

Triangles and Vitali sets*

Jindřich Zapletal
University of Florida
zapletal@ufl.edu

October 2, 2023

Abstract

It is consistent relative to an inaccessible cardinal that $\text{ZF}+\text{DC}$ holds, the hypergraph of equilateral triangles in Euclidean plane has countable chromatic number, while there is no Vitali set.

1 Introduction

Classification of analytic graphs and hypergraphs from various points of view is a subject matter frequently discussed in descriptive set theory and descriptive combinatorics. Geometric set theory adds its own angle: independence results in $\text{ZF}+\text{DC}$ choiceless set theory regarding the existence of various structures (e.g., colorings or perfect matchings) related to analytic (hyper)graphs. One can consider for example the following broad question:

Question 1.1. For which analytic hypergraphs Γ on Polish spaces is it consistent with $\text{ZF}+\text{DC}$ that Γ has countable chromatic number and no Vitali set exists?

Note that existence of Vitali set is equivalent in $\text{ZF}+\text{DC}$ to the countable chromatic number of the Vitali equivalence relation on \mathbb{R} . [?, Chapter 11] contains various results in this direction for locally countable graphs and hypergraphs. Difficulties quickly mount when one steps out of the locally countable realm. As one interesting and traditional class of cases, the algebraic graphs and hypergraphs on Euclidean spaces are combinatorially quite complex, but none of them seem to have any relationship with the Vitali equivalence relation. In an encouraging development, the question has been resolved in the affirmative for all algebraic graphs Γ containing no perfect clique [?]. In this paper, I resolve a quite special case in arity three, which requires several additional ideas.

Definition 1.2. Let $\langle X, \cdot \rangle$ be a standard Borel abelian group. A hypergraph Γ of arity three on X is a *slim linear hypergraph* if there are Borel homomorphisms $g_0, g_1, g_2: X \rightarrow X$ such that

*2020 AMS subject classification 03E35, 14P99, 05C15.

1. all of g_0, g_1, g_2 and $g_0 + g_1, g_1 + g_2$, and $g_0 + g_2$ are injections;
2. Γ is the set of all triples $\{x_0, x_1, x_2\}$ of pairwise distinct points such that under suitable enumeration, $g_0(x_0) + g_1(x_1) + g_2(x_2) = 0$.

In most interesting cases, the sum $g_0 + g_1 + g_2$ is equal to the zero homomorphism, and then the hypergraph Γ will be invariant under translations. Slim linear hypergraphs carry geometric meaning:

Example 1.3. Let $c \in \mathbb{C}$ be a non-real complex number, and consider the hypergraph on \mathbb{C} given by the arithmetic identity $c \cdot (x_1 - x_0) = x_2 - x_0$. It is the set of all triangles in the complex plane directly similar to the triangle with vertices $0, 1, c$.

Theorem 1.4. *Let $\langle X, + \rangle$ be a standard Borel abelian group, and let Γ be a finite union of slim linear hypergraphs of arity three on X . Then in a balanced forcing extension of the choiceless Solovay model, the following items hold:*

1. *ZF+DC holds and Γ has countable chromatic number;*
2. *there is no Vitali set;*
3. *there is no complete countable section for the \mathbb{E}_1 equivalence relation.*

Corollary 1.5. *Relative to an inaccessible cardinal, it is consistent with ZF+DC that the hypergraph of equilateral triangles in \mathbb{R}^2 has countable chromatic number, yet there is no Vitali set.*

The theorem understates the understanding of the resulting model. In particular, I prove that a certain graph simpler than the Vitali equivalence relation—the diagonal Hamming graph—has uncountable chromatic number in the resulting model.

In a number of natural cases, I can show that it is not entirely easy to color slim linear hypergraphs. No such coloring will appear in a compactly balanced forcing extension of the choiceless Solovay model as per the following theorem. The broad class of compactly balanced forcings was introduced in [?, Section 9.3]; it includes in particular the poset for adding a Ramsey ultrafilter, or the poset for adding a linear ordering of a quotient space of any Borel equivalence relation.

