A)\sigma\rangle$, where σ is the canonical name for generic enumeration of the set A. By Fact 4.3.17(1), it must be the case that $|A| < \aleph_{n+1}$. Thus τ has an $\bar{E}_{Rn}$-equivalent on a poset of size $\leq \aleph_n$ and so $\kappa(E_{Rn}) \leq \aleph_{n+1}$.

For (2), I will use the following relation R on $[2^\omega]^{<\omega}$. Define a Borel function $g : [\mathcal{P}(\omega)]^{<\aleph_0} \to [\omega]^{<\aleph_0}$ by $g(a) = \{\min(x \setminus m + 1) : x \in a\}$ if a is a set of size at least two and consists of pairwise almost disjoint sets and m is the largest number which appears in at least two of them; $g(a) = \min(x)$ if $a = \{x\}$ is a singleton; and otherwise $g(a) = 0$. Let R be the equivalence relation induced by the function g. I will show that $\mathrm{MA}_{\aleph_n}$ proves $\kappa(E_{Rn}) = \aleph_{n+1}$.

To this end, I will prove a simple coding result: Suppose that κ is a cardinal and T is an equivalence relation on $[\kappa]^{<\aleph_0}$ with countably many classes. Then there is a c.c.c. poset R adding an injection $\pi\colon \kappa \to \mathcal{P}(\omega)$ inducing a homomorphism from $\neg T$ to $\neg R$. Once this is done, use Martin's Axiom and Fact 4.3.17(2) to find an equivalence relation T on $[\kappa]^{<\aleph_0}$ such that every finite set $a \subset \omega_m$ can be written in at most $2^n - 1$ ways as $a = b \cup c$ such that $b \neq c$ and $b\ T\ c\}$. Use Martin's Axiom again to find an injection $\pi\colon \omega_n \to \mathcal{P}\omega$ inducing a homomorphism from $\neg T$ to $\neg R$, and consider the set $A = \mathrm{rng}(\pi)$. The $\mathrm{Coll}(\omega, A)$-name for the generic enumeration of the set A is an E_{Rn}-pinned name, and it has no $\bar{E}_{Rn}$-equivalent on a poset of size $< \aleph_n$ as it entails collapsing the size of A to $\aleph_0$. This means that $\kappa(E_{Rn}) \geq \aleph_{n+1}$ as required.

Towards the coding result, let $f\colon [\kappa]^{<\aleph_0} \to \omega$ be a map inducing the equivalence relation F. Let $\nu\colon [\omega]^{<\aleph_0} \to \omega$ be sufficiently generic map such that $\nu(g(0)) = f(0)$. Let P be the poset of all maps p such that

- $\mathrm{dom}(p) \subset \kappa$ is a finite set;
- for every $\alpha \in \mathrm{dom}(p)$ the value $p(\alpha)$ is a nonempty subset of ω;
- for every $\alpha \in \mathrm{dom}(p)$, $\nu(\min(p(\alpha))) = f(\alpha)$;
- for every set $a \subset \mathrm{dom}(p)$ of size at least 2 there is a number which belongs to at least two sets $p(\alpha), p(\beta)$ for $\alpha \neq \beta \in a$, and writing m for the largest such number, $p(\alpha) \setminus m + 1 \neq 0$ holds for every $\alpha \in a$, and $\nu(\{\min(p(\alpha) \setminus m + 1)\colon \alpha \in a\}) = f(a)$.

The ordering is defined by $q \leq p$ if for every $\alpha \in \mathrm{dom}(p)$, $q(\alpha)$ end-extends $p(\alpha)$, and the sets $q(\alpha) \setminus p(\alpha)$ are pairwise disjoint for $\alpha \in \mathrm{dom}(p)$. The following routine claims complete the proof.

Claim 4.3.18. *P has c.c.c.*

Proof. In fact, P is semi-Cohen in the sense of [1], but we will not need that fact here. By the usual Δ-system arguments, it is enough to show that any two conditions $p, q \in P$ such that $p \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q) = q \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q)$, are compatible. To find the lower bound, enumerate $\mathrm{dom}(p) \cup \mathrm{dom}(q)$ as β_i for $i \in k$, enumerate $(\mathrm{dom}(p) \setminus \mathrm{dom}(q)) \times (\mathrm{dom}(q) \setminus \mathrm{dom}(p))$ as u_j for $j \in l$. Use the genericity of the function ν to build numbers $m_0 < m_1 < \cdots < m_{l-1}$ and pairwise distinct numbers n_i^j for $i \in k$ and $j \in l$ so that

- $m_0 > \max(\bigcup \mathrm{rng}(p) \cup \bigcup \mathrm{rng}(q))$;
- $m_j < n_i^j < m_{j-1}$ for every $j \in l$;
- for every set $a \subset \mathrm{dom}(p) \cup \mathrm{dom}(q)$, $\nu(\{n_i^j\colon \beta_i \in a\}) = f(a)$.

The lower bound is then a function r defined by $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$, for $\alpha \in \mathrm{dom}(p)$, $\alpha = \beta_i$ set $r(\alpha) = p(\alpha) \cup \{n_i^j : j \in l\} \cup \{m_j : \alpha$ appears in the pair $u_j\}$. Similarly, for $\alpha \in \mathrm{dom}(q)$, $\alpha = \beta_i$ set $r(\alpha) = q(\alpha) \cup \{n_i^j : j \in l\} \cup \{m_j : \alpha$ appears in the pair $u_j\}$. It is not difficult to check that $r \leq p, q$ as required. □

Claim 4.3.19. *The set $D_\alpha = \{p \in P : \alpha \in \mathrm{dom}(p)\}$ is dense in P for every $\alpha \in \kappa$.*

Proof. Let $\alpha \in \kappa$ and $p \in P$; I must produce $q \leq p$ such that $\alpha \in \mathrm{dom}(q)$. Enumerate $\mathrm{dom}(p)$ as β_i for $i \in k$ and write $\alpha = \beta_k$. Use the genericity of the function ν to find numbers $m < m_0 < m_1 < \dots m_{k-1}$ and pairwise distinct n_i^j for $i \in k, j \in k+1$ so that

- $\nu(m) = f(\alpha)$ and $m_0 > \max \bigcup \mathrm{rng}(p)$;
- $m_i < n_i^j$ for every $j \in k+1$;
- for every nonempty set $a \subset \mathrm{dom}(p) \cup \{\alpha\}$ and every $i \in k$, $\nu(\{n_i^j : \beta_j \in a\}) = f(a)$.

Once this is done, just consider the function q defined by $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\alpha\}$, $s(\alpha) = \{m, m_i : i \in k, n_i^k : i \in k\}$, and for every $i \in k$, $q(\beta_i) = p(\beta_i) \cup \{m_i, n_j^i : j \in k\}$. It is not difficult to observe that $q \in P$ and $q \leq p$ as desired. □

If $G \subset P$ is a filter meeting all the dense sets D_α for $\alpha \in \kappa$ from the last claim, then let $\pi : \kappa \to \mathcal{P}(\omega)$ be defined by $\pi(\alpha) = \bigcup_{p \in G} p(\alpha)$. It is immediate that $f = \nu \circ g \circ \pi$, in particular π is a homomorphism of $\neg T$ to $\neg R$. □

4.3.5 Chang's Conjecture

As a last example I will present a simple analytic equivalence relation weakly reducible to $=^+$ whose pinned cardinal depends on the status of Chang's conjecture.

Definition 4.3.20. Let R be a Borel equivalence relation on $(2^\omega)^2$. E_R is the equivalence relation $=^+ \restriction B_R$ where $B_R = \{x \in (2^\omega)^\omega :$ there are no infinite sets $a, b \subset \mathrm{rng}(x)$ such that $a \times b$ is a subset of a single R-equivalence class$\}$.

One can again equip the resulting objects with a natural quasiorder. For Borel equivalence relations R, S on $(2^\omega)^2$ write $R \leq S$ if there is a Borel injection $\pi : 2^\omega \to 2^\omega$ which induces a homomorphism from $\neg R$ to $\neg S$. If $R \leq S$ then E_R is Borel reducible to E_S since the injection π induces the reduction $x \mapsto \pi \circ x$.

Theorem 4.3.21. *Let R be a Borel equivalence relation on $(2^\omega)^2$ with countably many classes.*

1. *if Chang's conjecture holds then $\kappa(E_R) \leq \aleph_2$;*

2. *if R is sufficiently high in the $\leq$-quasiorder then* $\mathrm{MA}_{\aleph_2} \vdash$*Chang's conjecture is equivalent with $\kappa(E_R) \leq \aleph_2$.*

Proof. Recall that the *Chang's conjecture* is the statement that every first order model of type $(\aleph_2, \aleph_1)$ has an elementary submodel of type $(\aleph_1, \aleph_0)$. The consistency of Chang's conjecture requires some modest large cardinals. We will need the following combinatorial fact:

Fact 4.3.22. (Todorcevic, [24])

1. *If Chang's conjecture holds, then for every partition of ω_2^2 into countably many pieces, one piece of the partition contains a product of infinite sets.*
2. *If* $\mathrm{MA}_{\aleph_2}$ *holds and Chang's conjecture fails, then there is a partition of ω_2^2 into countably many pieces such that no piece of the partition contains a product of infinite sets.*

To prove (1), suppose that R is a Borel equivalence relation on $(2^\omega)^2$ with countably many classes. Use Corollary 4.1.4 to see that for every E_R-pinned name τ on a poset P there is a set $A \subset 2^\omega$ such that no class of $R \restriction A^2$ contains a product of infinite sets, and $\langle P, \tau \rangle \; \bar{E}_R \; \langle \mathrm{Coll}(\omega, A), \sigma \rangle$, where σ is the name for the generic enumeration of the set A. Fact 4.3.22(1) implies that the set A has size at most $\aleph_1$. Therefore, the name τ has an $\bar{E}_R$-equivalent on a poset of size $\leq \aleph_1$. $\kappa(E_R) \leq \aleph_2$ immediately follows.

To prove (2), I will use a specific Borel equivalence R on $\mathcal{P}(\omega)^2$ Let $\omega = \bigcup_{n,m\in\omega} a_{n,m}$ be a partition of ω into infinite sets. For almost disjoint sets $b, c \subset \omega$ such that b is lexicographically less than c define $f(b, c) = n$ and $f(c, b) = m$ if $\max(b \cap c) \in a_{n,m}$, in other cases define $f(b, c) = 0$. Let R be the equivalence relation induced by the function f. I will show that $\mathrm{MA}_{\aleph_2}$ proves that the failure of Chang's conjecture implies $\kappa(E_R) > \aleph_2$.

To this end, I will prove a small coding result. For every cardinal κ and every equivalence relation T on κ^2 there is a c.c.c. poset R adding an injection $\pi\colon \kappa \to \mathcal{P}(\omega)$ which induces a homomorphism from $\neg T$ to $\neg R$. Once this is known, assume that $\mathrm{MA}_{\aleph_2}$ holds and Chang's conjecture fails. Use Fact 4.3.22(2) to produce an equivalence T on ω_2^2 with countably many classes such that neither class contains a product of two infinite sets. Use Martin's Axiom again to produce an injection $\pi\colon \omega_2 \to \mathcal{P}(\omega)$ which induces a homomorphism from $\neg T$ to $\neg R$. Let $A = \mathrm{rng}(\pi)$ and let τ be the $\mathrm{Coll}(\omega, A)$-name for the generic enumeration of the set A. This is an E_R-pinned name which has no $\bar{E}_R$-equivalent on a poset of size $\aleph_1$ since it entails the collapse of $|A|$ to $\aleph_0$. (2) follows.

Towards the proof of the coding result, fix a cardinal κ and a function $g\colon \kappa^2 \to \omega$ which induces the equivalence relation T. Define the poset P as the collection of all functions p such that

- $\mathrm{dom}(p) \subset \kappa$ is a finite set;
- $\mathrm{rng}(p)$ consists of finite subsets of ω such that neither of them is an initial segment of another;

- for every $\alpha \neq \beta$ such that $p(\alpha)$ is lexicographically smaller than $p(\beta)$, the set $p(\alpha) \cap p(\beta)$ is nonempty, and its maximum belongs to the set $a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$.

The ordering on P is defined by $q \leq p$ if $\mathrm{dom}(p) \subset \mathrm{dom}(q)$, for every $\alpha \in \mathrm{dom}(p)$ the set $p(\alpha)$ is an initial segment of $q(\alpha)$, and the sets $\{q(\alpha) \setminus p(\alpha) \colon \alpha \in \mathrm{dom}(p)\}$ are pairwise disjoint. The following routine claims complete the proof of the theorem.

Claim 4.3.23. *The poset P is c.c.c.*

Proof. By the usual Δ-system arguments it is only necessary to show that any two conditions $p, q \in P$ such that $p \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q) = q \restriction \mathrm{dom}(p) \cap \mathrm{dom}(q)$ are compatible in the poset P. Strengthening the conditions p, q on $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$ and $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$ respectively if necessary, I may assume that no set in $\mathrm{rng}(p) \cup \mathrm{rng}(q)$ is an initial segment of another. Enumerate $(\mathrm{dom}(p) \setminus \mathrm{dom}(q)) \times (\mathrm{dom}(q) \setminus \mathrm{dom}(p))$ as u_i for $i \in j$ and find pairwise distinct numbers m_i for $i \in j$ such that

- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically smaller than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\beta, \alpha) = n$;
- if $u_i = \langle \alpha, \beta \rangle$ and $p(\alpha)$ is lexicographically greater than $q(\beta)$ then $m_i \in a_{m,n}$ where $g(\alpha, \beta) = n$ and $g(\beta, \alpha) = m$;
- all numbers m_i are greater than $\max(\bigcup \mathrm{rng}(p) \cup \bigcup \mathrm{rng}(q))$.

In the end, let r be the function defined by $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$, for all $\alpha \in \mathrm{dom}(p)$ let $r(\alpha) = p(\alpha) \cup \{m_i \colon \alpha \text{ appears in } u_i\}$, and for all $\beta \in \mathrm{dom}(q)$ let $r(\beta) = q(\beta) \cup \{m_i \colon \beta \text{ appears in } u_i\}$. It is not difficult to check that r is a common lower bound of the conditions p, q as desired. □

Claim 4.3.24. *For every $\alpha \in \kappa$ the set $D_\alpha = \{p \in P \colon \alpha \in \mathrm{dom}(r)\}$ is dense in P.*

Proof. Let $\alpha \in \kappa$ be an ordinal and $p \in P$ be a condition; I must produce a condition $q \leq p$ such that $\alpha \in \mathrm{dom}(q)$. It will be the case that $q(\alpha) \cap \max(\bigcup \mathrm{rng}(p)) + 1 = 0$; this way, $q(\alpha)$ will be lexicographically smaller than all $q(\beta)$ for $\beta \in \mathrm{dom}(p)$. List $\mathrm{dom}(p)$ as β_i for $i \in j$, and find pairwise distinct numbers m_i for $i \in j$ so that

- $m_i \in a_{m,n}$ where $g(\alpha, \beta) = m$ and $g(\alpha, \beta) = n$;
- each m_i is greater than $\max(\bigcup \mathrm{rng}(p))$.

Then, let q be the function defined by $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{\alpha\}$ and $q(\beta_i) = p(\beta_i) \cup \{m_i\}$ and $q(\alpha) = \{m_i \colon i \in j\}$. It is immediate that the condition q works. □

Finally note that if $G \subset P$ is a filter meeting all the dense sets D_α for $\alpha \in \kappa$, the function $\pi \colon \kappa \to \mathcal{P}(\omega)$ defined by $\pi(\alpha) = \bigcup \{p(\alpha) : p \in G\}$ induces a homomorphism of $\neg T$ to $\neg R$ as desired. □

4.4 Characterization theorems

It seems to be quite difficult to characterize those analytic equivalence relations E such that $\kappa(E)$ attains a prescribed set of cardinal values (say $\kappa(E) \geq \mathfrak{c}^+$ or $\kappa(E) \leq \mathfrak{c}^+$) in descriptive set theoretic terms. It is a priori even not clear if the collections of equivalence relations defined in this way are say in $L(\mathbb{R})$.

4.4.1 Infinite pinned cardinal

In this subsection, I will provide such a characterization for the class of analytic equivalence relations with the largest possible value of the pinned cardinal.

Theorem 4.4.1. *Assume that there is a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. $\kappa(E) = \infty$;
2. $E_{\omega_1} \leq_{\mathrm{aB}} E$.

Proof. (2) implies (1) by Example 4.3.1 and Theorem 4.2.2. The large cardinal assumption is not needed for this direction. For the (1)→(2) implication, suppose that $\kappa(E) = \infty$. Let κ be a measurable cardinal.

