COLORING RIGHT TRIANGLES

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$\Sigma^2_1$ sentences.

Sentences of the form $\exists A \subset X \phi(A)$ where $X$ is a Polish space and $\phi$ quantifies only over natural numbers and elements of $X$. Maybe true, maybe false, provable in ZF or ZFC or ZFC+CH.

- If $G$ is a Borel graph on $X$: the statement $G$ has countable chromatic number;

- If $X$ is a vector space over a countable field: $X$ has a basis;

- the Continuum Hypothesis.
Chromatic numbers of hypergraphs.

A hypergraph $\Gamma$ on $X$ is just a subset of $[X]^k$ where $k \in \omega$. It has countable chromatic number if there is a decomposition $X = \bigcup_n A_n$ such that no $A_n$ contains a hyperedge.

- the graph of points of rational Euclidean distance in $\mathbb{R}^n$;
- the hypergraph of equilateral triangles in $\mathbb{R}^n$;
- the hypergraph of all squares in $\mathbb{R}^n$;
- the hypergraph of all right triangles in $\mathbb{R}^n$.

All algebraic. Countable chromatic number in ZFC except the last item.
Classifying hypergraph chromatic numbers.

**Theorem.** (Schmerl) There are computable sets $B_n: n \geq 1, C, D$ such that

- $G \in B_n$ iff (ZFC proves $G$ has countable chromatic number iff $2^{\aleph_0} \leq \aleph_n$);

- $G \in C$ iff ZFC proves $G$ has countable chromatic number;

- $G \in D$ iff ZFC proves $G$ does not have countable chromatic number;

- $\bigcup_n B_n \cup C \cup D =$ all algebraic hypergraphs.

**Task.** Find similar classification for ZF+DC. MUCH more complicated!
(the Geometric Set Theory book.) Given hypergraphs $G_0, G_1$, attempt to find a balanced generic extension of the Solovay model in which $G_0$ has countable chromatic number and $G_1$ does not.

Boils down to comparison of combi/topo properties of $G_0, G_1$. Sometimes impossible.

**Theorem.** (ZF) If the graph of right angle triangles in $\mathbb{R}^2$ has countable chromatic number then there is a linear pre-order of the reals with all proper initial segments countable.

**Corollary.** No way of getting balanced generic extensions where the right angle hypergraph has countable chromatic number.
Proof of theorem: basic tool.

Given $A \subset 2^\omega$ (or any other Polish space) and $x \in 2^\omega$, form $HOD_{A,x}$. Define $x \leq y$ if $x \in HOD_{A,y}$ (a preorder).

**Fact.** If $A$ is a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ then $\leq$ is wellfounded.

**Fact.** If $A$ is a coloring for the rectangle graph, then $\leq$ is well-founded of height $\omega_1$.

**Fact.** If $A$ is a coloring for the right triangles, then $\leq$ is a pre-well-ordering of height $\omega_1$. 
Aside.

$\text{HOD}_{A,x}$ is the class of all sets $z$ such that $z$ and every element of the transitive closure of $z$ has a definition from ordinal parameters and additional parameters $A, x$.

$\text{HOD}_{A,x}$ is a model of ZFC. It contains all ordinals, $x$, and $A \cap \text{HOD}_{A,x}$ (the shadow of $A$).

Shadow of a coloring is a coloring, shadow of an ultrafilter is an ultrafilter, etc.
Proof of theorem: linearity.

Let $c$ be the coloring of the right triangle hypergraph.

To prove linearity of $\leq$, suppose towards a contradiction that $x_0, x_1$ are mutually undefinable. Let $L_0$ be the vertical line in the plane with coordinate $x_0$, let $L_1$ be the horizontal line in the plane with coordinate $x_1$.

Let $n = c(x_0, x_1)$. This is not the only point on $L_0$ with color $n$, otherwise $x_1$ is definable from $x_0$. Same for $L_1$. So there is a monochromatic right triangle of color $n$. 
Proof of theorem: well-foundedness.

We will show that $HOD_{c,x_0} = HOD_{c,x_1}$ if and only if the two models have the same $\omega_1$.

Suppose towards a contradiction that $x_0 < x_1$ and the two models have the same $\omega_1$. Work in $HOD_{c,x_1}$. Find a countable subset of the line $L_1$ harvesting all possible colors. Find a point $x'_0 \in HOD_{c,x_0}$ which is not on this countable set. Then proceed as previously.
Proof of theorem: AC case.

We prove that in ZFC, existence of the coloring implies CH. This has been proved by Erdős and Komjáth earlier.

Suppose towards a contradiction that CH fails. Let $M_0$ be an elementary submodel of size $\aleph_1$ containing $c$ as element, let $x_0 \in \mathbb{R} \setminus M_0$ be any element. Let $M_1$ be a countable elementary submodel containing $M_0, x, c$ as elements, let $x_1 \in M_1 \setminus M_0$ be any element.

Now proceed as previously.
Proof of theorem: wrap-up.

Case 1. There is \( x \) such that \( \text{HOD}_{c,x} \) has the same \( \omega_1 \) as \( V \). Then it contains all reals, and it satisfies CH, giving a well-ordering of reals of ordertype \( \omega_1 \).

Case 2. If Case 1 fails, then all models \( \text{HOD}_{c,x} \) have countable \( \omega_1 \). Since they satisfy CH, they contain only countably many reals each.

In both cases, the theorem follows.