Theorem 1.6. *Let $\langle X, + \rangle$ be a Polish abelian group, let $n \geq 3$ be a number, and let $g_i: X \rightarrow X$ for $i \in n$ be continuous homomorphisms such that*

1. $\sum_{i \in n} g_i = 0$;
2. *every open neighborhood of 0 in X contains pairwise distinct nonzero points x_i for $i \in n$ such that $\sum_{i \in n} g_i(x_i) = 0$.*

Then, in every compactly balanced extension of the choiceless Solovay model, the hypergraph Γ on X consisting of all tuples $\{x_i: i \in n\}$ of pairwise distinct elements of X such that $\sum_{i \in n} g_i(x_i) = 0$ has uncountable chromatic number.

Corollary 1.7. *Relative to an inaccessible cardinal, it is consistent with ZF+DC that there is a non-principal ultrafilter on ω , yet the hypergraph of equilateral triangles in \mathbb{R}^2 has uncountable chromatic number.*

There are many open questions left. Since the equilateral triangle hypergraph in \mathbb{R}^3 is not linear, methods of this paper do not apply to it, and it is the simplest algebraic hypergraph for which I do not know if its countable chromatic number implies existence of a Vitali set in ZF+DC. I do not know whether the chromatic number of the Kechris–Solecki–Todorćević graph \mathbb{G}_0 [?] is countable or uncountable in the resulting model. Finally, one should be able to remove the inaccessible cardinal assumption from the proof, but I have not done the footwork necessary for that.

The architecture of the paper is probably somewhat surprising to a reader who is not familiar with the mysterious ways of geometric set theory [?]. In Section 2, I define the central combinatorial tool of the paper: the remainder graphs of a ternary hypergraph. This is an interesting object regardless of any forcing considerations, and the main point of this paper is that in the case of linear hypergraphs, the remainder graphs can be handled efficiently. In Section 3, I define a couple of c.c.c. finite condition coloring posets for various graphs and hypergraphs. A fine evaluation of their chain condition situation is necessary to exclude the Vitali set in the model for Theorem 1.4. This part of the paper is reminiscent of certain computations of Todorćević [?] regarding similar finite condition c.c.c. posets. In Section 4, I define a σ -closed balanced poset P adding a coloring of the hypergraph Γ of Theorem 1.4. The model for Theorem 1.4 is then a P -extension of a choiceless Solovay model derived from an inaccessible cardinal. Section 5 then checks the properties of this model, using the remainder graphs and c.c.c. coloring posets identified earlier. Finally, Section 6 shows that there are natural algebraic hypergraphs for which in ZF countable chromatic number has consequences contradicting the conclusion of Theorem 1.4.

The terminology of the paper follows the set theoretic standard of [?], in matters of descriptive set theory that of [?], in matters of geometric set theory that of [?]. \mathbb{E}_1 is the equivalence relation on the set \mathbb{R}^ω connecting points x, y if the set $\{n \in \omega : x(n) \neq y(n)\}$ is finite. The *diagonal Hamming graph* $\mathbb{H}_{<\omega}$ is the graph on the set of all functions $x \in \omega^\omega$ such that $\forall n x(n) \leq n$, connecting points x, y if there is exactly one $n \in \omega$ such that $x(n) \neq y(n)$. HF denotes the set of all hereditarily finite sets.

2 Remainder graphs

The consistency result depends on several combinatorial observations. The first seems critical and irreplaceable in the argument, and its limitations seem to be the main obstacle to natural generalizations of Theorem 1.4. Consider the following definition.

Definition 2.1. Let Γ be a hypergraph on a Polish space X , and let $A \subset X$

be a set. The *remainder graph* is the graph Γ_A on $X \setminus A$ connecting distinct points x, y if there is $z \in A$ such that $\{x, y, z\} \in \Gamma$ holds.

The most natural case of remainder graphs appears when A is a suitably algebraically closed subset of X , possibly uncountable. I will be interested in the chromatic number of the remainder graphs. This looks like a completely intractable issue, but some cases do allow for informative insight. Consider the following examples.

Example 2.2. Let Γ be the hypergraph on \mathbb{R} of all triples $\{x, y, z\}$ of pairwise distinct elements which (in some ordering) satisfy the identity $x - 2y + z = 0$. Let $A \subset \mathbb{R}$ be a subfield. Then the remainder graph contains no circuits of odd length, and it contains an $|A| \times |A|$ -biclique.

Proof. First, I will show that there are no circuits of odd length in the remainder graph. For this, consider the group G of all non-constant linear functions from \mathbb{R} to itself with the composition operation, acting on \mathbb{R} by application. For each such a function $g(u) = au + b$, call a the *slope* of the function. The slope function is obviously a homomorphism from the group G to the multiplicative group of nonzero real numbers. Each element of G except for the identity has at most one fixed point.