Claim 4.4.2. *There is a poset P of size κ and an E-pinned P-name τ such that τ is not $\bar{E}$-related to any name on a poset of size $< \kappa$.*

Proof. Let M be an elementary submodel of a large structure such that $E, \kappa \in M$, $V_\kappa \subset M$, and $|M| = \kappa$. There must be a poset Q and an E-pinned name $\sigma \in M$ such that Q is not $\bar{E}$-equivalent to any name on a poset of size $< \kappa$. Consider $P = Q \cap M$ and $\tau = \sigma \cap M$. The transitive collapse of the model M is correct about pinned names and so τ is indeed an E-pinned name on P. The transitive collapse of the model M is also correct about the $\bar{E}$-equivalence by Theorem 5.1.1 and it contains all E-pinned names on posets of size $< \kappa$. Therefore $\langle P, \tau \rangle$ is not $\bar{E}$-equivalent to any poset of size $< \kappa$. As $|P| = \kappa$, the proof is complete. $\square$

Choose a poset P of size κ and an E-pinned name τ as in the claim. Let M be a countable elementary submodel of a large enough structure. Let Y be the space of binary relations on ω, so $Y = \mathrm{dom}(E_{\omega_1})$. By Lemma 6.2.9 and 6.2.2, there are Borel functions $f\colon Y \to Y$, $g\colon Y \times M \to \omega$, $h\colon Y \to \mathcal{P}(\omega)$ and $k\colon y \to X$ such that whenever $y \in Y$ is a wellorder then $f(y)$ is an isomorph of the iteration of the model M of length y, $g(y)$ is the iteration elementary embedding of M into $f(y)$, $h(y)$ is a filter on $g(y)(P)$ generic over $f(y)$, and $k(y) = g(y)(\tau)/h(y)$. It will be enough to show that k is a Borel reduction of E_{ω_1} to E on the set of $y \in Y$ which code well-orders.

Suppose first that $y, z \in Y$ are well-orders of the same length. Then $f(y), f(z)$ are wellfounded and isomorphic. Write N for their common transitive isomorph, $j : m \to N$ for the iteration map, and let $Q = j(P)$ and $\sigma = j(\tau)$. By the elementarity of the embedding j, $N \models \sigma$ is an E-pinned Q-name. Identify $h(y), h(z)$ with filters on Q separately generic over N, so $k(y) = \sigma/h(y)$ and $k(z) = \sigma/h(z)$. Now $k(y)\ E\ k(z)$ must hold by Proposition 4.1.1 applied to the model N.

Suppose now that $y, z \in Y$ are well-orders of different lengths; say that y is shorter than z. Let N_y be the transitive isomorph of $f(y)$, $j_y \colon m \to N_y$ the iteration map, $Q_y = j_y(P)$, $\sigma_y = j_y(\tau)$; similarly for N_z, j_z, Q_z, σ_y. Identify $h(y), h(z)$ with filters on Q_y, Q_z separately generic over N_y, N_z, so $k(y) = \sigma/h(y)$ and $k(z) = \sigma/h(z)$. Now $k(y)\ E\ k(z)$ must fail by Proposition 4.1.1 applied to the model N_z. $\square$

4.4.2 Pinned cardinal below $=^+$

Among the equivalence relations Borel reducible to $=^+$, $=^+$ is the only one whose pinned cardinal can be proved in ZFC to be equal to $\mathfrak{c}^+$. This is the contents of the following theorem:

Theorem 4.4.3. *Let $\kappa > \beth_{\omega_1}$ be a regular cardinal such that $\kappa^{\aleph_0} = \kappa$. Let P be the usual poset for adding κ Cohen reals. In the P-extension, exactly one of the following holds for every Borel equivalence relation E Borel-reducible to $=^+$:*

1. *E is bireducible with $=^+$;*
2. *$\kappa(E) \leq \mathfrak{c}$.*

Proof. It is clear that (1) implies the negation of (2). If $=^+ \leq_{\mathrm{B}} E$ then $\kappa(E) \geq \kappa(=^+) = \mathfrak{c}^+$ holds in every generic extension, and so (2) fails. I must show that the negation of (2) implies that $=^+$ is Borel-reducible to $=^+$. For simplicity assume that E is a Borel equivalence relation on $X = 2^\omega$ with a code in the ground model, and fix a Borel reduction $h : 2^\omega \to (2^\omega)^\omega$ of E to $=^+$.

Let $G \subset P$ be a filter generic over the ground model and work in $V[G]$. Since $V[G] \models \kappa(E) = \mathfrak{c}^+$, There must be a poset Q and an E-pinned name σ on the poset Q such that σ is not $\bar{E}$-equivalent to any name on a poset of size $< \mathfrak{c}$. The Q-name $\dot{h}(\sigma)$ is $=^+$-pinned and so there is a set $A \subset 2^\omega$ such that $Q \Vdash \check{A} = \mathrm{rng}(\dot{h}(\sigma))$ by Corollary 4.1.4. By Mostowski absoluteness, there is a $\mathrm{Coll}(\omega, A)$-name χ for an element of X such that $\mathrm{rng}(\dot{h}(\chi)) = A$. The name χ is $\bar{E}$-related to σ. Since σ was chosen to have no equivalents on posets of size $< \mathfrak{c}$, it follows that $|A| = \mathfrak{c}$.

Back to the ground model. The previous paragraph shows that there must be a P-name $\dot{A}$ for a subset of 2^ω of size $\mathfrak{c} = \kappa$ such that in some further forcing extension (by Mostowski absoluteness it is enough to take $\mathrm{Coll}(\alpha, \kappa)$-extension) there is an element $x \in X$ such that $\mathrm{rng}(h(x)) = \dot{A}$.

Since $\kappa > \beth_{\omega_1}$, essentially by Theorem ??? there is a countable model M elementarily equivalent to ??? such that the integers of M are isomorphic to ω and M has an infinite collection of ordered indiscernibles below κ^M. Let N be a model obtained as a Skolem hull of indiscernibles of ordertype 2^ω with the usual lexicographic ordering from M. More precisely, let $Y = \{\langle t, \vec{a}\rangle$ where for some

$n \in \omega$, t is an n-ary Skolem term and $\vec{a} \in (2^\omega)^n\}$. Let F be the equivalence relation on Y defined by $\langle t, \vec{a}\rangle \; F \; \langle s, \vec{b}\rangle$ if for some sequences $\vec{a}'\vec{b}'$ of indiscernibles in the model M ordered in the same way as $\vec{a}, \vec{b}$, $M \models t(\vec{a}') = t(\vec{b}')$. Build the model N on the collection of F-classes in the usual way: ???

The following piece of notation will be useful below. For a pair $y = \langle t, \vec{a}\rangle \in Y$, write $\text{supp}(y)$ for the intersection of all sets $b \subset 2^\omega$ such that for some Skolem term s and some sequence $\vec{c}$ with $\text{rng}(\vec{c}) \subseteq b$, $\langle s, \vec{c}\rangle \; F \; \langle t, \vec{a}\rangle$. An easy indiscernibiity argument shows that $y \mapsto \text{supp}(y)$ is a Borel function in a natural sense.

Claim 4.4.4. *N is (can be presented as) a Borel model. The natural numbers of N are isomorphic to ω.*

Proof. The equivalence relation F is clearly Borel; I will show that it is smooth. ??? □

Claim 4.4.5. *There is a filter $G \subset P^N$ generic over the model N which is moreover a Borel subset of N.*

Let $G \subset P^N$ be a filter generic over the model N. For any point $z \in (2^\omega)^\omega$, let N_z be the Skolem hull of the N-indiscernibles in the set $\text{rng}(z)$, naturally presented as a structure on ω. Note that $G_z = G \cap N_z$ is a filter generic over the model N_z: this follows from the fact that N_z is an elementary submodel of N, $N \models P^N$ is c.c.c., and all natural numbers of N belong to N_z. Let $a_z = \dot{A}^{N_z}/G_z$; this is a subset of 2^ω.

Claim 4.4.6. *For $z_0, z_1 \in (2^\omega)^\omega$, $z_0 =^+ z_1 \leftrightarrow a_{z_0} = a_{z_1}$ holds.*

Use Claim ??? to find a Borel map $z \mapsto H_z$ assigning to every point $z \in (2^\omega)^\omega$ a filter $H_z \subset \text{Coll}(\omega, \kappa)^{N_z}$ generic over the model $N_z[G_z]$. Use ??? to find a Borel map $z \mapsto x_z$ such that $x_z \in X^{N_z[G_z][H_z]}$ such that $N_z[G_z][H_z] \models \text{rng}(h(x_z)) = a_z$. By the Borel absoluteness between V and the model $N_z[G_z][H_z]$, the statement $\text{rng}(h(x_z) = a_z$ holds even in V. Finally, Claim ??? shows that the map $z \mapsto x_z$ is the desired Borel reduction of $=^+$ to E. □

4.4.3 Pinned equivalence relations in the absence of AC

Kechris [12, Question 17.6.1] conjectured that a Borel equivalence relation is pinned if and only if $=^+$ does not reduce to it. This turns out to be false under the axiom of choice, as Theorem 4.3.5 and remarks following it show. In a partial validation of the conjecture, I can prove that $=^+$ is the simplest Borel equivalence relation which can be proved to be unpinned in the theory ZF+DC. This is the contents of the following theorem and corollary:

Theorem 4.4.7. *The following holds in the Solovay model derived from a measurable cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is unpinned;*

2. $=^+ \leq_{\mathrm{B}} E$ *or* $E_{\omega_1} \leq_{\mathrm{aB}} E$.

Corollary 4.4.8. *The following holds in the Solovay model derived from a measurable cardinal. Let E be a Borel equivalence relation on a Polish space X. E is unpinned if and only if* $=^+ \leq_{\mathrm{B}} E$.

Proof. This follows from Theorem 4.4.7 once I show that the option $E_{\omega_1} \leq_{\mathrm{aB}} E$ is not available for any Borel equivalence relation E. This in turn follows easily from results on the pinned cardinal $\kappa(E)$ obtained in Section 4.2: $\kappa(E) < \infty$ by Theorem 4.2.3(1), $\kappa(E_{\omega_1}) = \infty$ by Example 4.3.1, and the pinned cardinal is monotone with respect to the reducibility ordering $\leq_{\mathrm{aB}}$–Theorem 4.2.2. □

Proof of Theorem 4.4.7. Let κ be a measurable cardinal. Let $G \subset \mathrm{Coll}(\omega, < \kappa)$ be a filter generic over V, and let $V[[G]]$ be the derived Solovay model. In $V[[G]]$, (2) certainly implies (1) as the proofs that $=^+, E_{\omega_1}$ are unpinned work in ZF, and the proof that pinned equivalence relations persist downwards in the orderings $\leq_{\mathrm{B}}$ and $\leq_{\mathrm{aB}}$ works in ZF+DC.

For the implication (1)→ (2), assume that $V[[G]] \models E$ is unpinned. There must be a poset P and an E-pinned E-nontrivial name τ on the poset P, both in $V[[G]]$. Both P and τ must be definable in $V[G]$ from a ground model parameter and a real. For simplicity assume that these reals belong to the ground model V. Return to V. Let Q be the two-step iteration $\mathrm{Coll}(\omega, < \kappa) * \dot{P}$, and write σ for the Q-name obtained from the $\dot{P}$-name τ. There are two cases.

Case 1. There is a condition $q \in Q$ such that the $Q \restriction q$-name σ is E-pinned. In this case, I will conclude that $E_{\omega_1} \leq_{\mathrm{aB}} E$ and use the Shoenfield absoluteness to transfer the almost reducibility to the Solovay model. To simplify the notation assume that q is the largest element of the poset Q.

First, observe that the name σ cannot be $\bar{E}$-equivalent to any name on a poset of size $< \kappa$. Suppose for contradiction that R is a poset of size $< \kappa$ and χ an E-pinned R-name such that $\langle R, \chi\rangle \ \bar{E} \ \langle Q, \sigma\rangle$. Let $H \subset R$ be a generic filter over V in the model $V[G]$. Proposition 4.1.1 applied to V then shows that $V[G] \models P \Vdash \tau \ E \ \chi/H$. This contradicts the assumption that $V[G] \models P \Vdash \tau$ is not E-related to any point in $V[G]$.

Now, it follows that $\kappa(E) \geq \kappa$. By Theorem 4.2.3(2), since κ is a measurable cardinal, $\kappa(E) = \infty$. By Theorem 4.4.1, $E_{\omega_1} \leq_{\mathrm{aB}} E$ as desired.

Case 2. For every condition $q \in Q$, the $Q \restriction q$-name σ is not E-pinned. In this case, I will conclude that $=^+ \leq_{\mathrm{B}} E$. Then, a Shoenfield absoluteness argument shows that the Borel reduction of $=^+$ to E remains a Borel reduction also in the Solovay model.

Fix a countable elementary submodel M of a large structure containing the code for E, the posets P, Q and the name τ. I will start with an auxiliary lemma. A collection $\langle g_i : i \in I\rangle$ of filters on $\mathrm{Coll}(\omega, < \kappa) \cap M$ is called *mutually generic* over M if for every finite set $a \subset I$ the filter $\prod_{i \in a} g_i \subset \mathrm{Coll}(\omega, < \kappa)^{|a|}$ is generic over M. For every set $a \subset I$ write $2^\omega_a = \bigcup\{2^\omega \cap M[\prod_{i \in b} g_i] : b \subset a$ finite$\}$, $M_a = M(2^\omega_a)$, P_a and τ_a for the poset and name in M_a defined in the model $M(2^\omega_a)$ by the formulas ϕ_P and ϕ_τ. Similar usage will prevail for functions $y : \omega \to I$, writing $P_y = P_{\mathrm{rng}(y)}$ etc.

Lemma 4.4.9. *Suppose that $\{g_i : i \in I\}$ is a mutually generic collection of filters on* $\mathrm{Coll}(\omega, < \kappa)$.

1. *whenever $a \subset I$ is a nonempty set then in there is a filter h $\mathrm{Coll}(\omega, < \kappa)$-generic over M such that $2^\omega \cap M[h] = 2^\omega_a$;*

2. *whenever $a, b, c \subset I$ are pairwise disjoint countable nonempty sets then there is a filter $h_a \times h_b \times h_c \subset \mathrm{Coll}(\omega, < \kappa)^3$ generic over M such that $2^\omega_a = 2^\omega \cap V[h_a]$ and similarly for 2^ω_b and 2^ω_c;*

3. *whenever a, b are distinct countable subsets of I then $P_a \times P_b \Vdash \neg\tau_a \ E \ \tau_b$.*

Proof. For (1), let $R = \{k : \exists\alpha \in \kappa \ \exists b \subset a$ finite $k \subset \mathrm{Coll}(\omega, < \alpha)$ is a filter generic over M and $k \in M[g_i : i \in b]\}$ and order R by inclusion. Let $K \subset R$ be a sufficiently generic filter; I claim that $h = \bigcup K$ works as desired. Indeed, a simple density argument shows that $h \subset \mathrm{Coll}(\omega, < \kappa)$ is an ultrafilter all of whose proper initial segments are generic over M. By the κ-c.c. of $\mathrm{Coll}(\omega, < \kappa)$, the filter h is in fact generic over M itself. A straightforward genericity argument then shows that $2^\omega_a = 2^\omega \cap M[h]$ as desired.

(2) follows easily from (1). Let $h_a, h_b, h_c \subset \mathrm{Coll}(\omega, < \kappa)$ be any filters obtained from (1); I will show that these filters are in fact mutually generic over the model V. Since $\mathrm{Coll}(\omega, < \kappa)^3$ has κ-c.c.c., it is enough to show that for every ordinal $\alpha \in \kappa$, the filters $h^\alpha_a = h_a \cap \mathrm{Coll}(\omega, < \alpha)$, $h^\alpha_b = h_b \cap \mathrm{Coll}(\omega, < \alpha)$, and $h^\alpha_c = h_c \cap \mathrm{Coll}(\omega, < \alpha)$ are mutually generic over V. Since the filters h^α_a, h^α_b and h^α_c are coded by reals in the models $M[h_a]$, $M[h_b]$, and $M[h_c]$,there are finite sets a', b', c' of a, b, c respectively such that $h^\alpha_a \in M[\prod_{i \in a'} g_i]$ etc. The mutual genericity now follows from the general Lemma 6.1.9 about product forcing.

