COLORING RIGHT TRIANGLES

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Σ^2_1 sentences.

Sentences of the form $\exists A \subset X\ \phi(A)$ where X is a Polish space and ϕ quantifies only over natural numbers and elements of X. Maybe true, maybe false, provable in ZF or ZFC or ZFC+CH.

- If G is a Borel graph on X: the statement G has countable chromatic number;

- If X is a vector space over a countable field: X has a basis;

- the Continuum Hypothesis.

Chromatic numbers of hypergraphs.

A *hypergraph* Γ on X is just a subset of $[X]^k$ where $k \in \omega$. It has *countable chromatic number* if there is a decomposition $X = \bigcup_n A_n$ such that no A_n contains a hyperedge.

- the graph of points of rational Euclidean distance in $\mathbb{R}^n$;

- the hypergraph of equilateral triangles in $\mathbb{R}^n$;

- the hypergraph of all squares in $\mathbb{R}^n$;

- the hypergraph of all right triangles in $\mathbb{R}^n$.

All algebraic. Countable chromatic number in ZFC except the last item.

Classifying hypergraph chromatic numbers.

Theorem. (Schmerl) There are computable sets $B_n\colon n \geq 1$, C, D such that

- $G \in B_n$ iff (ZFC proves G has countable chromatic number iff $2^{\aleph_0} \leq \aleph_n$;
- $G \in C$ iff ZFC proves G has countable chromatic number;
- $G \in D$ iff ZFC proves G does not have countable chromatic number;
- $\bigcup_n B_n \cup C \cup D$ =all algebraic hypergraphs.

Task. Find similar classification for ZF+DC. MUCH more complicated!

Technology.

(the Geometric Set Theory book.) Given hypergraphs G_0, G_1, attempt to find a balanced generic extension of the Solovay model in which G_0 has countable chromatic number and G_1 does not.

Boils down to comparison of combi/topo properties of G_0, G_1. Sometimes impossible.

Theorem. (ZF) If the graph of right angle triangles in $\mathbb{R}^2$ has countable chromatic number then there is a linear pre-order of the reals with all proper initial segments countable.

Corollary. No way of getting balanced generic extensions where the right angle hypergraph has countable chromatic number.

Proof of theorem: basic tool.

Given $A \subset 2^\omega$ (or any other Polish space) and $x \in 2^\omega$, form $HOD_{A,x}$. Define $x \leq y$ if $x \in HOD_{A,y}$ (a preorder).

Fact. If A is a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ then $\leq$ is wellfounded.

Fact. If A is a coloring for the rectangle graph, then $\leq$ is well-founded of height ω_1.

Fact. If A is a coloring for the right triangles, then $\leq$ is a pre-well-ordering of height ω_1.

Aside.

$HOD_{A,x}$ is the class of all sets z such that z and every element of the transitive closure of z has a definition from ordinal parameters and additional parameters A, x.

$HOD_{A,x}$ is a model of ZFC. It contains all ordinals, x, and $A \cap HOD_{A,x}$ (the *shadow* of A).

Shadow of a coloring is a coloring, shadow of an ultrafilter is an ultrafilter, etc.

Proof of theorem: linearity.

Let c be the coloring of the right triangle hypergraph.

To prove linearity of $\leq$, suppose towards a contradiction that x_0, x_1 are mutually undefinable. Let L_0 be the vertical line in the plane with coordinate x_0, let L_1 be the horizontal line in the plane with coordinate x_1.

Let $n = c(x_0, x_1)$. This is not the only point on L_0 with color n, otherwise x_1 is definable from x_0. Same for L_1. So there is a monochromatic right triangle of color n.

Proof of theorem: well-foundedness.

We will show that $HOD_{c,x_0} = HOD_{c,x_1}$ if and only if the two models have the same ω_1.

Suppose towards a contradiction that $x_0 < x_1$ and the two models have the same ω_1. Work in HOD_{c,x_1}. Find a countable subset of the line L_1 harvesting all possible colors. Find a point $x_0' \in HOD_{c,x_0}$ which is not on this countable set. Then proceed as previously.

Proof of theorem: AC case.

We prove that in ZFC, existence of the coloring implies CH. This has been proved by Erdős and Komjáth earlier.

Suppose towards a contradiction that CH fails. Let M_0 be an elementary submodel of size $\aleph_1$ containing c as element, let $x_0 \in \mathbb{R} \setminus M_0$ be any element. Let M_1 be a countable elementary submodel containing M_0, x, c as elements, let $x_1 \in M_1 \setminus M_0$ be any element.

Now proceed as previously.

Proof of theorem: wrap-up.

Case 1. There is x such that $HOD_{c,x}$ has the same ω_1 as V. Then it contains all reals, and it satisfies CH, giving a well-ordering of reals of ordertype ω_1.

Case 2. If Case 1 fails, then all models $HOD_{c,x}$ have countable ω_1. Since they satisfy CH, they contain only countably many reals each.

In both cases, the theorem follows.