Now, let $n \in \omega$ be a number and $\langle x_i : i \leq n \rangle$ be a path in the remainder graph with $x_n = x_0$; I must show that n is even. To this end, for each $i \in n$ find $z_i \in V$ such that $\{x_i, x_{i+1}, z_i\} \in \Gamma$. There are three possible mutually exclusive cases how this can occur:

- $x_i - 2x_{i+1} + z_i = 0$. In this case, let $g_i : X \rightarrow X$ be the function defined by $g_i(u) = (u + z_i)/2$;
- $x_i - 2z_i + x_{i+1} = 0$. In this case, let $g_i : X \rightarrow X$ be the function defined by $g_i(u) = 2z_i - u$;
- $z_i - 2x_i + x_{i+1} = 0$. Here, let $g_i(u) = 2u - z_i$.

All the functions g_i for $i \in n$ are coded in F_0 , and $g_i(x_i) = x_{i+1}$. The composition of all the elements $g_i \in G$ for $i \in n$ is a linear function coded in A , and it has $x_0 \notin A$ as a fixed point, therefore it has to be the identity. Now consider the slopes of the functions g_i : in the first case, it is $1/2$, in the second case, it is -1 , in the third case, it is 2 . The product has to give 1 , the slope of the identity. Thus, the first and third case must be used equal number of times, and the second case must be used even number of times. In conclusion, n is even as desired.

Now, to produce the required biclique in the remainder graph, let $B_0, B_1 \subset A$ be disjoint sets of reals of cardinality $|A|$. Let $\varepsilon > 0$ be a real number in $\mathbb{R} \setminus A$. Consider the set $C_0 = B_0 + \varepsilon$ and the set $C_1 = (B_1 + \varepsilon)/2$. These are disjoint subsets of $\mathbb{R} \setminus A$, and they form an $|A| \times |A|$ bi-clique in the remainder graph. To see this, let $x_0 \in B_0$ and $x_1 \in B_1$ be points. Then $(x_0 + \varepsilon) - (2(x_1 + \varepsilon)/2) = x_0 - x_1 \in A$, so $\{x_0 + \varepsilon, (x_1 + \varepsilon)/2, x_0 - x_1\} \in \Gamma$ and $x_0 + \varepsilon, (x_1 + \varepsilon)/2$ are connected in the remainder graph as desired. \square

Example 2.3. Let Γ be the hypergraph of equilateral triangles in \mathbb{R}^3 . Let F be any real closed subfield of \mathbb{R} properly smaller than \mathbb{R} , and let $A = F^3$. Then in the remainder graph, one can find a triangle. Just let $\varepsilon > 0$ be a real number $\mathbb{R} \setminus F$, let $z \in A$ be an arbitrary point, and find any regular tetrahedron with one vertex equal to z and length of all sides equal to ε . The other three vertices cannot belong to A since together with z each of them computes ε , and they form a triangle in the remainder graph.

Example 2.4. Let Γ be the hypergraph of isosceles triangles on \mathbb{R}^2 . Let F be any real closed subfield of \mathbb{R} properly smaller than \mathbb{R} , and let $A = F^3$. Then the remainder graph contains a perfect clique. To see this, let $\varepsilon > 0$ be a real number which is not in F , and let $z \in A$ be an arbitrary point. Consider the set C of all points in \mathbb{R}^2 which are at a distance ε from z . It is immediate to see that C contains no points from A (since every point in C together with z computes ε) and C is a clique in the remainder graph.

Finally, we get to the hypergraph of immediate interest to the present paper.

Definition 2.5. Let $\langle X, + \rangle$ be a standard Borel abelian group. Let $\Gamma = \bigcup_{i \in n} \Gamma_i$ be a finite or countable union of slim linear hypergraphs of arity three, for $n \leq \omega$. Let Γ_i be generated by continuous homomorphisms g_j^i for $j \in 3$. A set $A \subset X$ is Γ -closed if it is a subgroup of X , it is closed under each homomorphism g_j^i and its inverse for $i \in n$ and $j \in 3$, and also closed under the inverses of the homomorphisms $g_j^i + g_k^i$ for $i \in \omega$ and j, k distinct indices in 3.