(3) is proved in several parallel cases depending on the mutual position of the sets a, b vis-a-vis inclusion. I will treat the case in which all three sets $a \cap b$, $a \setminus b$, $b \setminus a$ are nonempty. Suppose for contradiction that $P_a \times P_b \Vdash \tau_a \ E \ \tau_b$. From (2), it follows that in V, the triple product $\mathrm{Coll}(\omega, < \kappa)^3$ forces $\dot{P}_{\{0,1\}} \times \dot{P}_{\{1,2\}} \Vdash \tau_{\{0,1\}} \ E \ \tau_{\{1,2\}}$. Then, the quadruple product $\mathrm{Coll}(\omega, < \kappa)^4$ forces in V that $\dot{P}_{\{0,1\}} \times \dot{P}_{\{1,2\}} \times \dot{P}_{\{2,3\}} \Vdash \tau_{\{0,1\}} \ E \ \tau_{\{1,2\}} \ E \ \tau_{\{2,3\}}$, in particular $\dot{P}_{\{0,1\}} \times \dot{P}_{\{2,3\}} \Vdash \tau_{\{0,1\}} \times \tau_{\{2,3\}}$. In view of (2) again, this means that the product $\mathrm{Coll}(\omega, < \kappa) \times \mathrm{Coll}(\omega, < \kappa)$ forces $\dot{P}_{\mathrm{left}} \times \dot{P}_{\mathrm{right}} \Vdash \tau_{\mathrm{left}} \ E \ \tau_{\mathrm{right}}$. In other words, the name σ is pinned on the poset Q, contradicting the case assumption. ☐

Use Lemma 6.2.8 to find a continuous map $f : 2^\omega \to \mathcal{P}(\mathrm{Coll}(\omega, < \kappa) \cap M)$ such that its range consists of mutually generic filters over M. Write $Y = (2^\omega)^\omega = \mathrm{dom}(=^+)$. It is easy to find a Borel map $g : y \to (2^\omega)^\omega$ such that for every $y \in Y$, $g(y)$ enumerates the set 2^ω_y. Use Lemmas 4.4.9(1) and 6.2.4 to find a Borel map $h : y \to \mathcal{P}(Q \cap M)$ such that for every $y \in Y$, $h(y) \subset Q$ is a filter generic over M and $\mathrm{rng}(g(y)) = 2^\omega \cap V[h_0(y)]$, where $h_0(y) \subset \mathrm{Coll}(\omega, < \kappa)$ is the filter generic over M obtained from $h(y)$. Let $k : y \to X$ be given by $k(y) = \tau / h(y)$; this is a Borel map by Lemma 6.2.3. I will show that k is a reduction of $=^+$ to E.

First, assume that $y_0, y_1 \in Y$ are $=^+$-related. Then $\mathrm{rng}(y_0) = \mathrm{rng}(y_1)$, $\mathrm{rng}(g(y_0)) = \mathrm{rng}(g(y_1))$, and so $M_{y_0} = M_{y_1}$, $P_{y_0} = P_{y_1}$ and $\tau_{y_0} = \tau_{y_1}$. Let $H \subset P_{y_0}$ be a filter generic over both countable models $M_{y_0}[k(y_0)]$ and $M_{y_1}[k(y_1)]$ and let $x = \tau_{y_0}/H$. By the forcing theorem applied in the model $M_{y_0} = M_{y_1}$ and the fact that τ_{y_0} is an E-pinned name, conclude that $x \mathrel{E} k(y_0)$ and $x \mathrel{E} k(y_1)$ and so $k(x_0) \mathrel{E} k(x_1)$ as desired.

Second, assume that $y_0, y_1 \in Y$ are not $=^+$-related. Choose a sufficiently generic filter $H_0 \times H_1 \subset P_{y_0} \times P_{y_1}$ so that H_0 is generic over $M_{y_0}[k(y_0)]$ and H_1 is generic over $M_{y_1}[k(y_1)]$. As the names τ_{y_0} and τ_{y_1} are E-pinned, the forcing theorem in the models M_{y_0} and M_{y_1} implies that $k(y_0) \mathrel{E} \tau_{y_0}/H_0$ and $k(y_1) \mathrel{E} \tau_{y_1}/H_1$. Now, $\tau_{y_0}/H_0 \mathrel{E} \tau_{y_1}/H_1$ fails by Lemma 4.4.9(3), and so $k(y_0) \mathrel{E} k(y_1)$ must fail as well. This completes the proof.

□

4.5 Restrictions on forcings

It is now natural to ask which forcings can carry nontrivial E-pinned names. After all, every pinned name encountered in Section 4.1 lived on a collapse poset, leading to a natural conjecture that almost no forcing sophistication is really necessary in finding pinned names. However, this conjecture is false in general, and the complete analysis of the question depends on the equivalence relation E in question. I start with a standard definition and a general theorem.

Definition 4.5.1. (Foreman, Magidor [4]) A poset P is *reasonable* if for every ordinal λ and for every function $f\colon \lambda^{<\omega} \to \lambda$ in the P-extension there is a set $a \subset \lambda$ which is closed under f, belongs to the ground model, and it is countable in the ground model.

In particular, all c.c.c. and all proper forcings are reasonable. Good examples of unreasonable forcings are posets which collapse $\aleph_1$, Namba forcing and Prikry forcing.

Theorem 4.5.2. *Let E be an analytic equivalence relation on a Polish space X.*

1. *If E is not pinned then there is a nontrivial E-pinned name on every poset collapsing $\aleph_1$ to $\aleph_0$;*
2. *if $V = L$ then there are no nontrivial E-pinned names on $\aleph_1$-preserving posets;*
3. *there are no nontrivial E-pinned names on reasonable posets;*
4. *if E is an orbit equivalence relation then there are no nontrivial E-pinned names on $\aleph_1$-preserving posets.*

Proof. For (1), let τ be a nontrivial E-pinned name on some poset P. Let $\langle M_\alpha \colon \alpha \in \omega_1\rangle$ be a continuous $\in$-tower of countable elementary submodels of a large structure containing X and E. Let $M_{\omega_1} = \bigcup_\alpha M_\alpha$, let $Q = P \cap M_{\omega_1}$ and let $\sigma = \tau \cap M_{\omega_1}$.

Observe that in any generic extension, whenever $G, H \subset Q$ are generic filters over M, then $\sigma/G\ E\ \sigma/H$ and their equivalence class contains no ground model elements. The first part of this statement follows from Proposition 4.1.1 applied to the transitive collapse of the model M_{ω_1}. For the second part of the statement, it is enough to show that in V, $Q \Vdash \sigma$ is not E-equivalent to any element of the ground model. Suppose for contradiction that there is a point $x \in X$ such that $Q \Vdash \sigma\ E\ \check{x}$. I will show that there then must be $y \in M_{\omega_1} \cap X$ which is E-related to x. Then $Q \Vdash \sigma\ E\ \check{y}$, by the Mostowski absoluteness between the Q-extensions of M_{ω_1} and V $M_{\omega_1} \models P \Vdash \tau\ E\ \check{y}$, and this contradicts the elementarity of the model M_{ω_1} and the nontriviality of the name τ.

To find the point $y \in M_{\omega_1} \cap X$, let N be a countable elementary submodel of a large structure containing $\langle M_\alpha \colon \alpha \in \omega_1\rangle, Q, x$. Since the tower of models $\langle M_\alpha \colon \alpha \in \omega_1\rangle$ is continuous, there is a limit ordinal $\alpha \in \omega_1$ such that $M_\alpha = N \cap M_{\omega_1}$. Let $Q_\alpha = Q \cap M_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \sigma \cap M_\alpha = \tau \cap M_\alpha$. By elementarity of the model N and analytic absoluteness between the Q_α-extension of N and V, $Q_\alpha \Vdash \sigma_\alpha\ E\ \check{x}$. Since $Q_\alpha = P \cap M_\alpha$ and $\sigma_\alpha = \tau \cap M_\alpha$, both Q_α, τ_α belong to the model $M_{\alpha+1}$. By the elementarity of the model $M_{\alpha+1}$, there must be a point $y \in X \cap M_{\alpha+1}$ such that $Q_\alpha \Vdash \sigma_\alpha\ E\ \check{y}$ (since the point x is such). By the transitivity of E, it follows that $x\ E\ y$. The point $y \in M_{\alpha+1} \subset M_{\omega_1}$ works.

Now, suppose that R is a poset collapsing $\aleph_1$. Since $|M_{\omega_1}| = \aleph_1$, in the R-extension there is a filter Q-generic over M_{ω_1}. Let $\dot{H}$ be an R-name for such a filter and let ν be the R-name for $\sigma/\dot{H}$. I have just proved that ν is a nontrivial E-pinned name on R, proving (1).

For (2), assume that $V = L$ and let P be an $\aleph_1$-preserving poset and τ an E-pinned name on P. Let $G \subset P$ be a filter generic over V and work in $V[G]$. I must find a point $x_1 \in X \cap V$ which is E-related to $x_0 = \tau/G$.

Let M be a countable elementary submodel of a large structure containing P, τ. Let N be the transitive isomorph of $M \cap V$. Let $\pi : m \cap V \to N$ be the transitive collapse map. Note that $N = L_\alpha$ for some countable ordinal α, in particular $N \in V$ and it is countable there. By elementarity, $N \models \pi(\tau)$ is an E-pinned name on the poset $\pi(P)$. The filter $H_0 = \pi''G \subset \pi(P)$ is generic over N and $x_0 = \pi(\tau)/H_0 \in X$. Also, there is a filter $H_1 \subset \pi(P)$ generic over N which belongs to V, and the point $x_1 = \pi(\tau)/H_1 \in X$ also belongs to V. By Proposition 4.1.1 applied to the model N, $x_0\ E\ x_1$ as desired.

For (3), suppose that P is a reasonable poset and τ is an E-pinned name on P. I will produce a condition $p \in P$ and a point $x \in X$ such that $p \Vdash \tau\ E\ \check{x}$. Towards this end, choose a large structure and use the reasonability of P to find a countable elementary submodel M of it containing P, E and τ and a condition $p \in P$ such that $p \Vdash \dot{G} \cap \check{M}$ is generic over $\check{M}$, where $\dot{G}$ is the canonical P-name for its generic ultrafilter. As M is countable, there is a filter $H \subset P \cap M$ generic over M in the ground model V. Let $x = \tau/H \in X$. Proposition 4.1.1 applied to the model M and the filters H and $\dot{G} \cap M$ now says that $p \Vdash \check{x}\ E\ \tau$, completing the proof of (3).

(4) is much more difficult, and it is proved in Corollary 4.5.8. □

Theorem 4.5.12 provides a consistent example of a (simple) Borel equivalence relation E such that there is a nontrivial E-pinned name on Namba forcing. The following remains open though.

Question 4.5.3. Can there be an analytic equivalence relation E and a poset P such that every countable set of ordinals in the P-extension is covered by a countable set of ordinals in the ground model, and there is a nontrivial E-pinned name on P?

Now, the stage is set for the investigation of the suspicion that every pinned name is in fact a collapse name.

Theorem 4.5.4. *Suppose that E is an equivalence relation on a Polish space X, almost reducible to an orbit equivalence relation. Suppose that τ is an E-pinned name on a poset P. The following are equivalent for every poset Q:*

1. $Q \Vdash |\kappa(\tau)| = \aleph_0$;
2. *there is an E-pinned name on Q which is $\bar{E}$-related to τ.*

Thus, all orbit equivalence relations behave a little bit like the model-theoretic examples of Section 4.1–their pinned names seem to be connected to the collapse of the cardinality of a certain structure to $\aleph_0$.

Proof. Without loss of generality, assume that $|P| = |\kappa(\tau)|$. The proof of the (1)→(2) implication is much easier and does not use the orbit equivalence assumption. Suppose that $V[H]$ is a generic extension in which $|\kappa(\tau)| = \aleph_0$; it will be enough to find a point $x \in V[H]$ such that $V[H] \models P \Vdash \tau\ E\ \check{x}$. Back in the ground model, find an elementary submodel M of a large enough structure containing P, τ such that $|M| = \kappa$ and $P \subset M$. In the model $V[H]$, there is a filter $K \subset P$ generic over M. Let $x = \tau/K$. It follows from Proposition 4.1.1 applied to M that $P \Vdash \tau\ E\ \check{x}$.

The implication (2)→(1) is the heart of the matter. The proof starts with a preliminary general claim.

Claim 4.5.5. *Suppose that R is a poset such that $R \Vdash \check{R}$ has a countable dense subset. Then R is in the forcing sense equivalent to* $\mathrm{Coll}(\omega, \kappa)$ *where κ is the minimum size of a dense subset of R.*

Proof. Let $\dot{x}$ be a R-name for an enumeration of a countable dense subset of R. Then $D = \{q \in R \colon \exists p \in R\ \exists n \in \omega\ p \Vdash \check{q} = \dot{x}(n)\}$ must be a dense subset of R and as such has size at least κ. In the R-extension, the set D is equal to $\{q \in R \colon \exists p \in \mathrm{rng}(x)\ \exists n \in \omega\ p \Vdash \check{q} = \dot{x}(n)\}$ which is clearly countable. Thus, $R \Vdash |\kappa| = \aleph_0$. Now, every poset collapsing its own density character to $\aleph_0$ is in the forcing sense equivalent to the collapse poset by a classical theorem of McAloon [11, Lemma 26.6] □

Now, some notation. Let $\Gamma \curvearrowright X$ be a Polish group action inducing E as its orbit equivalence relation. Write P_Γ for the Cohen poset on the acting group Γ, adding a generic group element $\dot\gamma$. Write $\hat\tau$ for the $P \times P_\Gamma$-name $\dot\gamma \cdot \tau$ and $\hat P$ for the complete subalgebra of $RO(P \times P_\Gamma)$ generated by the name $\hat\tau$. Note that $\langle \hat P, \hat\tau\rangle \; \bar E \; \langle P, \tau\rangle$.

Claim 4.5.6. *Suppose that $V[H]$ is a generic extension containing a point $x \in X$ such that $P \Vdash \tau \; E \; \check x$. In $V[H]$, $P_\Gamma \Vdash \dot\gamma \cdot \check x$ is $\hat P$-generic over $V[H]$. Moreover, for every condition $\hat p \in \hat P$ there is a condition $q \in P_\Gamma$ forcing the filter generated by $\dot\gamma \cdot \check x$ to contain $\hat p$.*

Proof. Let $p \in P$ and $q \in P_\Gamma$ be arbitrary conditions. Let $K \subset P$ be a generic filter over $V[H]$ and write $y = \tau/K$. It will be enough to find points $\gamma, \delta \in \Gamma$ P_Γ-generic over the model $V[H][K]$, such that $\delta \in q$ and $\gamma \cdot x = \delta \cdot y$. For then, γ is P_Γ-generic over $V[H]$ and $\gamma \cdot x$ is a $\hat P$-generic point over $V[H]$ meeting the projection of the condition $\langle p, q\rangle$ into $\hat P$.

By the assumption on the point x, it is the case that $V[H][K] \models x \; E \; y$, and there is an element $\varepsilon \in \Gamma \cap V[H][K]$ such that $\varepsilon \cdot x = y$. Let $\delta \in q$ be a P_Γ-generic point over $V[H][K]$ and let $\gamma = \delta\varepsilon$. As the topology on the group Γ is invariant under translations, γ is also P_Γ-generic over $V[H][K]$, and $\gamma \cdot x = \delta\varepsilon \cdot x = \delta \cdot y$ as required. □

Claim 4.5.7. *$\hat P$ is in the forcing sense equivalent to* $\mathrm{Coll}(\omega, \kappa)$ *for some cardinal κ.*

Proof. Since τ is an E-pinned name, $\hat P \Vdash P \Vdash \hat\tau \; E \; \tau$. Thus, by Claim 4.5.6, in the $\hat P$-extension the algebra $RO(\hat P)$ is completely embedded into $RO(P_\Gamma)$ and therefore $\hat P$ contains a countable dense set. The proof is completed by a reference to Claim 4.5.5. □

Now suppose that $V[H]$ is a generic extension containing a point $x \in X$ such that $P \Vdash \tau \; E \; \check x$. By Claim 4.5.6, in $V[H]$ the poset P_Γ adds a $\hat P$-generic and therefore by Claim 4.5.7, it collapses κ to $\aleph_0$. Since P_Γ is just the Cohen poset, this can happen only if $|\kappa| = \aleph_0$ holds already in $V[H]$. □

The theorem has interesting corollaries.

Corollary 4.5.8. *If E is an orbit equivalence relation and P is an $\aleph_1$-preserving poset, then every E-pinned name on P is trivial.*

Proof. Let τ be an E-pinned name. Since P preserves $\aleph_1$, the cardinal $\kappa(\tau)$ must be equal to $\aleph_0$. Thus, $\mathrm{Coll}(\omega, \omega)$ must contain an E-pinned name $\bar E$-related to τ. However, $\mathrm{Coll}(\omega, \omega)$ is just the Cohen forcing, therefore reasonable, and so all E-pinned names in it are trivial by Theorem 4.5.2. □

To conclude this section, I isolate a simple Borel equivalence relation which may have a nontrivial pinned name on an $\aleph_1$-preserving poset. This feature leads to a strong ergodicity result of this equivalence relation vis-a-vis all orbit equivalence relations.

Definition 4.5.9. The *mutual domination* Borel equivalence relation E on $X = (\omega^\omega)^\omega$ connects points $x, y \in X$ if for every $n \in \omega$ there is $m \in \omega$ such that $y(m)$ modulo finite dominates $x(n)$ and vice versa, for every $n \in \omega$ there is $m \in \omega$ such that $x(m)$ modulo finite dominates $y(n)$.

The main properties of the equivalence E are easy to verify.

Theorem 4.5.10. *Let E be the mutual domination equivalence relation.*

1. $\kappa(E) = \mathfrak{c}^+$*;*
2. *in some forcing extension, E has a nontrivial pinned name on an $\aleph_1$-preserving forcing.*
3. (Kechris, Macdonald) E_{K_σ} *is Borel reducible to E,*

Proof. For (1), to show that $\kappa(E) \geq \mathfrak{c}^+$, let $C \subset \mathcal{P}(\omega)$ be an almost disjoint family of size continuum and let $A \subset \omega^\omega$ be the set of characteristic functions of finite unions of sets in C. Consider the poset $\text{Coll}(\omega, A)$ and its name σ for an enumeration of the set A. I will show that σ is not $\bar{E}$-related to any name on a poset of size $< \mathfrak{c}$. Towards a contradiction, let Q be such a poset and τ be such a name. Each entry of τ is dominated by a characteristic function of a finite union of some elements of c, and by the small size of Q there is $c \in C$ which is forced by C not to appear in any of these finite unions. Then the characteristic function of c is forced to be not dominated by any element of τ, contradicting the assumption $\tau \mathrel{\bar{E}} \sigma$.

To show that $\kappa(E) \leq \mathfrak{c}^+$, I will classify the E-pinned names. It will be enough to show that for every E-pinned name τ on a poset P there is a set $A \subset \omega^\omega$ closed downwards in the modulo finite ordering such that $\langle P, \tau \rangle \mathrel{\bar{E}} \langle \text{Coll}(\omega, A), \sigma \rangle$, where σ is the name for a generic enumeration of the set A. Since $|\text{Coll}(\omega, A)| \leq \mathfrak{c}$, this will complete the proof.

Thus, suppose that τ is a pinned name on some poset P. Let $A = \{z \in \omega^\omega : \exists p\ p \Vdash \exists n\ \check{z}$ is modulo finite dominated by $\tau(n)\}$. Since the name τ is E-pinned, we have that in fact $A = \{z \in \omega^\omega : P \Vdash \exists n\ \check{z}$ is modulo finite dominated by $\tau(n)\}$. To see that the set A works as required, it is enough to show that $P \Vdash \forall n\ \exists z \in a\ \tau(n)$ is modulo finite dominated by z. Suppose for contradiction that $p \in P$ is a condition and $n \in \omega$ is a number such that $p \Vdash \tau(n)$ is not modulo finite dominated by any function in a. Use the assumption that τ is pinned to find $m, l \in \omega$ and $q \in P$ and strengthen p if necessary so that $\langle p, q \rangle \Vdash \forall k > l\ \tau_{\text{left}}(n)(k) \leq \tau_{\text{right}}(m)(k)$. Let $z \in \omega^\omega$ be a function that assigns to each $k \in \omega$ the maximal number $h \in \omega$ such that there is $p' \leq p$ forcing $\tau(n)(k) = h$ if such number exists. Note that for all $k > l$, such number must exist as otherwise it would be possible to find $p', q' \leq \langle p, q \rangle$ such that $\langle p', q' \rangle \Vdash \tau_{\text{left}}(n)(k) > \tau_{\text{right}}(m)(k)$. Now, $z \notin A$ since p forces $\tau(n)$ to be modulo finite dominated by z. This means that $q \Vdash \tau(m)$ does not dominate $\check{z}$ modulo finite and so there is $q' \leq q$ and $k > l$ such that $q' \Vdash \tau(m)(k) < z(k)$. Find $p' \leq p$ forcing $\tau(n)(k) = \check{z}(k)$; then $\langle p', q' \rangle \Vdash \tau_{\text{left}}(n)(k) > \tau_{\text{right}}(m)(k)$, contradicting the assumed properties of $\langle p, q \rangle$.

For (2), let $V[G]$ be some forcing extension in which there is a modulo finite increasing sequence $z = \langle z_\alpha \colon \alpha \in \omega_2 \rangle$ of elements of ω^ω. Work in $V[G]$. Let P be the Namba forcing, adding a cofinal sequence $\pi \colon \omega \to \omega_2^V$ and let τ be the P-name for the composition $z \circ \pi$. It is well-known that Namba forcing preserves $\aleph_1$. The P-name τ is E-pinned: for any two functions $f, g \colon \omega \to \omega_2$ with cofinal rangle, the compositions $z \circ f$ and $z \circ g$ are E-related. Finally, the name τ is nontrivial. If $P \Vdash \tau \mathrel{E} \check{x}$ for some $x \in X$, then the function $g \colon \omega \to \omega_2$ given by $g(n)$ =the least $\alpha \in \omega_2$ such that z_α modulo finite dominates $x(n)$ would have to have cofinal range in ω_2.

For (3), let $m \mapsto (m_0, m_1)$ be a bijection between ω and $\omega \times \omega$. Let $h \colon \omega^\omega \to X$ be the Borel function defined by $h(y)(n)(m) = 1$ if $y(m_0) + n > m_1$, and $h(y)(n)(m) = 0$ otherwise. I claim that this is a reduction of E_{K_σ} to E.

Indeed, if $y, z \in \omega^\omega$ are E_{K_σ}-equivalent, then there is a number n' such that $y + n'$ at all entries dominates the function z and $z + n'$ at all entries dominates the function y. For every number $n \in \omega$ then, $h(y)(n + n')$ dominates $h(z)(n)$ and $h(z)(n + n')$ dominates $h(y)$, and therefore $h(y) \mathrel{E} h(z)$. On the other hand, if $y, z \in \omega^\omega$ are not E_{K_σ}-equivalent, then either there is no n such that $y + n$ dominates z, or there is no n such that $z + n$ dominates y. Suppose for definiteness that the former is the case, and observe that $h(z)(0)$ is not modulo finite dominated by any $h(y)(n)$ for any $n \in \omega$. □

Towards the statement of the ergodicity result, I will first identify a σ-ideal naturally associated with the equivalence relation E.

Definition 4.5.11. Let $A \subset X$ be a set. The game $G(A)$ is played by Players I and II alternately choosing points $y_n \in \omega^\omega$ for $n \in \omega$ so that y_{n+1} modulo finite dominates y_n for every $n \in \omega$. Player I wins if the sequence of Player II's points belongs to A. The *mutual domination ideal* is generated by sets $X \setminus A$ where $A \subset X$ is an analytic set such that Player I has a winning strategy in $G(A)$.

It is immediate that countably many strategies for Player I can be compounded into one which transcends them all, and so the mutual domination ideal is a σ-ideal.

Theorem 4.5.12. *Let E be the mutual domination equivalence relation on the space $X = (\omega^\omega)^\omega$. If F is an orbit equivalence relation on a Polish space Y and $h \colon x \to Y$ is a Borel homomorphism from E to F, then there is an $y \in Y$ such that $X \setminus h^{-1}[y]$ belongs to the mutual domination ideal.*

Proof. I will first prove the theorem under the additional assumption that there is a scale of length ω_2. Suppose that $z = \langle z_\alpha \colon \alpha \in \omega_2 \rangle$ is a scale: a modulo finite increasing, dominating sequence in ω^ω. Let P be the Namba forcing adding an increasing map $\pi \colon \omega \to \omega_2^V$ with cofinal range. Let τ be the P-name for $z \circ \pi$.

First, observe that the E-invariant coanalytic sets in the ideal $I_\tau = \{C \subset X \colon P \Vdash \tau \in \dot{C}\}$ belong to the mutual domination ideal. To see this, note that whenever $A \subset X$ is E-invariant analytic such that $P \Vdash \tau \in \dot{A}$, then Player I has a winning strategy in the game $G(A)$. As the play of the game proceeds, on the side he will produce a sequence of countable elementary submodels M_n of some large structure such that the n-th moves of both players belong to M_n and the $n+1$-st moves dominate all elements of $\omega^\omega \cap M_n$ modulo finite. In the end, let $M = \bigcup_n M_n$ and let $H \subset P \cap M$ be a generic filter. By the forcing theorem in the model M, $M \models \tau/H \in A$. By the Mostowski absoluteness between $M[H]$ and V, $\tau/H \in A$ holds even in V. Now, both $\mathrm{rng}(\tau/H)$ and the sequence $x \in X$ obtained by Player II are cofinal in $\omega^\omega \cap M$ and therefore E-equivalent. As the set $A \subset X$ is E-invariant, $x \in A$ and Player I won.

Second, the name τ is E-pinned. Since F is an orbit equivalence relation and P preserves $\aleph_1$, by Corollary 4.5.8 the relation F is P-pinned. By Theorem 2.3.1 the homomorphism h stabilizes on a set whose complement is in the ideal I_τ and so in the mutual domination ideal.

This completes the proof under the assumption that there is a scale of length ω_2. The rest of the proof consists of an absoluteness argument pulling this proof into pure ZFC context. This requires insight into the definability of the mutual domination ideal as described in the following two general claims.

Claim 4.5.13. *Let $A \subset X$ be an analytic E-invariant set. The game $G(A)$ is determined.*

Proof. Let $A \subset X$ be an analytic E-invariant set. Fix a continuous function $f\colon \omega^\omega \to X$ such that $\mathrm{rng}(f) = A$. Let $H(A)$ be the unraveled version of the game $G(A)$: with each move y_{2n} of the game $G(A)$ Player I indicates a number $m_n \in \omega$, and he wins if $f(m_n : n \in \omega) = (y_{2n} : n \in \omega)$. The game $H(A)$ is closed for Player I, therefore determined by the Gale–Stewart theorem. If Player I has a winning strategy in $H(A)$, the same strategy without revealing the additional numbers will win for Player I in $G(A)$, since the set A is E-invariant and so $\langle y_{2n} : n \in \omega\rangle \in A$ iff $\langle y_{2n+1} : n \in \omega\rangle \in A$. It will be enough to show that if Player II has a winning strategy in $H(A)$ then he has a winning strategy in the game $G(A)$ as well.

Indeed, if σ is a winning strategy for Player II in the game $H(A)$, he can transform it into a winning strategy τ which disregards the numbers m_n. Just let τ answer a given finite sequence of moves of Player I in $G(A)$ with a function in ω^ω that modulo finite dominates all answers the strategy σ can give to the same sequence enriched with some choices of natural numbers in $H(A)$. Since there are only countably many natural numbers, this is possible. Such a strategy τ must be winning for Player II. Indeed, if $\langle y_n : n \in \omega\rangle$ is any counterplay against τ that Player I won, then there would have to be an element $z \in \omega^\omega$ such that $f(z) = \langle y_{2n} : n \in \omega\rangle$, and then Player I would also win against the strategy σ with the moves $y_{2n+1} : n \in \omega$ and $z(n) : n \in \omega$. Now observe that τ must be a winning strategy for Player II not only in the game $H(A)$ but also in $G(A)$ by the E-invariance of the set A. □

Claim 4.5.14. *The membership of E-invariant coanalytic sets in the mutual domination ideal is evaluated correctly by transitive models of ZFC.*

Proof. Assume that $A \subset X$ is an E-invariant analytic set and M is a transitive model of ZFC containing a code for A. Assume for contradiction that M and V

evaluate the membership of $X \setminus A$ in the mutual domination ideal differently, and by Proposition 4.5.13 this means that different players have winning strategies in M and V. For definiteness assume that $M \models$Player I has a winning strategy σ in $G(A)$ and $V \models$Player II has a winning strategy τ in $G(A)$.

Within the model M, find a countable elementary submodel N of a large structure containing the code for the set $A \subset X$. Within N, consider the finite support iteration P of length ω of Hechler forcing, adding a generic sequence $\dot{x} \in X$ and ask whether $P \Vdash \dot{x} \in \dot{A}$ or not. The statement has to be decided by the largest condition in P as the set A is E-invariant and therefore closed under finite modifications of its elements. I will produce two filters $G, H \subset P$ over N such that $\dot{x}/G \in A$ and $\dot{x}/H \notin A$. Then, by Mostowski absoluteness $M[G] \models \dot{x}/G \in A$ and $M[H] \models \dot{x}/H \notin A$, and this contradicts the forcing theorem.

Towards the construction of the filter G, work in the model M. Let $\langle D_n : n \in \omega\rangle$ enumerate all open dense subsets of P in the model N, and by induction on $n \in \omega$ build filters g_n and conditions $q_n \in P$ such that:

- writing $N_n = N[g_i : i \in n]$, the filter g_n is Hechler-generic over N_n;
- $p = q_0 \geq q_1 \geq \dots$ are conditions in P such that $q_{n+1} \in D_n$ and $p_n \restriction n \in g_0 * g_1 * \dots * g_{n-1}$;
- writing x_n for the n-th Hechler real derived from g_n, x_n modulo finite dominates the function $\sigma(x_i : i \in n)$.

This is not difficult to do. Given p_n and the filters g_i for $i \in n$, use the fact that Hechler forcing is a Suslin forcing to find a filter g_n Hechler-generic over N_n such that $p_n(n) \in g_n$ and its generic real dominates the function $\sigma(x_i : i \in n)$. By the genericity of the filter $g_0 * \dots * g_n$ on the iteration of Hechler forcing of length $n+1$, there must be a condition $q_{n+1} \leq q_n$ in the model N such that $q \restriction n+1 \in g_0 * \dots * g_n$ and $q_{n+1} \in D_{n+1}$. This completes the induction step.

In the end, the filter $G \subset P$ generated by the conditions q_n for $n \in \omega$ is P-generic over M. The sequence $\dot{x}$ is a legal counterplay of Player II against the strategy σ by the last item of the induction hypothesis. As the strategy σ was winning for Player I and the set A is E-invariant, $\dot{x}/G \in A$.

The construction of the filter $H \subset P$ is exactly the same, except now the work is performed in V against the strategy τ. This completes the proof. $\square$

Now, back to the proof of the theorem. Argue in ZFC. Let M be a countable elementary submodel of a large enough structure containing the codes for Y, F, h. Let $M[g]$ be a generic extension of M in which there is a scale of length ω_2; e.g. $M[g]$ is an extension of M by a finite support iteration of length ω_2 of Hechler forcing. By the Shoenfield absoluteness between M and $M[g]$, $M[g] \models h$ is a homomorphism from E to F. By the initial work with a scale, there is $y \in Y \cap M[g]$ such that $M[g] \models X \setminus h^{-1}[y]_F$ is in the mutual domination ideal. Now, the wellfounded model $M[g]$ is correct about membership of the complement of the E-invariant analytic set $h^{-1}[y]_F$ in the mutual domination ideal I by Claim 4.5.14 and therefore even in V, $X \setminus h^{-1}[y]_F \in I$ holds. The proof is complete. $\square$

Chapter 5

Absoluteness

The concepts introduced in this book depend not just on the Polish spaces and equivalence relations, but on the set theoretic universe as a whole. Therefore, a natural question arises: to which extent are the concepts surrounding trimness absolute between transitive models of set theory? While the previous sections did not really need any resolution to this question, it is to be expected that the absoluteness concerns will surface sooner or later as the subject grows.

5.1 Names

In this section, I will provide several theorems showing that many basic properties of names introduced in this book are absolute.

Theorem 5.1.1. *Suppose that E is an analytic equivalence relation on a Polish space X, M is a transitive model of set theory containing the code for X and E as well as posets P, Q and a P-name τ and a Q-name σ for elements of the space X. The following statements are absolute between M and V:*

1. *τ is E-symmetric;*
2. *$\langle P, \tau\rangle \bar{E} \langle Q, \sigma\rangle$.*

Proof. For (1), suppose first that $V \models \tau$ is E-symmetric. Let $p_0, p_1 \in P$ be conditions. Pass to a generic extension $V[K]$ in which $\mathcal{P}^2(P) \cap M$ is countable and the formula ϕ holds, where ϕ says "there are filters $G_0, G_1 \subset P$ meeting all open dense subsets of P in the model M such that $p_0 \in P_0$, $p_1 \in G_1$ and $\tau/G_0\ E\ \tau/G_1$". Such an extension exists by the assumption. In the model $V[K]$, find a filter $H \subset \mathrm{Coll}(\omega, \mathcal{P}(P) \cap M)$ generic over the model M. By the analytic absoluteness between the transitive models $V[K]$ and $M[H]$, the model $M[H]$ satisfies ϕ as well. This shows that $M \models \tau$ is E-symmetric.

Suppose now that $M \models \tau$ is E-symmetric. Let $p_0, p_1 \in P$ be conditions. In the model M, there is a poset Q and Q-names $\dot{G}_0, \dot{G}_1$ for filters on P generic over M such that Q forces that $\check{p}_0 \in \dot{G}_0$, $\check{p}_1 \in \dot{G}_1$, and $\tau/\dot{G}_0\ E\ \tau/\dot{G}_1$ all hold.