Note that all the homomorphisms mentioned in the above definitions are injective per the definition of a slim linear hypergraphs, so their inverses are well-defined partial functions. The definition of Γ -closed set seems to depend on the choice of the decomposition of Γ into slim hypergraphs and their generating homomorphisms; I will keep such decomposition and its generating homomorphisms always fixed. In this way, every infinite subset of X has a smallest Γ -closed superset which is of the same cardinality, and an increasing union of Γ -closed sets is Γ -closed.

Example 2.6. Let $\langle X, + \rangle$ be a standard Borel abelian group. Let Γ be a finite (resp. countable) union of slim linear hypergraphs of arity three. Let $A \subset X$ be a Γ -closed set. Then there is a homomorphism of the remainder graph Γ_A to a locally finite (resp. locally countable) graph.

Proof. Let $\Gamma = \bigcup_{i \in n} \Gamma_i$ be a finite or countable union of slim linear hypergraphs of arity three, for $n \leq \omega$. Let Γ_i be generated by continuous homomorphisms g_j^i for $j \in 3$. Let E be the equivalence relation on $X \setminus A$ connecting distinct points x_0, x_1 if $x_0 - x_1 \in A$. Let Y be the set of all E -classes, and define the graph Θ on Y which connects classes $c \neq d$ if there are representatives $x \in c$ and $y \in d$ which are connected in the remainder graph. I will show that that the graph Θ is locally finite (resp. locally countable), and the map $f: X \rightarrow Y$, $f(x) = [x]_E$ is a homomorphism of the remainder graph to Θ .

To show that the map f is a homomorphism, it is only necessary to show that if distinct points $x_0, x_1 \in X \setminus A$ are connected in the remainder graph, then

they are not E -equivalent. Suppose towards a contradiction that they are. Let $i \in n$ and $x_2 \in A$ be such that $g_0^i(x_0) + g_1^i(x_1) + g_2^i(x_2) = 0$, and let $y \in A$ be such that $x_0 + y = x_1$. Since A is a subgroup of X closed under the homomorphisms, plugging in for x_1 it becomes clear that $(g_0 + g_1)(x_0) \in A$. Since A is closed under the inverse of $g_0 + g_1$, it follows that $x_0 \in A$, contradicting the initial choice of x_0 .

To show that the graph Θ is locally finite (resp. locally countable), it will be enough to show that if c, d, d' are E -classes, $i \in n$ is an index, and there are points $x_0, x'_0 \in c$, $x_1 \in d_0$, $x'_1 \in d'$, and $x_2, x'_2 \in A$ such that $g_0^i(x_0) + g_1^i(x_1) + g_2^i(x_2) = 0$ and $g_0^i(x'_0) + g_1^i(x'_1) + g_2^i(x'_2) = 0$ both hold, then $d = d'$. To see that, subtract the latter equality from the former and get $g_0^i(x_0 - x'_0) + g_1^i(x_1 - x'_1) \in A$. The first summand here belongs to A as $x_0 E x'_0$ holds. As A is closed under the inverse of g_1^i , it follows that $x_1 - x'_1 \in A$ holds. Thus $d = d'$ as desired. \square

Corollary 2.7. *Let $\langle X, + \rangle$ be a standard Borel abelian group. Let Γ be a union of countably many slim linear hypergraphs of arity three on X . Whenever $A \subset X$ is a Γ -closed set, the chromatic number of $\Gamma(A)$ is countable.*

One can use the chromatic number of remainder graphs to prove that the chromatic number of Γ itself is countable. In addition, the mechanics of this proof will drive the consistency arguments in Section 5. For a notational convenience, my (hyper)graph colorings of hypergraphs will often have the countable set HF of hereditarily finite sets as a co-domain.

Definition 2.8. Let $\langle X, + \rangle$ be a Polish abelian group. Let Γ be a countable union of slim linear hypergraphs. A *coherent sequence of colorings* is a tuple $\langle I, \leq, A_i, c_i, d_i \rangle$ where

1. $\langle I, \leq \rangle$ be a linearly ordered set;
2. $\langle A_i : i \in I \rangle$ is an inclusion-increasing sequence of Γ -closed subsets of X ;
3. for every $x \in \bigcup_i A_i$, there is a smallest index $i \in I$ such that $x \in A_i$;
4. $c_i : A_i \rightarrow \text{HF}$ is a Γ -coloring;
5. if i is the smallest element of I then $d_i = 0$; otherwise, write Δ_i for the remainder graph $\Delta(\bigcup_{j < i} A_j)$, and $d_i : A_i \setminus \bigcup_{j < i} A_j \rightarrow \text{HF}$ is a $\Gamma_{\bigcup_{j < i} A_j}$ -coloring.