Let $K \subset Q$ be a filter generic over V. The filters $H_0 = \dot{G}_0/K$, $H_1 = \dot{G}_1/K$ are both generic over V, $p_0 \in H_0$, $p_1 \in H_1$ holds, and by the analytic absoluteness between the transitive models $M[K]$ and $V[K]$, $\tau/H_0\ E\ \tau/H_1$ holds as well. This proves that $V \models \tau$ is E-symmetric.

(2) now follows immediately from (1) and the fact that the statement $\langle P, \tau\rangle\ \bar{E}\ \langle Q, \sigma\rangle$ is equivalent to $\tau \dot{\cup} \sigma$ being E-symmetric on $P \dot{\cup} Q$ by Corollary 2.1.8. □

Certain trimness properties of names are absolute between transitive models of set theory, but others are not. This is explained by the following theorems and examples.

Theorem 5.1.2. *Suppose that E is an analytic equivalence relation on a Polish space X, M is a transitive model of set theory containing the code for X and E as well as a poset P and a P-name τ for an element of the space X. The statement "τ is E-pinned" is absolute between M and V.*

Proof. The statement "τ is E-pinned" is equivalent to the statement that τ is E-symmetric and $P \times P \Vdash \tau_{\text{left}}\ E\ \tau_{\text{right}}$ by Proposition 4.1.1. The first conjunct is absolute between M and V by Theorem 5.1.1, and the second item is absolute between M and V by the analytic absoluteness between their $P \times P$ extensions. □

Theorem 5.1.3. *Suppose that E is an analytic equivalence relation on a Polish space X, M is a transitive model of set theory containing the code for X and E as well as a poset P and a P-name τ for an element of the space X. The statement "τ is E-trim" is absolute between M and V.*

Proof. Suppose first that $V \models \tau$ is E-trim. Let $p_0, p_1 \in P$ be arbitrary conditions. By the assumption, in some generic extension $V[H]$ there are filters $G_0, G_1 \subset P$ separately generic over V such that $p_0 \in G_0, p_1 \in G_1, \tau/G_0\ E\ \tau/G_1$, and $V[G_0] \cap V[G_1] = V$. The two filters G_0, G_1 are also separately generic over M, and I claim that $M[G_0] \cap M[G_1] = M$. To see this, let $\sigma \in M$ be any P-name and observe the following chain of implications: $\sigma/G_0 \notin M$ implies $\sigma/G_0 \notin V$ (by the genericity of the filter G_0 over V), which implies $\sigma/G_0 \notin V[G_1]$ (as $V[G_0] \cap V[G_1] = V$), and that implies $\sigma/G_0 \notin M[G_1]$ (as $M \subset V$).

Now, let θ be a large enough cardinal in the model M and let $K \subset \text{Coll}(\omega, \theta)$ be a filter generic over $V[H]$. Use the Mostowski absoluteness between the model $M[K]$ and $V[H][K]$ to see that in the model $M[K]$, there are filters $\bar{G}_0, \bar{G}_1 \subset P$ which are separately generic over M, $p_0 \in \bar{G}_0, p_1 \in \bar{G}_1$, $\tau/\bar{G}_0\ E\ \tau/\bar{G}_1$, and $M[\bar{G}_0] \cap M[\bar{G}_1] \cap \mathcal{P}(P) = M \cap \mathcal{P}(P)$. Such filters exist in $M[K]$ since they exist in $V[H][K]$–the filters G_0, G_1 are such as I just checked. However, this implies that $M \models \tau$ is E-trim as desired.

The reverse implication requires a claim of independent pure forcing interest.

Claim 5.1.4. *Let C be a complete Boolean algebra and A, B its complete subalgebras. Let $\dot{G}$ be the C-name for the generic filter. The statement $C \Vdash V[\dot{G} \cap A] \cap V[\dot{G} \cap B] = V$ is equivalent to a first order statement about a dense Boolean subalgebra of C closed under the projection functions into A and B.*

Proof. I will first show that the following are equivalent:

1. $C \Vdash V[\dot{G} \cap A] \cap V[\dot{G} \cap B] = V$;

2. for densely many $c \in C$, the algebra $A_c \cap B_c$ contains an atom, where $A_c = \{a \wedge c\colon a \in A\}$ and similarly for B_c.

Observe that the intersection of two complete subalgebras is again a complete subalgebra, so $A_c \cap B_c$ is in fact a complete subalgebra of C_c.

Suppose first that (2) fails; i.e. there is $c \in C$ such that the algebra $A_c \cap B_c$ is atomless. Let $\dot{H}$ be the name for $\dot{G} \cap A_c \cap B_c$. Then $c \Vdash_C \dot{H} \notin V$, since $\dot{H}$ is a filter on an atomless Boolean algebra generic over V. Also, $c \Vdash_C \dot{H} \in V[\dot{G} \cap A]$, since it can be reconstructed there as the set $\{a \wedge c\colon a \in \dot{G} \cap \check{A}, a \cap c \in \check{B}_c\}$. For the same reason $c \Vdash_C \dot{H} \in V[\dot{G} \cap B]$ and so (1) fails.

Suppose on the other hand that (2) holds and $c \in C$ and τ is an A-name for a set of ordinals and σ is a B-name for a set of ordinals and $c \Vdash_C \tau = \sigma$. To prove (1), I must find a condition $d \leq c$ that decides the membership of all ordinals in τ; this will show that $d \Vdash \tau \in V$ and by the obvious density and $\in$-minimalization arguments it will prove the independence of A and B. Note that for every ordinal α, the Boolean values $|\check{\alpha} \in \tau|$ and $|\check{\alpha} \in \sigma|$ are the same in C_c. Since τ is an A_c-name, it must be the case that $|\check{\alpha} \in \tau| \in A_c$; since σ is an B_c-name, it must be the case that $|\check{\alpha} \in \sigma| \in B_c$ and so these Boolean values belong to $A_c \cap B_c$. Thus, if $d \leq c$ is an atom of $A_c \cap B_c$, d must decide the membership of every ordinal in τ (and σ) and so $d \Vdash \tau \in V$ as desired.

The proof of the claim will be complete if I show that the statement "$A \cap B$ has an atom" is equivalent to a first order statement about a dense subalgebra of C. A definition will be helpful here. For a nonzero element $c \in C$, let $f(c) \subset C$ be the collection of conditions $\{c_n\colon n \in \omega\}$ where $c_0 = c$, c_{2n+1} =projection of c_{2n} into A, and c_{2n+2} =the projection of c_{2n+1} into B. Note that $f(c)$ is a decreasing sequence of nonzero conditions in C, in which elements of A and B alternate, and so its infimum (possibly zero) is in the algebra $A \cap B$ and it is the largest element of that algebra below c.

Suppose that $C' \subset C$ is a dense subalgebra closed under the projection functions. The statement "$A \cap B$ has an atom" is equivalent to the statement ϕ ="there is a nonzero $d \in C'$ such that for every $c \in C'$ either the infimum of $f(c)$ is above d or else it is incompatible with d". To see this, if $A \cap B$ has an atom then any nonzero $d \in C'$ below that atom will witness ϕ. On the other hand, suppose that some nonzero element $d \in C'$ witnesses ϕ. Form $b = \prod \bigcup \{f(c)\colon c \in C'$ and all elements of $f(c)$ are above $d\} \in C$. This is some element of C above d; by the definitions of the sets $f(c)$, $d \in A \cap B$ holds. It will be enough to argue that b is an atom of $A \cap B$.

If this failed, then $b = b_0 \vee b_1$ for some disjoint elements $b_0, b_1 \in A \cap B$. Then, there would have to be some $a \in C'$ such that a is below b_0 and b_1 is compatible with d (or vice versa). Let $c = 1 - a$ and note that the conditions of $f(c)$ are all above b_1, since $b_1 \in A \cap B$ and $b_1 \leq c$. It follows that the infimum of $f(c)$ is compatible with d. Since d witnesses the statement ϕ, all elements of $f(c)$ are above d, and by the definition of b, they are also above b. This is impossible since already c is not above b. $\square$

Now, suppose that $M \models \tau$ is E-trim. Let $p_0, p_1 \in P$ be arbitrary conditions. By the assumption, in the model M there is a poset Q and names $\dot{G}_0$ and $\dot{G}_1$ such that Q forces $\check{p}_0 \in \dot{G}_0, \check{p}_1 \in \dot{G}_1$, $\dot{G}_0, \dot{G}_1 \subset P$ are separately generic over M, $\tau/\dot{G}_0 \ E \ \tau/\dot{G}_1$, and $M[\dot{G}_0] \cap M[\dot{G}_1] = M$. The claim shows that Q forces $\dot{G}_0, \dot{G}_1 \subset P$ to be separately generic over V and $V[\dot{G}_0] \cap V[\dot{G}_1] = V$. This proves that $V \models \tau$ is E-trim as desired. $\square$

The theorem has numerous consequences generally justifying the common preoccupation with the Cohen poset and/or the meager ideal for proving ergodicity results.

Corollary 5.1.5. *Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is Cohen-trim;*
2. *E is P-trim for every proper poset P.*

Proof. Only (1)→(2) needs proof. Suppose that (2) fails as witnessed by some proper poset P and a nontrivial E-trim name τ on P. Let M be a countable elementary submodel of a large enough structure containing E, P, τ. Let $\bar{M}$ be the transitive collapse of M and $\bar{P}, \bar{\tau}$ the images of P, τ under the transitive collapse. By Theorem 5.1.3, $\bar{\tau}$ is an E-trim name on the poset $\bar{P}$, which is countable, and therefore in the forcing sense isomorphic to Cohen forcing. It will be enough to show that the name $\bar{\tau}$ is nontrivial.

To this end, suppose that there is a condition $\bar{p}$ forcing $\bar{\tau} \ E \ \check{x}$ for some $x \in X$. This means that in the model $\bar{M}$, the name $\bar{\tau}$ is an E-pinned name–all generics come from the equivalence class $[x]_E$. Now, in the model $\bar{M}$, the poset $\bar{P}$ is proper, it can carry no nontrivial E-pinned names by Theorem 4.5.2 and therefore the name $\bar{\tau}$ is E-trivial in $\bar{M}$. This contradicts the initial assumption that the name τ was E-nontrivial and the elementarity of the model M. $\square$

Corollary 5.1.6. *Let E be an analytic equivalence relation on a Polish space X, almost reducible to an orbit equivalence relation. The following are equivalent:*

1. *E is Cohen-trim;*
2. *E is P-trim for every $\aleph_1$-preserving poset P.*

Proof. This is the same argument as in Corollary 5.1.5 with the additional piece of information that $\aleph_1$-preserving posets do not carry any nontrivial E-pinned names by Corollary 4.5.8. $\square$

Corollary 5.1.7. *Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is trim;*

2. *E is Cohen-trim and pinned.*

Proof. For (1)→(2), suppose that E is trim. Then E is Cohen-trim by the definitions. To show that E is pinned, just note that every E-pinned name is in fact an E-trim name by the product forcing theorem. Thus, every E-pinned name must be trivial and E is pinned.

For (2)→(1), suppose that (2) holds and τ is an E-trim name on a poset P; I must show that τ is trivial. Let M be a countable elementary submodel of a large enough structure containing E, P, τ. Let $\bar{M}$ be the transitive collapse of M and $\bar{P}, \bar{\tau}$ the images of P, τ under the transitive collapse. By Theorem 5.1.3, $\bar{\tau}$ is an E-trim name on the poset $\bar{P}$, which is countable, and therefore in the forcing sense isomorphic to Cohen forcing. As E is Cohen-trim, the name $\bar{\tau}$ is E-trivial, forced to be E-related to some $\check{x}$. This means that the name $\bar{\tau}$ is E-pinned in the model $\bar{M}$. Since the equivalence relation E is pinned in $\bar{M}$, there must be a point $y \in \bar{M} \cap X$ such that $\bar{P} \Vdash \bar{\tau} \ E \ \check{y}$. By the elementarity of the model M, $P \Vdash \tau \ E \ \check{y}$ holds and the name τ is E-trivial as desired. □

Not all natural trimness properties of names are absolute between models of set theory with choice. The simplest example concerns the σ-trimness concept.

Example 5.1.8. There are generic extensions $V[G] \subset V[H]$ and a poset P and a P-name τ in the model $V[G]$ so that

1. $V[G] \models \tau$ is E_1-symmetric, σ-trim name;

2. $V[H] \models \tau$ is not E_1-σ-trim.

Proof. This follows from the results of Section 3.3.3. There is a generic extension $V[G]$ in which E_1 is not σ-trim, as witnessed by some E_1-symmetric σ-trim name τ on some poset P by Theorem 3.3.15. Pass to a larger generic extension $V[H]$ in which the poset P has size $\aleph_0$. On posets of size $\aleph_0$ there are no E_1-σ-trim names by Theorem 3.3.18. This completes the proof. □

A more complex situation may appear when the poset P is understood as a definition as opposed to a fixed set. This occurs commonly throughout the book–various trim names are found, say, on the random poset. It turns out that as long as the poset has a sufficiently simple definition, the trimness of names found on it is suitably absolute.

Theorem 5.1.9. *Suppose that E is an analytic equivalence relation on a Polish space X, P is a Suslin c.c.c. poset, M is a transitive model of set theory containing all ordinals, the code for X, E, P as well as a P-name τ for an element of the space X. The statements "τ is an E-symmetric name", "τ is an E-trim name" and "τ is an E-σ-trim name" are absolute between M and V.*

Proof. First of all, note that τ remains a name for an element of Polish space X since all of the maximal antichains mentioned in it remain maximal by the Suslinness of P and analytic absoluteness between M and V. I will show that "τ is an E-symmetric name on P" is equivalent to the statement "for every countable wellfounded model M containing the codes for E, P, τ, it is the case that $M \models \tau$ is E-symmetric" (and similarly for trimness and σ-trimness). The latter statement is clearly $\mathbf{\Pi}^1_2$ and so the theorem will follow by a reference to the Shoenfield absoluteness.

To prove the equivalence, first assume that τ is not E-symmetric. In such a case, just let M be a countable elementary submodel of a large enough structure containing E, P, τ and note that $M \models \tau$ is not E-symmetric. For the other direction, assume that τ is E-symmetric and M is a wellfounded model containing the codes for E, P, τ; I must show that $M \models \tau$ is E-symmetric (and similarly for trimness and σ-trimness).

The argument for E-symmetricity is easier. Suppose that $p_0, p_1 \in P \cap M$ are conditions. By the E-symmetricity assumption in V, in some generic extension $V[K]$ there are filters $G_0, G_1 \subset P$ generic over V, containing the respective conditions p_0, p_1, and such that $\tau/G_0 \ E \ \tau/G_1$. Note that the filters $G_0 \cap M$ and $G_1 \cap M$ are generic over M by the Suslinness of the poset P. Now, let $\kappa = \mathfrak{c}^M$, and let $H \subset \mathrm{Coll}(\omega, \kappa)$ be a filter generic over $V[K]$. By the Mostowski absoluteness between the models $M[H]$ and $V[K][H]$ there are in the model $M[H]$ filters $\bar{G}_0, \bar{G}_1 \subset P \cap M$, separately generic over M, such that $p_0 \in \bar{G}_0$, $p_1 \in \bar{G}_1$, $\tau/\bar{G}_0 \ E \ \tau/\bar{G}_1$, since the filters $G_0 \cap M$ and $G_1 \cap M$ are such. This, however, verifies the E-symmetricity of the name τ in the model M as desired.

The argument for trimness and σ-trimness starts with two general claims.

Claim 5.1.10. *Suppose that P is a Suslin poset and σ is a P-name for a set of ordinals and $p \in P$ is a condition such that $p \Vdash \sigma \notin V$. Then for every poset Q, $Q \Vdash p \Vdash \sigma \notin V[\dot{G}]$.*

Here, $\dot{G}$ is a canonical name for the generic filter on Q. As P is a Suslin poset, σ is again in the Q-extension a name for a set of ordinals.