Note that the linearly ordered set I must have a smallest element, namely the smallest i such that $1 \in F_i$. One way to satisfy item (3) is to insist that \leq is a well-ordering; however, I will have an opportunity to use an ill-founded linear order as well.

Definition 2.9. The *amalgamation* of a coherent sequence $\langle I, \leq, A_i, c_i, d_i \rangle$ of colorings is the function e on $\bigcup_i A_i$ defined as follows. For every $x \in \bigcup_i A_i$ find the smallest $i \in I$ such that $x \in A_i$. If i is the smallest element of I , then let $e(x) = f_i(x)$; otherwise, let $e(x) = \langle f_i(x), g_i(x) \rangle$.

Note that the amalgamation extends f_i for the smallest element $i \in I$.

Proposition 2.10. *The amalgamation of a coherent sequence of colorings is a Γ -coloring.*

Proof. Let $\langle I, \leq, A_i, c_i, d_i \rangle$ be a coherent sequence of colorings. Write $Y = \bigcup_i A_i$. For each $x \in X$ write $i(x)$ for the least $i \in I$ such that $x \in A_i$. Let e be the amalgamation of the sequence. Let $a \subset X$ be a Γ -hyperedge and work to show that $e \upharpoonright a$ is not constant. Let i be the \leq -largest element of the finite set $\{i(x) : x \in a\}$. There are three configurations to consider; their discussion completes the proof.

Case 1. There is exactly one point $x \in a$ such that $i(x) = i$. This is in fact impossible, since the subgroup $\bigcup_{j < i} A_j$ is Γ -closed.

Case 2. There are exactly two points $x \in a$ such that $i(x) = i$. In this case, these two points are Δ_i -connected and they receive distinct d_i -colors, as d_i is a Δ_i -coloring. In consequence, they also receive distinct e -colors.

Case 3. All three points $x \in a$ have $i(x) = i$. In this case, a is not c_i -monochromatic, since c_i is a Γ -coloring. As a result, a is not e -monochromatic either. \square

Corollary 2.11. *Let $\langle X, + \rangle$ be a Polish abelian group. Let Γ be a countable union of countably many slim linear hypergraphs of arity three on X . The chromatic number of Γ is countable.*

Proof. By induction on the cardinality of a Γ -closed set $A \subset X$ I will argue that there is a Γ -coloring from A to HF. This is clear if $|A| = \aleph_0$, since one can choose any injection as the coloring. Suppose now that κ is an uncountable cardinal, $|A| = \kappa$, and the statement has been proved for all Γ -closed sets of cardinality smaller than κ . Express $F = \bigcup_{\beta \in \alpha} F_\beta$ as an increasing union of Γ -closed sets of cardinality smaller than κ . Use the induction hypothesis to find a Γ -coloring $c_\beta : A_\beta \rightarrow \text{HF}$ and Example 2.6 to find a $\Delta(\bigcup_{\gamma \in \beta} A_\gamma)$ -coloring $d_\beta : A_\beta \setminus \bigcup_{\gamma \in \beta} A_\gamma \rightarrow \text{HF}$, this for every $\beta \in \alpha$. Let $c : A \rightarrow \text{HF}$ be the amalgamation of $\langle c_\beta, d_\beta : \beta \in \alpha \rangle$. Proposition 2.10 confirms that this is a Γ -coloring, completing the induction step and the proof. \square

3 C.c.c. coloring posets

Another issue seemingly unrelated to the consistency result in question is the possibility of forcing colorings of given (hyper)graphs by finite conditions. If Γ is a hypergraph on a set X of countable chromatic number, there may not be a coloring definable from Γ —this is the basic difficulty of descriptive combinatorics. However, there still may be a definable poset adding a coloring which has useful regularity properties. The poset I use in this paper is the simplest one:

Definition 3.1. Let Γ be a hypergraph on a set X . The *finite approximation coloring poset* $R(\Gamma)$ is the poset of all partial finite colorings $p : X \rightarrow \omega$, ordered by reverse inclusion.