Proof. Suppose for contradiction that some condition $q \in Q$ forces the opposite. Let $G_0, G_1 \subset Q$ be filters mutually generic over V, both containing the condition q. Let $H \subset P$ be a filter generic over $V[G_0, G_1]$, containing the condition p. Since P is a Suslin forcing, the filters $H, H \cap V[G_0], H \cap V[G_1]$ are generic over the models $V, V[G_0], V[G_1]$, still containing the condition p. Write $z = \sigma/H = \sigma/H \cap V[G_0] = \sigma/H \cap V[G_1]$. The forcing theorem applied in V now shows that $z \notin V$, and the forcing theorem applied in $V[G_0]$ and $V[G_1]$ shows that $z \in V[G_0]$ and $z \in V[G_1]$. By the product forcing theorem though, $V[G_0] \cap V[G_1] = M$, a contradiction. □

Claim 5.1.11. *Suppose that $p \in P \cap M$, $\sigma \in M$ is a P-name for a set of ordinals, and $M \models p \Vdash \sigma \notin V$. Then $V \models p \Vdash \sigma \notin V$.*

Proof. Suppose for contradiction that there is a condition $q \leq p$ forcing $\sigma \in V$. Strengthen q if necessary to find a particular $z \in V$ such that $q \Vdash \sigma = \check{z}$. Let

$G_0 \subset P$ be a filter generic over V containing q, and let $G_1 \subset P$ be a filter generic over $V[G_0]$ containing q. By the Suslinity of P, $z = \sigma/G_0 = \sigma/G_1$. By the Suslinity of P again, the filter $G_0 \cap M$ is P-generic over M, and the filter $G_0 \cap M[G_0 \cap M]$ is P-generic over $M[G_0 \cap M]$. Also, $z = \sigma/G_1 \in M[G_0 \cap M]$, contradicting Claim 5.1.10 in the model M. □

I will now continue the proof for simple trimness, the case of σ-trimness being entirely parallel. As τ is trim in V, in some generic extension $V[K]$, there are filters $G_0, G_1 \subset P$ separately generic over V such that $\tau/G_0 \; E \; \tau/G_1$, and $V[G_0] \cap V[G_1] = V$.

Claim 5.1.12. *$G_0 \cap M$ and $G_1 \cap M$ are filters separately generic over $M[H]$. Moreover, $M[G_0 \cap M] \cap M[G_1 \cap M] = M$.*

Proof. The first sentence follows from the fact that P is a Suslin forcing. The second sentence follows from Claim 5.1.11. Whenever $\sigma \in M$ is a P-name for a set of ordinals, it is either the case that $\sigma/G_0 \in M$ or else $\sigma/G_0 \notin V$. In the latter case, $\sigma/G_0 \notin V[G_1]$ as $V[G_0] \cap V[G_1] = V$, and consequently $\sigma/G_0 \notin M[G_1 \cap M]$. □

Now let $\kappa = \mathfrak{c}^M$, and let $H \subset \mathrm{Coll}(\omega, \kappa)$ be a filter generic over $V[K]$. By the Mostowski absoluteness between the models $M[H]$ and $V[K][H]$ there are in the model $M[H]$ filters $\bar{G}_0, \bar{G}_1 \subset P \cap M$, separately generic over M, such that $p_0 \in \bar{G}_0$, $p_1 \in \bar{G}_1$, $M[\bar{G}_0] \cap M[\bar{G}_1] \cap \mathcal{P}(P \cap M) = (\mathcal{P}(P))^M$ and $\tau/\bar{G}_0 \; E \; \tau/\bar{G}_1$, since by the claim the filters $G_0 \cap M$ and $G_1 \cap M$ are such. This, however, verifies the E-trimness of the name τ in the model M as desired. □

5.2 Equivalence relations

In this section, I will show that certain trimness properties of equivalence relations are absolute between models of set theory with choice. The most interesting case concerns the pinned property.

Theorem 5.2.1. *Suppose that E is a Borel equivalence relation on a Polish space X. Let M be a transitive model of set theory with choice containing the codes for E and X. The statement "E is pinned" is absolute between M and V.*

Proof. For simplicity assume $X = \omega^\omega$. I will show that in ZFC, the statement "E is pinned" is equivalent to "for every countable ω-model M of ZFC containing the code for E, $M \models E$ is pinned". The latter statement is clearly $\mathbf{\Pi}^1_1$. The proof of the theorem is then concluded by a reference to Mostowski absoluteness.

Now, if E is unpinned, then any countable elementary submodel of a large enough structure will satisfy that E is unpinned. The opposite direction is more difficult. Fix a countable ω-model M of ZFC containing the code for E such that $M \models E$ is unpinned. In the model M, find a nontrivial E-pinned name τ_0 on some poset Q_0. Form a transfinite sequence of models and a commuting system of elementary embeddings $\langle M_\alpha, Q_\alpha, \tau_\alpha, j_{\beta\alpha} \colon \beta < \alpha \leq \omega_1 \rangle$ so that

1. $M_0 = M$, for every countable α M_α is a countable ω-model of ZFC, and $j_{\beta\alpha}(Q_\beta, \tau_\beta) = Q_\alpha, \tau_\alpha$;
2. for limit α the model M_α is obtained as a direct limit of the earlier models;
3. if there is $x \in X$ such that $Q_\alpha \Vdash \tau_\alpha\ E\ \check{x}$ then there is such an x in the model $M_{\alpha+1}$.

After the induction is performed, I will show that τ_{ω_1} is a nontrivial E-pinned name on Q_{ω_1}, and therefore E is indeed unpinned and (1) fails.

The successor step of the induction is arranged through the following theorem of ZFC applied in the model M_α:

Claim 5.2.2. *Whenever P is a poset, then in some generic extension there is an elementary embedding $j : V \to W$ into a possibly illfounded ω-model W such that W contains $j''P$ as well as some subset of $j''P$ whose j-preimage is a P-generic filter over V.*

Proof. Consider the set $Y = [P \cup \mathcal{P}(P)]^{\aleph_0}$ and functions f, g with domain Y such that $f(a) = a \cap P$ and $g(a)$ is some filter on $a \cap P$ which meets all open dense subsets of $a \cap P$ in the set a. Let I be the σ-ideal of nonstationary subsets of the set Y and consider the poset $R = \mathcal{P}(Y)$ modulo I and the associated generic ultrapower $j : V \to W$. It is easy to see that the functions f, g represent the desired elements in the generic ultrapower: $[f] = j''P$ and $[g]$ is a filter on $j''P$ meeting $j(D)$ for every open dense subset $D \subset P$ in the ground model. $\square$

Now working in M_α, find a poset R forcing the existence of the elementary embedding as above for $P = Q_\alpha$. Let R' be the disjoint union of R and Q_α with the ordering defined by $r \leq q$ if $r \Vdash j(q) \in \dot{g}$. Then Q_α is a regular subposet of R' and R is a dense subset of R'. Now suppose that $Q_\alpha \Vdash \tau_\alpha\ E\ \check{x}$; perhaps the point x is not in the model M_α. Let N be a countable elementary submodel of a large enough structure and let $h \subset R'$ be generic over N. Then $h \subset R'$ is also generic over M_α. Let $j = j_{\alpha\alpha+1} : m_\alpha \to M_{\alpha+1}$ be the generic embedding obtained by an application of the claim in M_α. Let $g = h \cap Q_\alpha$. Then $\tau/g\ E\ x$ by the forcing theorem applied in the model N and the Mostowski absoluteness between $N[h]$ and V. Also, $\tau/g \in M_{\alpha+1}$ since $j''g \in M_{\alpha+1}$ and τ/g is reconstructed as the unique point $y \in \omega^\omega$ such that for every $n \in \omega$, $y(n) = m$ if there is $p \in j''g$ which forces in the poset jQ_α that $j(\tau)(n) = m$.

Once the induction is performed, consider the poset $Q = Q_{\omega_1}$ as well as the name $\tau = \tau_{\omega_1}$ from the point of view of V as opposed to the model M_{ω_1}. Theorem 5.1.2. shows that τ is an E-pinned name on Q. I will complete the proof by showing that τ is nontrivial–again as viewed in V as opposed to M_{ω_1}.

Suppose that there is a point $x \in X$ such that $Q \Vdash \tau\ E\ \check{x}$. Let N be a countable elementary submodel of a large enough structure containing the iteration and the point x and write $\alpha = \omega_1 \cap N$. Since the model M_{ω_1} is a direct limit of the tower of earlier models, the transitive collapse π of $N \cap M_{\omega_1}$ is an isomorphism of Q_α, τ_α with $Q \cap N, \tau \cap N$. Whenever $g \subset Q \cap N$ is a filter generic over V, it must be the case that $N[g] \models x\ E\ \tau/g$ by the forcing theorem applied in N, and $V[g] \models x \mathrel{E} \tau_\alpha/\pi''g = \tau/g$ by the Mostowski absoluteness between $V[g]$ and $N[g]$. Thus, $Q_\alpha \Vdash \tau_\alpha \mathrel{E} \check{x}$, and by the inductive assumption there is a point $y \in X \cap M_{\alpha+1}$ which is E-related to x. Then, $Q \Vdash \tau \mathrel{E} \check{y}$. By the Borel absoluteness between the Q-extension of V and M_{ω_1}, it must be the case that $M_{\omega_1} \models Q \Vdash \tau \mathrel{E} \check{y}$. This contradicts the fact that $M_0 \models \tau_0$ is a nontrivial E-pinned name together with the elementarity of the embedding $j_{0\omega_1}$. □

The absoluteness of the pinned status of an analytic equivalence relation is a considerably more difficult question. It does not hold in ZFC alone as the following rather primitive constructible example shows.

Example 5.2.3. In the constructible universe, there is an analytic equivalence relation E which is pinned, while in some generic extension it becomes unpinned.

Proof. The domain of E consists of structures with universe ω and language including one binary relation $\in$ and one ternary relation symbol R. The equivalence relation is defined by the following formula: $x \mathrel{E} y$ if either x, y both fail to be wellfounded models of the $L_{\omega_1\omega}$ sentence $\phi \wedge \psi$ where ϕ says "$V = L_\alpha$ and $\mathcal{P}(\omega)$ exists" and ψ says "for every infinite ordinal β, $R(\beta, \cdot, \cdot)$ orders the ordinals smaller than β in ordertype ω", or x is isomorphic to y. It is not difficult to see that E is indeed an analytic equivalence relation.

Now in L, there are no uncountable wellfounded models of $\phi \wedge \psi$, since ψ implies that such model would have to be L_{ω_1}, and $L_{\omega_1} \models \mathcal{P}(\omega)$ does not exist. This means that in L, E is pinned. On the other hand, in the $\mathrm{Coll}(\omega, \omega_1)$-extension of L, the relation E becomes unpinned, as $\phi \wedge \psi$ has an uncountable wellfounded model ($L_{\omega_2^L}$ with an appropriate relation R) and the $\mathrm{Coll}(\omega, \omega_2^L)$-name for an isomorph of this model with universe ω is a nontrivial E-pinned name. □

Still, in the presence of sufficiently large cardinals the pinned status of every analytic equivalence relation is absolute:

Theorem 5.2.4. *Assume that there is a measurable cardinal larger than a Woodin cardinal. Let E be an analytic equivalence relation on a Polish space X. The following are equivalent:*

1. *E is pinned;*
2. *for every wellfounded, linearly iterable model M of ZFC+there is a measurable cardinal larger than a Woodin cardinal containing the code for E, $M \models E$ is pinned.*

Proof. The argument is entirely parallel to the proof of Theorem 5.2.1 and I will only indicate the minor changes. In the proof of (1)→(2), From a failure of (2) extract a countable wellfounded linearly iterable model $M_0 \models E$ is not pinned, and build the iteration so that the encountered models M_α are wellfounded. This is possible by using the stationary tower forcing associated with the Woodin cardinal at successor stages of the construction [17, Theorem 2.7.7]. In the end, the model M_{ω_1} is wellfounded as well, and by the Mostowski absoluteness it and its generic extensions are correct about the analytic equivalence relation E. This is all that is necessary to push the essentially identical argument through. $\square$

Corollary 5.2.5. *Suppose that there are class many Woodin cardinals. Let E be an analytic equivalence relation on a Polish space X. Then E is pinned if and only if every poset forces E to be pinned.*

Proof. The truth value of the statement (2) in Theorem 5.2.4 does not change if one considers countable models only. This follows from a downward Löwenheim-Skolem argument. The countable version of (2) is $\mathbf{\Sigma}^1_3$ and so the corollary follows from $\mathbf{\Sigma}^1_3$-absoluteness between the ground model and any forcing extension under the given large cardinal assumption. $\square$

Finally, it is possible to move towards absoluteness of trimness.

Theorem 5.2.6. *Let E be an analytic equivalence relation on a Polish space X. The truth value of the statement "E is Cohen-trim" is the same in all forcing extensions.*

Proof. Theorem 5.1.3 immediately implies that the statement "E is Cohen-trim" is equivalent to the statement "for every countable wellfounded model M of ZFC, $M \models E$ is Cohen-trim". This latter statement is clearly $\mathbf{\Pi}^1_2$ and therefore absolute among all forcing extension by the Shoenfield absoluteness. $\square$

Corollary 5.2.7. *Let E be a Borel equivalence relation on a Polish space X. The truth value of the statement "E is trim" is the same in all forcing extensions.*

Proof. The statement "E is trim" is equivalent to the conjunction of E being Cohen-trim and pinned by Corollary 5.1.7. The Cohen-trimness is absolute by Theorem 5.2.6 and the pinnedness is absolute by Theorem 5.2.1. $\square$

Absoluteness of other trimness variations is open to question. Note that Theorem 3.3.15 shows that E_1 fails to be σ-trim in some forcing extension, and at the same time I do not know if E_1 fails to be σ-trim provably in ZFC.

Question 5.2.8. Let E be a Borel equivalence relation on a Polish space X. Is the truth value of the statement "E is σ-trim" the same in all forcing extensions?

5.3 The pinned cardinal

The computation of the cardinal $\kappa(\tau)$ for an E-symmetric name τ depends on many circumstances and as such it is not absolute between transitive models of set theory. The only absoluteness feature I can see is contained in the following theorem.

Theorem 5.3.1. *Let E be an analytic equivalence relation on a Polish space X almost reducible to an orbit equivalence relation. Let M be a transitive model of large portion of set theory containing the codes for E and the almost reduction and let $P, \tau \in M$ be a poset and an E-symmetric name on it. If $M \models \tau$ is pinned then $(\kappa(\tau))^V \geq |(\kappa(\tau))^M|$.*

Proof. The proof uses the transitivity of the model M in one fairly tricky abstract point. Let $M \models B$ is a complete Boolean algebra, completely generated by a set $A \subset B$. Then $RO(B)$ is a complete Boolean algebra completely generated by A again. To see this, observe that every element of B is obtained by a transfinite series of Boolean operations from the set A in the model M, which is also a transfinite series of operations in the algebra $RO(B)$ in V, and these operations are evaluated in the same way in B as in $RO(B)$ as B is dense in $RO(B)$. It follows that A generates a dense subset of $RO(B)$ in V, and therefore it generates $RO(B)$ as well.

Let $\Gamma \curvearrowright Y$ be a Polish group action, F its orbit equivalence relation, and $h\colon X \to Y$ a Borel almost reduction of E to F such that the codes of these objects belong to the model M. Suppose that $\langle P, \tau \rangle$ is an E-pinned name in the model M and let $\kappa = \kappa(\tau)^M$; note that $\kappa = \kappa(h\tau)^M$ as well by Theorem 4.2.2(1). I have to argue that $\langle P, \tau \rangle$ does not have an $\bar{E}$-equivalent on a poset of size $< |\kappa|$ even in V.

Suppose for contradiction that $\langle Q, \eta \rangle$ is such an equivalent. Let $\chi = h\eta$; this is a name $\bar{F}$-equivalent to $h\tau$. Let P_Γ be the Cohen poset on the group Γ and $\dot{\gamma}_{\text{gen}}$ its name for a generic point in Γ. It follows from Claim 4.5.6 that the point $\dot{\gamma}_{\text{gen}} \cdot h\eta$ added by $Q \times P_\Gamma$ is $P \times P_\Gamma$-generic over V and therefore also over the smaller model M. By Theorem 4.5.4 applied in the model M, $|\kappa|$ is collapsed to $\aleph_0$ in the $Q \times P_\Gamma$-extension. This contradicts the fact that $|Q \times P_\Gamma| < |\kappa|$. $\square$

Corollary 5.3.2. *Let E be an analytic equivalence relation on a Polish space X almost reducible to an orbit equivalence relation. Let M be a transitive model of large portion of set theory containing the codes for E and the almost reduction. Then $\kappa(E)^V \geq |\kappa(E)^M$.*

Chapter 6

Appendix

6.1 Forcing basics

I use the standard textbook [11] as a reference for basic forcing terminology and facts. I will start with a definition of a Cohen forcing associated with a specific Polish space.

Definition 6.1.1. Let X be a Polish space. The *Cohen* poset P_X consists of nonempty open subsets of X ordered by inclusion.

Note that for any choice of countable basis for X, the basis is dense in P_X and therefore the poset P_X has countable density. In the common case of a perfect space X, the poset P_X has no atoms and it is therefore in forcing sense equivalent to Cohen forcing. It adds a single element of the space X, typically denoted by $\dot{x}_{gen}$, which belongs to the intersection of all open sets in the generic filter. A point $x \in X$ is P_X-generic over a model of ZF if and only if it belongs to all open dense subsets of X coded in the model.

Definition 6.1.2. Let A be a nonempty set. The *collapse* $\mathrm{Coll}(\omega, A)$ is the poset consisting of all finite partial functions from ω to A ordered by reverse inclusion.

The following fact summarizes the commonly known properties of the collapse used in this book.