The countable set used as the target of the colorings is of course immaterial; I may use the set HF instead of ω . The property of the coloring posets critical for this paper is the following.

Definition 3.2. Let R be a poset.

1. A set $A \subset R$ is *Ramsey-centered* if for every $n \in \omega$ there is $m \in \omega$ such that for every m -tuple $\langle r_i : i \in m \rangle$ of elements of A there is a set $b \subset m$ of cardinality n such that the set $\{r_i : i \in b\}$ has a common lower bound in R .
2. if z is a set, then R is *z -definably σ -Ramsey-centered* if R can be covered by an ω -sequence of Ramsey-centered sets which is defined from z

There are several situations in which I can show that the finite approximation coloring poset is suitably definably Ramsey-centered. Some are exhibited in [?, Section 11.6]. The cases I need in this paper are recorded in the following two similar theorems with similar proofs.

Theorem 3.3. *Let X be a set. Let Γ be a hypergraph of arity three on X such that there is a number $d \in \omega$ such that for every $x_0, x_1 \in X$ the set $\{x_2 \in X : \{x_0, x_1, x_2\} \in \Gamma\}$ has cardinality at most d . Then the finite approximation coloring poset for Γ is X, Γ -definably σ -Ramsey centered.*

Proof. Let $A_{kl} \subset R(\Gamma)$ be the set of all conditions whose domain has cardinality $k \in \omega$ and the range is a subset of $l \in \omega$. It will be enough to show that A_{kl} is Ramsey-centered.

To this end, let $n \in \omega$ be a number. Increasing n if necessary I may assume that $n > d \cdot [2k]^2$ and $n > 3l$. Let m be a number such that $m \rightarrow (n)_{9k+1}^3$. It will be enough to show that among any m many elements of A_k one can find n many with a common lower bound. To this end, let $\{p_i : i \in m\} \subset A_k$ be a set. For each $i \in m$, fix an enumeration $\{x_j^i : j \in k\}$ of its domain. Consider the following map f on $[m]^3$. If $u \in [m]^3$ is a set, let $i_0 < i_1 < i_2$ be its elements in increasing order and define $f(u)$ by a split into cases:

- if there is an index $b \in 3$ and an index $j \in k$ such that $x_j^{i_b}$ belongs to at least one of $\text{dom}(p_{i_c})$ for $c \neq b$ and it is not equal to at least one $x_j^{i_c}$ for $c \neq b$, then the functional value will be $f(u) = \langle 0, b, j \rangle$ for some such b and j ;
- if the previous item fails and there are $b \in 3$ and $j \in k$ such that for some $c \in 3$ distinct from b , $x_j^{i_b} = x_j^{i_c}$ and $p_{i_b}(x_j^{i_b}) \neq p_{i_c}(x_j^{i_c})$, then $f(u) = \langle 1, b, j \rangle$ for some such b, j ;
- if the previous items fail and there are $b \in 3$ and $j \in k$ such that there is a hyperedge $e \in \Gamma$ which is contained in $\bigcup_{i \in u} \text{dom}(p_i)$ and such that $x_j^{i_b}$ is the only element of e which is not in $\text{dom}(p_{i_c})$ for any $c \neq b$, then let $f(u) = \langle 2, b, j \rangle$ for some such b, j ;

- if all of the previous items fail then $f(u) = \text{OK}$.

By the Ramsey assumption on the number m , there is a set $a \subset m$ of cardinality n which is homogeneous for the coloring f . I will argue that the homogeneous color is OK, the sets $\text{dom}(p_i)$ for $i \in a$ form a Δ -system, $\bigcup_{i \in a} p_i$ is a function, and every Γ -edge in $\text{dom}(\bigcup_{i \in a} p_i)$ is a subset of $\text{dom}(p_i)$ for some $i \in a$. That edge will not be monochromatic since p_i is a Γ -coloring. In conclusion, $\bigcup_{i \in a} p_i$ is a common lower bound in $R(\Gamma)$ of the conditions p_i for $i \in a$. This will conclude the proof of the theorem. I proceed in a sequence of claims.