Fact 6.1.3. [11, Lemma 26.7, Corollary 26.8] *Let κ be an infinite cardinal.*

1. *P is up to forcing equivalence the only poset of size κ which forces $|\kappa| = \aleph_0$;*
2. *if P is a partial order of size $\leq \kappa$ then P can be regularly embedded into $RO(\mathrm{Coll}(\omega, \kappa))$;*
3. *if P is any partial order of size $< \kappa$ regularly embedded in $RO(\omega, \kappa)$ then the remainder forcing is isomorphic to* $\mathrm{Coll}(\omega, \kappa)$.

Definition 6.1.4. Let κ be an ordinal. The *Lévy collapse* $\mathrm{Coll}(\omega, <\kappa)$ is the finite support product $\prod_{\alpha\in\kappa}\mathrm{Coll}(\omega,\alpha)$. If κ is a strongly inaccessible cardinal and $G \subset \mathrm{Coll}(\omega, <\kappa)$ is generic over V, then $V[[G]]$ denotes the *Solovay model*, which is the collection of all sets hereditarily definable in $V[G]$ from elements of V and reals.

Fact 6.1.5. *Let κ be a strongly inaccessible cardinal.*

1. $\mathrm{Coll}(\omega, <\kappa)$ *is κ-c.c. and it forces $\check{\kappa} = \aleph_1$;*
2. *the truth value of every formula with parameters in V is decided by the largest condition in* $\mathrm{Coll}(\omega, \kappa)$*;*
3. *whenever $G \subset \mathrm{Coll}(\omega, <\kappa)$ is a filter generic over V and $H \in V[G]$ is a countable set of ordinals then $V[G]$ is a $\mathrm{Coll}(\omega, <\kappa)$ extension of $V[H]$;*
4. *whenever $G_0, G_1 \subset \mathrm{Coll}(\omega, <\kappa)$ are filters separately generic over V and a is a transitive set which belongs to both $V[G_0]$ and $V[G_1]$ and is countable in both, and ϕ is a formula with parameters in V and a, then the truth value of ϕ is the same in $V[G_0]$ and $V[G_1]$;*
5. $V[[G]] \models$ *ZF+DC.*

To see why (4) holds, note that each model $V[G_0]$, $V[G_1]$ is an extension of the (possibly choiceless) model $V(a)$ by the two-step iteration $\mathrm{Coll}(\omega, a) * \mathrm{Coll}(\omega, <\kappa)$ by (3). Now, the first step of that iteration is a homogeneous poset, and the other is a homogeneous poset in the ground model. Therefore, the truth value of any formula with parameters in $V(a)$ is decided by the largest condition in the iteration. Ergo, if ϕ is a formula with parameters in $V \cup a$, then $V[G_0] \models \phi$ iff $V(a) \models \mathrm{Coll}(\omega, a) * \mathrm{Coll}(\omega, \kappa) \Vdash \phi$ iff $V[G_1] \models \phi$, proving (4).

The regular embeddings of posets and basic properties of names for generic filters are used throughout the book as well.

Fact 6.1.6. *Suppose that P, Q are posets and $p \in P$ and τ is a P-name such that $p \Vdash \tau \subset \check{Q}$ is a filter generic over V. Then*

1. *for every $p' \leq p$ there is $q \in Q$ such that for every $q' \leq q$ in Q there is $p'' \leq p'$ such that $p'' \Vdash \check{q}' \in \tau$;*
2. *if $p' \in P$ and $q \in Q$ are as in item (1) and, in some generic extension, $H \subset Q$ is a filter generic over V containing q, then in some further extension there is a filter $G \subset P$ generic over V containing p such that $H = \tau/G$.*

In the product forcing, if P, Q are posets and, in some forcing extension, $G \subset P$ and $H \subset Q$ are filters such that $G \times H$ is a filter generic over V, then the filters G, H are called *mutually generic*. The following fact characterizes mutual genericity:

Fact 6.1.7. (Product forcing theorem, [11, Lemma 15.9]) *Let P, Q be posets and, in some generic extension, $G \subset P$ and $H \subset Q$, be filters. The following are equivalent:*

1. *$G \times H \subset P \times Q$ is a filter generic over V;*
2. *$P \subset G$ is a filter generic over V and $H \subset Q$ is generic over $V[G]$.*

If either (1) or (2) occurs then $V[G] \cap V[H] = V$.

A small less well-known variation of the above fact is used in several places in the book:

Proposition 6.1.8. *Suppose that M_0, M_1 are transitive models of ZFC and $P_0 \in M_0$ and $P_1 \in M_1$ are posets and $G_0 \times G_1 \subset P_0 \times P_1$ is a filter generic over V. Then $M_0[G_0] \cap M_1[G_1] = M_0 \cap M_1$ holds.*

Proof. Suppose that $\tau_0 \in M_0$ is a P_0-name for a set of ordinals and $\tau_1 \in M_1$ is a P_1-name for a set of ordinals, and $p_0 \in P_0$ and $p_1 \in P_1$ are conditions such that in the product $\langle p_0, p_1 \rangle \Vdash \tau_0 = \tau_1$. It is enough to show that p_0 decides the membership of all ordinals in τ_0 and p_1 decides the membership of all ordinals in τ_1, since then the decided values of τ_0 and τ_1 belong to M_0 and M_1 respectively and so to $M_0 \cap M_1$.

To see that p_0 decides all values of τ_0, suppose this fails and find an ordinal α and conditions $p_0^0, p_0^1 \leq p_0$ such that $p_0^0 \Vdash \check{\alpha} \notin \tau_0$ and $p_0^1 \Vdash \check{\alpha} \in \tau_0$. Find a condition $p_1^0 \leq p_1$ in the poset P_1 deciding the statement $\check{\alpha} \in \tau_1$; for definiteness assume that the decision is affirmative. Then the condition $\langle p_0^0, p_1^0 \rangle \leq \langle p_0, p_1 \rangle$ forces $\check{\alpha} \in \tau_0 \Delta \tau_1$, contradicting the assumptions.

The proof that p_1 decides all values of τ_1 is symmetric. □

Mutual genericity is not really a property of filters but of the corresponding extensions:

Lemma 6.1.9. *Suppose that P_0, P_1 are posets and $G_0 \times G_1 \subset P_0 \times P_1$ is a filter generic over V. Suppose that Q_0, Q_1 are posets in V and $H_0 \subset Q_0$ in $V[G_0]$ and $H_1 \subset Q_1$ in $V[G_1]$ are filters generic over V. Then $H_0 \times H_1 \subset Q_0 \times Q_1$ is a filter generic over V.*

Proof. To this end, suppose that $p_0 \in P_0$ and $p_1 \in P_1$ are conditions and $D \subset Q_0 \times Q_1$ is an open dense set. It is necessary to find conditions $p_0' \leq p_0$ and $p_1' \leq p_1$ such that $\langle p_0', p_1' \rangle \Vdash \tau_0 \times \tau_1 \cap \check{D} \neq 0$. To this end, by Fact 6.1.6 there must be a condition $q_0 \in Q_0$ such that for every $q_0' \leq q_0$ there is $p_0' \leq p_0$ such that $p_0' \Vdash \check{q}_0' \in \tau_0$. Similarly, there must be a condition $q_1 \in Q_1$ such that for every $q_1' \leq q_1$ there is $p_1' \leq p_1$ such that $p_1' \Vdash \check{q}_1' \in \tau_1$. Use the density of the set D to find conditions $q_0' \leq q_0$ and $q_1' \leq q_1$ such that $\langle q_0', q_1' \rangle \in D$. Find conditions $p_0' \leq p_0$ and $p_1' \leq p_1$ such that $p_0' \Vdash \check{q}_0' \in \tau_0$ and $p_1' \Vdash \check{q}_1' \in \tau_1$ in P_0 and P_1 respectively. Clearly, $\langle p_0', p_1' \rangle$ forces in the product $P_0 \times P_1$ that $\tau_0 \times \tau_1 \cap \check{D} \neq 0$ as desired. □

The following simple technical tool is used in several places of the book.

Definition 6.1.10. Let P be a partial ordering. A *graphing of* P is a graph Γ on P such that

1. if $p_0, p_1 \in P$ are conditions stronger than some p, then there is a finite Γ path which starts with a condition stronger than p_0, ends with a condition stronger than p_1, and uses only elements stronger than p;

2. if p_0, p_1 are Γ-related points and $q_0 \leq p_0$, then there is a condition $q_1 \leq p_1$ Γ-related to q_0.

Proposition 6.1.11. *Let P be a partial ordering and Γ its graphing. Suppose that τ is a P-name for a subset of the ground mode and $p \in P$ is a condition. Either $p \Vdash \tau = \check{a}$ for some set a, or there are Γ-related conditions $p_0, p_1 \leq p$ and a set x such that $p_0 \Vdash \check{x} \notin \tau$ and $p_1 \Vdash \check{x} \in \tau$.*

Proof. If the value of a is not decided by p then there is some x and conditions $r_0, r_1 \in P$ such that $r_0 \Vdash \check{x} \notin \tau$ and $r_1 \Vdash \check{x} \in \tau$. Use (1) of the definition of graphing to find a Γ-path starting with a condition stronger than r_0, ending with a condition stronger than r_1, and using only condition stronger than p. Use (2) of the definition of graphing to strengthen the conditions on the path repeatedly to obtain a Γ-path such that the conditions on it all decide the membership of $\check{x}$ in τ. Plainly, there must be neighboring conditions on the path that decide the statement $\check{x} \in \tau$ in opposite ways, and they will exemplify the second alternative in the proposition. $\square$

Definition 6.1.12. Suppose that Γ is graph on a poset P. The natural extension of Γ to the space of all filters on P connects filters $G_0, G_1 \subset P$ if for every condition $p_0 \in G_0$ and $p_1 \in G_1$ there are conditions $q_0 \leq p_0$ and $q_1 \leq p_1$ in G_0, G_1 respectively such that $q_0 \; \Gamma \; q_1$.

Corollary 6.1.13. *Suppose that Γ is a graphing of a poset P. Suppose that $G_0 \subset P$ is a filter generic over V. For every P-name τ for a subset of V, either $\tau/G_0 \in V$ or in some further generic extension there is a filter $G_1 \subset P$ generic over V, Γ-related to G_0, and such that $\tau/G_0 \neq \tau/G_1$.*

Proof. Suppose that τ is a P-name for a subset of V and $p \in P$ is a condition such that $p \Vdash \tau \notin V$; it will be enough to show that in some generic extensions there are Γ-related filters $G_0, G_1 \subset P$ separately generic over V such that $\tau/G_0 \neq \tau/G_1$. Just pass to a generic extension $V[K]$ in which $\mathcal{P}(P) \cap V$ is countable. List all open dense subsets of P in V by $\{D_n \colon n \in \omega\}$. Use Proposition 6.1.11 to find Γ-related conditions $p^0_0, p^1_0 \leq p$ and a set x such that $p^0_0 \Vdash \check{x} \notin V$ and $p^1_0 \Vdash \check{x} \in V$. Now, use (2) in the definition of graphing to produce two decreasing sequences $\langle p^0_n \colon n \in \omega\rangle$ and $\langle p^1_n \colon n \in \omega\rangle$ of conditions such that p^0_n and p^1_n are Γ-related, and p^0_{2n+1} and p^1_{2n+2} are both in D_n. It is clear that the filters $G_0, G_1 \subset P$ generated by the respective sequences are as desired. $\square$

Claim 6.1.14. *Suppose that E is an analytic equivalence relation on a Polish space Y. Suppose that $x \in 2^\omega$ is a point generic or random over V and $y \in Y$ is a point in $V[x]$. The set $F = \{a \subset \omega \colon a \in V$ and y has an E-equivalent in the model $V[x \restriction a]$ is closed under finite intersections.*

Proof. I will treat the case of the generic point; the proof for random point is the same, replacing the reference to Kuratowski-Ulam theorem with a reference to the Fubini theorem.

I will first show that if $\omega = a_0 \cup a_1$ is a partition of ω then every E-class represented in both $V[x \restriction a_0]$ and $V[x \restriction a_1]$ is represented already in V. To see this, go back to the ground model and view 2^ω as the product $2^{a_0} \times 2^{a_1}$. Suppose that $f_0 \colon 2^{a_0} \to Y$ and $f_1 \colon 2^{a_1} \to Y$ are Borel functions and $B \subset 2^{a_0} \times 2^{a_1}$ is a Borel nonmeager set such that for every point $\langle x_0, x_1 \rangle \in B$, $f_0(x_0)\ E\ f_1(x_1)$ holds. That is, B forces in the Cohen poset that the points $f_0(x \restriction a_0)$ and $f(x \restriction a_1)$ represent the same E-class. I must find a point $y \in Y$ such that the set $C = \{\langle x_0, x_1 \rangle \in C\ f(x_0)\ E\ y\}$ is nonmeager; i.e. the condition $C \subset B$ forces y to represent the E-class in question. To this end, first thin out the set B if necessary so that all of its vertical sections are nonmeager. Then, use the Kuratowski–Ulam theorem to find a point $z \in 2^{a_1}$ such that the z-th horizontal section of B is nonmeager. By the Kuratowski–Ulam theorem again, the point $y = f_1(z)$ works as required.

Now for the full statement of the claim. Suppose that a_0, a_1 are sets in V such that $V[x \restriction a_0]$ and $V[x \restriction a_1]$ both contain an E-equivalent of the point y. Let $b = a_0 \cap a_1$ and apply the previous paragraph in the model $V[x \restriction b]$ to see that $V[x \restriction b]$ contains an E-equivalent of the point y. Note that the point $x \restriction (a_0 \cup a_1) \setminus b$ is generic over the model $V[x \restriction b]$. This completes the proof of the claim. $\square$

6.2 Definability of forcing

In this section, I will show that various operations which are the essence of the forcing method are Borel in a suitable sense. The resulting lemmas are perhaps more difficult to state properly than they are to prove. Nevertheless, they are quite useful in many complexity computations.

For the notation in this section, let $X = 2^{\omega \times \omega}$ be the Polish space of all binary relations on ω. Each element of X is understood as a model for a language with a single binary relational symbol. The following lemma is standard.

Lemma 6.2.1. *Suppose that $f \colon 2^\omega \to X$ is a Borel function, ϕ is a formula of the language with n free variables, and $g_i \colon 2^\omega \to \omega$ are Borel functions for every $i \in n$. The set $\{x \in X \colon f(x) \models \phi(g_0(x), g_1(x) \dots g_{n-1}(x))\}$ is Borel.*

Proof. By induction on complexity of the formula ϕ. Left to the reader. $\square$

Now, if M is a countable model of ZF and $P \in M$ is a poset, one may want to produce a filter $G \subset P$ generic over M and construct a forcing extension $M[G]$. This is a Borel procedure, as the next two lemmas show.

Lemma 6.2.2. *Suppose $M\colon 2^\omega \to X$ and $P\colon 2^\omega \to \omega$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a model of ZF and $M(y) \models P(y)$ is a poset. Then, there is a Borel function $G\colon 2^\omega \to \mathcal{P}(\omega)$ such that for every $y \in 2^\omega$, $G(y)$ is a filter on $P(y)$ which is generic over $M(y)$.*

Proof. By induction on $n \in \omega$ define Borel functions $f_n\colon 2^\omega \to \omega$ so that

- for every $y \in 2^\omega$, $M(y) \models f_n(y)$ is the largest element of the poset $P(y)$;
- for every $y \in 2^\omega$ and $n \in \omega$, if $M(y) \models n$ is an open dense subset of the poset $P(y)$, then $f_{n+1}(y)$ is the smallest number m such that $M(y) \models m$ is an element of n and it is smaller that $f_n(y)$ in the poset $P(y)$; otherwise, $f_{n+1}(y) = f_n(y)$.

The functions defined in this way are Borel by Lemma 6.2.1. Let $G\colon 2^\omega \to \mathcal{P}(\omega)$ be defined so that for every $y \in \omega$, $G(y)$ is the set of all m such that $M(y) \models m$ is an element of the poset $P(y)$ and for some $n \in \omega$, $M(y) \models f_n(y)$ is smaller than m in the poset $P(y)$. It is clear that the function G works. $\square$

Lemma 6.2.3. *Suppose that $M\colon 2^\omega \to X$, $P\colon 2^\omega \to \omega$, $G\colon 2^\omega \to \mathcal{P}(\omega)$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a model of ZF, $M(y) \models P(y)$ is a poset, and $G(y) \subset P(y)$ is a filter generic over $M(y)$. Then, there are Borel functions $M[G]\colon 2^\omega \to X$ and $\mathrm{Re}\colon 2^\omega \times \omega \to \omega$ such that for every $y \in 2^\omega$, $M[G](y)$ is a generic extension of $M(y)$ by $G(y)$ and for every $n \in \omega$, $\mathrm{Re}(y)(n) = n/G(y)$ whenever $M(y) \models n$ is a $P(y)$-name.*

Proof. By induction on $n \in \omega$ build Borel functions $f_n\colon 2^\omega \to \omega$ such that for every $y \in 2^\omega$, $f_n(y)$ is the smallest number m such that $M(y) \models m$ is a $P(y)$-name, and for every $n' < n$ it is not the case that the filter $G(y)$ contains a condition p such that $M(y) \models p \Vdash_{P(y)} f_{n'}(y) = f_n(y)$. These functions are Borel by Lemma 6.2.1. Let $M[G]\colon 2^\omega \to X$ be the Borel function defined by $\langle n, m\rangle \in M[G](y)$ if the filter $G(y)$ contains a condition p such that $M[y] \models p \Vdash_{P(y)} f_n(y) \in f_m(y)$. Let $\mathrm{Re}\colon 2^\omega \times \omega \to \omega$ be the function defined by $\mathrm{Re}(y, n) = m$ if there is a condition p in the filter $G(y)$ such that $M(y) \models p \Vdash_{P(y)} f_m(x) = n$. These functions $M[G], \mathrm{Re}$ work by basic theorems on forcing applied in the models $M[y]$ for $y \in 2^\omega$. $\square$

If M is a transitive countable model of set theory, $P \in M$ is a poset, $\tau \in M$ is a P-name for a transitive set, and a is a transitive set, one may ask whether there is a filter $G \subset P$ which is generic over M such that $a = \tau/G$, and attempt to produce such a filter if it exists. The following lemma shows that this is a Borel procedure. An important nontrivial case arises in applications where $P \Vdash V(\tau)$ fails the axiom of choice. The lemma apparently only works for wellfounded models as opposed to arbitrary (perhaps illfounded) models.

Lemma 6.2.4. *Suppose $M\colon 2^\omega \to X$, $P, \tau\colon 2^\omega \to \omega$ and $a\colon 2^\omega \to X$ are Borel functions such that for every $y \in 2^\omega$, $M(y)$ is a wellfounded model of ZFC, $M(y) \models P(y)$ is a poset, $\tau(y)$ is a $P(y)$-name for a transitive set. Then,*

1. *the set $B = \{y \in Y$:there is a filter $G \subset P(y)$ generic over the model $M(y)$ such that $\langle \tau(y)/G, \in\rangle$ is isomorphic to $a\}$ is Borel;*

2. *there is a Borel function $G\colon B \to \mathcal{P}(\omega)$ such that for every $y \in B$, $G(y) \subset P(y)$ is a filter generic over $M(y)$ such that $\langle \tau(y)/G(y), \in\rangle$ is isomorphic to $\langle a(y), \in\rangle$.*

Proof. Use Lemma 6.2.1 to find Borel functions $P_0, \sigma, \nu, P_1, P_2\colon 2^\omega \to \omega$ so that for every $y \in 2^\omega$, $M(y)$ satisfies the following rest of this paragraph. $\sigma(y)$ is some $P(y) * \mathrm{Coll}(\omega, |\tau(y)|)$-name for an isomorph of $\langle \tau(y), \in\rangle$ with domain ω. Write Q be the poset of nonempty open subsets of the infinite permutation group S_∞, adding a Cohen-generic element $\dot\pi \in S_\infty$. Then $P_0(y)$ is the three step iteration $P(y) * \mathrm{Coll}(\omega, |\tau(y)|) * \dot Q$, $\nu(y)$ is the $P_0(y)$-name for the binary relation $\sigma \circ \dot\pi$ on ω. $P_1(y)$ is the complete Boolean algebra generated by the name $\nu(y)$, a complete subalgebra of the completion of the poset $P_0(y)$. Let $\dot P_2(y)$ be the P_1-name for the remainder poset $P_0(y)/P_1(y)$.

Let $D = \{\langle y, \pi, G\rangle \in 2^\omega \times S_\infty \times \mathcal{P}(\omega)$: $G \subset P_1(y)$ is a filter generic over the model $M(y)$ such that $\nu(y)/G = a(y) \circ \pi\}$. This is a Borel set by Lemma 6.2.3. The projection of D into the 2^ω coordinate is the set B by definitions. Write $C \subset 2^\omega \times S_\infty$ for the projection of D into the first two coordinates.

Claim 6.2.5. *1. The $\mathcal{P}(\omega)$-sections of D are either empty or else singletons.*

2. The S_∞-sections of C are either empty or else comeager in S_∞.

Proof. To simplify the notation, fix $y \in 2^\omega$ and omit the argument y from the expressions like $M(y), \sigma(y) \dots$.

(1) uses the wellfoundedness of the model M. Suppose that the $\mathcal{P}(\omega)$-section $D_{y,\pi}$ is nonempty. As $M \models$"P_1 is completely generated by the name ν", the filter G can be recovered by transfinite induction using infinitary Boolean expressions in M applied to $a \circ \pi$, and therefore it is unique.

(2) is more difficult, and it is the heart of the proof. Suppose that the S_∞-section C_y is nonempty. Thus, there is a filter $G_0 \subset P$ be a filter generic over M such that $\langle \tau/G_0, \in\rangle$ is isomorphic to the binary relation a on ω. Let $G_1 \subset \mathrm{Coll}(\omega, \tau/G_0)$ be a filter generic over $M[G_0]$, and let $z' = \sigma/(G_0 * G_1)$. Thus, z, z' are isomorphic binary relations on ω, an there is a permutation $\pi_0 \in S_\infty$ such that $z = z' \circ \pi_0$. Recall that Q is the poset of nonempty open subsets of S_∞. Let N be a countable elementary submodel of a large enough structure containing M, G_0, G_1, g_0. It will be enough to show that every element of S_∞ which is Q-generic over N belongs to the set C_y, since there are comeagerly many such points. Let $\pi \in S_\infty$ be a point Q-generic over N. Since the meager ideal on S_∞ is translation invariant, even the point $\pi_0^{-1}\pi$ is Q-generic over N and therefore over the smaller model $M[G_0][G_1]$ as well. Let $G_2 \subset Q$ be the filter generic over $M[G_0][G_1]$ associated with $\pi_0^{-1}\pi$. Now, $z \circ \pi = z' \circ \pi_0 \circ \pi_0^{-1} \circ \pi = z' \circ \pi_0^{-1}\pi$. Therefore, $z \circ \pi$ is equal to the point $\nu/(G_0 * G_1 * G_2)$and $\pi \in C_y$ as required. □

Now, as one-to-one projections of Borel sets are Borel [14, Theorem 15.1], the set C is Borel by Claim 6.2.5(1). As the category quantifier yields Borel sets [14, Theorem 16.1], the set B as the projection of C into the first coordinate is Borel. Borel sets with nonmeager vertical sections allow Borel uniformizations [14, Theorem 18.6], and so there is a Borel uniformization $f : B \to S_\infty$ of C. As the set D has singleton vertical sections, it is itself its uniformization $g : C \to \mathcal{P}(\omega)$. Let $G_1 : B \to \mathcal{P}(\omega)$ be the function defined by $G_1(y) = g(y, f(y))$. Thus, for every $y \in B$, $G_1(y) \subset P_1(y)$ is a filter generic over $M(y)$ such that $\nu/G_1(y)$ is isomorphic to $a(y)$.

The argument is now in its final stage. Let $P_2 \colon 2^\omega \to \omega$ be a Borel function such that for every $y \in 2^\omega$, $M(y) \models P_2(y)$ is a name for the quotient poset $P_0(y)/P_1(y)$. Use Lemmas 6.2.2 and 6.2.3 to find a Borel function $G_2 : B \to \mathcal{P}(\omega)$ such that $G_2(y) \subset P_2/G_1(y)$ is a filter generic over the model $M(y)[G_1(y)]$. Let $G_0 : B \to \mathcal{P}(\omega)$ be a Borel function indicating a filter on $P_0(y)$ which is the composition of G_1 and G_2. Let $G : B \to \mathcal{P}(\omega)$ be the function which indicates the first coordinate of the filter $G_0(y) \subset P_0(y)$. Recall that the poset $P_0(y)$ is a three stage iteration of which the first stage is $P(y)$, so for every $y \in B$, $G(y) \subset P(y)$ is a filter generic over the model $M(y)$. The function G has the required properties. $\square$

There is an interesting corollary:

Corollary 6.2.6. *Suppose that X, Y are Polish spaces, E, F are analytic equivalence relations on X, Y respectively, E is Borel reducible to an orbit equivalence relation, $B \subset X$ is a Borel set and $h \colon B \to Y$ is a Borel homomorphism from E to F. If μ is a Borel probability measure on X, there is a Borel homomorphism $k \colon X \to Y$ such that the set $\{x \in B \colon \neg h(x)\ F\ k(x)\}$ is of zero μ-mass.*

Proof. Without loss of generality assume that $\mu(B) > 0$. Let Z be a Polish space and Γ a Polish group acting continuously on Z, and let $g \colon X \to Z$ be a Borel reduction of E to the orbit equivalence relation of the action of Γ. Let Q be the poset of μ-positive Borel subsets of B ordered by inclusion with generic point $\dot{x}$, let P be the poset of nonempty open subsets of Γ ordered by inclusion with a generic point $\dot\gamma$, and let $\dot{z}$ be the $P \times Q$-name for the point $\dot\gamma \cdot g(\dot{x})$. Let R be the poset generated by the name $\dot{z}$, an let $\dot{y}$ be an R-name for an element of Y such that $V[\dot{z}] \models \exists x \in B\ h(x) = \dot{y}$ and $g(x)$ is orbit equivalent to $\dot{z}$. Such an element of Y must exist in $V[\dot{z}]$ by the Mostowski absoluteness between $V[\dot{z}]$ and $V[\dot{x}, \dot\gamma]$ since $h(\dot{x})$ is such a point there.

Now, let M be a countable elementary submodel of a large structure containing all the above information. Let $A = \{z \in Z \colon z$ is R-generic over the model $M\} \subset Z$. This set is Borel by Lemma 6.2.4(3). Let $C = \{x \in X \colon$ for co-meagerly many $\gamma \in \Gamma$, $\gamma \cdot h(x) \in A\} \subset X$.

Claim 6.2.7. *The set $C \subset X$ is Borel, it is E-invariant, and it contains all elements of the set B except perhaps for a μ-null set.*

Proof. The Borelness follows from the definability properties of the category quantifier [14, Theorem 16.1]. The E-invariance follows from the invariance of the meager ideal on Γ under translations. Finally, every element of B which is

Q-generic over the model M belongs to the set C by the definitions of P, Q and R. □

Use the large section uniformization theorem [14, Corollary 18.7] to find a Borel function $f\colon C \to \Gamma$ such that for every $x \in C$, $f(x) \cdot x \in A$. Define the function k by $k(x) = y$ for some fixed element $y \in Y$ if $x \notin C$, and $k(x) = \dot{y}/f(x) \cdot h(x)$ if $x \in C$. The function k is a homomorphism of E to F by the definitions. The Borelness of k follows from Lemma 6.2.4. Finally, if $x \in B$ is such that $h(x)\ E\ k(x)$ fails, then $x \notin C$ must occur. Now, $\mu(B \setminus C) = 0$ by the claim, and the corollary follows. □

It is also possible to produce many filters mutually generic over a single countable model:

Lemma 6.2.8. *Whenever M is a countable model of ZF and $P \in M$ is a partial order, then there is a continuous map $G\colon 2^\omega \to \mathcal{P}(P)$ such that for every finite tuple $\langle z_i \colon i \in n\rangle$ of pairwise distinct elements of 2^ω the sets $\{G(z_i)\colon i \in n\}$ are filters on P mutually generic over the model M.*

Proof. Let $\langle a_n, D_n \colon n \in \omega\rangle$ enumerate with repetitions all pairs $\langle a, D\rangle$ such that $a \subset 2^{<\omega}$ is a finite sequence of pairwise distinct binary sequences of the same length, and $D \subset P^{|a|}$ is an open dense set in M. By induction on $n \in \omega$ build conditions $p_s \in P$ for $s \in 2^n$ so that

- $s \subset t$ implies $p_t \leq p_s$;
- whenever $\langle s_i \colon i \in |a_n|\rangle$ is a sequence of binary strings of length n such that $a(i) \subset s_i$, then $\langle p_{s_i} \colon i \in |a|\rangle \in D_n$.

This is easily done. Once the induction has been performed, let the function $G\colon 2^\omega \to \mathcal{P}(P)$ assign to every binary sequence $x \in 2^\omega$ the filter on P generated by the conditions $\{p_{x\restriction n}\colon n \in \omega\}$. This is easily seen to work. □

If M is a countable model of ZFC, $j : m \to N$ is an elementary embedding which is a class in M, and L is a linear ordering, one may form the iteration of j along the linear ordering L. This is a Borel operation as the following lemma shows.

Lemma 6.2.9. *Let $M\colon 2^\omega \to X$ and $U\colon 2^\omega \to \omega$ and $L\colon 2^\omega \to X$ be Borel functions. There is a Borel function $N\colon 2^\omega \to X$ and $j\colon 2^\omega \times \omega \to \omega$ such that for every $y \in 2^\omega$, if $M(y)$ is a model of ZFC, $M(y) \models U(y)$ is a normal measure on an uncountable cardinal, and $L(y)$ is a wellordering, then $N(y)$ is a model of ZFC isomorphic to the iteration of the $U(y)$ ultrapower of the model $M(y)$ along $L(y)$, and $j(y) : m(y) \to N(y)$ is the iteration embedding.*

Proof. To simplify the notation, assume that for every $y \in 2^\omega$, $M(y)$ is a model of ZFC and $M(y) \models U(y)$ is a normal measure on an uncountable cardinal. I have to show that the usual direct limit description of the iteration is Borel.

First, consider the construction of the usual ultrapower. Let $\kappa(y)$ be the cardinal on which $U(y)$ is normal measure in $M(y)$. For every $m \in \omega$ let $U^m(y)$ be the m-th Fubini product of $U(y)$; thus $M(y) \models U^m(y)$ is a $\kappa(y)$-complete ultrafilter on $\kappa^m(y)$. For $a \subset \omega$ and $y \in 2^\omega$, let $E_a(y)$ be the equivalence relation on ω defined by $n\ E_a\ m$ if $M(y) \models \{x \colon n(x) = m(x)\} \in U^{|a|}(y)$. Let $\in_a (y)$ be the binary relation on ω defined by $n\ E_a\ m$ if $M(y) \models \{x \colon n(x) \in m(x)\} \in U^{|a|}(y)$. Thus, $E_a, \in_a$ are Borel relations on $2^\omega \times \omega \times \omega$ and moreover, $\in_a (y)$ respects the equivalence E_a.

Now look at the direct limit of the ultraproducts constructed in the previous paragraph. For $a \subset b \subset \omega$ let $j_{ab}(y) \colon \omega \to \omega$ by the function defined by $j_{ab}(y)(n) = m$ if $M(y) \models n$ is a function with domain $\kappa^{|a|}(y)$ and m is a function with domain $\kappa^{|b|}(y)$ and $n = m \circ \phi$ where $\phi : |a| \to |b|$ is the unique map such that $\phi \circ \psi_0 = \psi_1 \circ i$, where $i : a \to b$ is the identity map, ψ_0 is the order preserving map from a with the $L(y)$-order to $|a|$ with the usual natural number order and ψ_1 is the order preserving map from b with the $L(y)$-order to $|b|$ with the usual natural number order. Thus, $j_{ab} \colon 2^\omega \times \omega \to \omega$ is a Borel map, $j_{ab}(y)$ respects the equivalences $E_a(y)$ and $E_b(y)$, and $j_{ac}(y) = j_{bc}(y) \circ j_{ab}(y)$ whenever $a \subset b \subset c$. Let $E(y)$ be an equivalence relation on $\omega \times [\omega]^{<\aleph_0}$ defined by $\langle n, a\rangle\ E\ (y)\langle m, b\rangle$ if $j_{a,a\cup b}(y)(n)\ E_{a\cup b}\ j_{b,a\cup b}(y)(m)$. Let $\in (y)$ be a relation on $\omega \times [\omega]^{<\aleph_0}$ defined by $\langle n, a\rangle \in (y)\langle m, b\rangle$ if $j_{a,a\cup b}(y)(n) \in_{a\cup b} (y) j_{b,a\cup b}(y)(m)$.

Now, the final considerations. It is now easy to find a Borel map $\pi \colon 2^\omega \times \omega \times [\omega]^{<\aleph_0}$ such that $\pi(y)$ is constant on $E(y)$-equivalence classes and distinct $E(y)$-equivalence classes are mapped to distnict numbers. Let $N \colon 2^\omega \to X$ be the Borel map defined by $(n, m) \in N(y)$ if $\pi^{-1}(y)(n) \in (y)\ \pi^{-1}(y)(m)$. Let $j \colon 2^\omega \times \omega \to \omega$ be the Borel map defined by $j(y)(n) = m$ if $\pi(y)(n, 0) = m$. This works. $\square$

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