Claim 3.4. *The homogeneous color is not $\langle 0, b, j \rangle$ for any $b \in 3$ and $j \in k$.*

Proof. Suppose towards a contradiction that this occurs; for definiteness, assume that $b = 2$. Then there are only $2k$ many options for x_j^i for $i \in a$ (they all must come from the first two elements of the set a), so they must repeat, and there must be a set $u \in [a]^3$ such that the value of x_j^i for $i \in u$ does not depend on i . This contradicts the assumption that $f(u) = \langle 0, b, j \rangle$. \square

It follows that writing $G = \{j \in k : \text{the value of } x_j^i \text{ does not depend on } i \in a\}$, the sets $\text{dom}(p_i)$ for $i \in a$ form a Δ -system with heart $H = \{x_j^i : i \in a, j \in G\}$. To see this, note that if $i_0 \neq i_1$ were two distinct elements of a and $j \in k$ were such that $x_j^{i_0} \in \text{dom}(p_{i_1}) \setminus \{x_j^{i_1}\}$, then this would trigger the first item above for any $u \in [a]^3$ containing both elements i_0, i_1 , contradicting the claim.

Claim 3.5. *The homogeneous color is not $\langle 1, b, j \rangle$ for any $b \in 3$ and $j \in k$.*

Proof. If it were, $j \in G$ would have to occur. There are only l many colors available for $p_i(x_j^i)$ for $i \in a$, which means that one of them has to repeat at least three times. Let $u \in [a]^3$ be such that $p_i(x_j^i)$ for $i \in u$ does not depend on i . Then $f(u) = \langle 1, b, j \rangle$ must fail, a contradiction. \square

It follows that $\bigcup_{i \in a} p_i$ is a function. Any disagreement of distinct conditions on the common part H of their domain would trigger the second item above for some set $u \in [a]^3$, contradicting the claim.

Claim 3.6. *The homogeneous color is not $\langle 2, b, j \rangle$ for any $b \in 3$ and $j \in k$.*

Proof. Suppose towards a contradiction that this occurs; for definiteness, assume that $b = 2$. Then there are at most $d \cdot [2k]^2$ many options for the value of x_j^i as i varies over all elements of a by the assumption on the hypergraph Γ . This means that these values have to repeat, and there will be a set $u \in [a]^3$ such that x_j^i is the same for all $i \in u$. Such u cannot have $f(u) = \langle 2, b, j \rangle$. \square

Now, let e be a Γ -hyperedge in $\text{dom}(\bigcup_{i \in a} p_i)$; I need to argue that there is a single $i \in a$ such that $e \subset \text{dom}(p_i)$. Otherwise, there would be $i \in a$ such that $e \cap \text{dom}(p_i) \setminus H$ is a singleton, which would trigger the third item for any set $u \in [a]^3$ containing i and the other two indices needed to reconstruct e , contradicting the claim. This completes the proof of the theorem. \square

Theorem 3.7. *Let X be a set and let Γ be a graph such that for every $x_0 \in X$, the set $\{x_1 \in X : \{x_0, x_1\} \in \Gamma\}$ is finite. Then $R(\Gamma)$ is X, Γ -definably Ramsey- σ -centered.*

Proof. For each $k, l, d \in \omega$ let $A_{kld} = \{p \in R(\Gamma) : |\text{dom}(p)| = k, \text{rng}(p) \subset l, \text{ and each point in } \text{dom}(p) \text{ has at most } d\text{-many neighbors}\}$. It will be enough to show that each set A_{kld} is Ramsey-centered.

To this end, let $k, l, d, n \in \omega$ be arbitrary. Increasing the number n if necessary, I may assume that $n > d + 2, k + 2$. Let $m \in \omega$ be a number such that $m \rightarrow (n)_{6k+1}^2$. It will be enough to show that among any m many elements of A_{kld} one can find n many with a common lower bound, i.e. such that their union is still a partial Γ -coloring.

To do that, let $\{p_i : i \in m\} \subset A_{kld}$ be any set. For each $i \in m$, fix an enumeration $\langle x_j^i : j \in k \rangle$ of $\text{dom}(p_i)$. Consider the following map f on $[m]^2$. If $u \in [m]^2$ is a set, let $i_0 < i_1$ be its elements in increasing order and define $f(u)$ by a split into cases:

- if there is an index $b \in 2$ and an index $j \in k$ such that $x_j^{i_0} \neq x_j^{i_1-b}$ yet $x_j^{i_0} \in v_{i_1-b}$, then the functional value will be $f(u) = \langle 0, b, j \rangle$ for some such b and j ;
- if the previous item fails and there is $j \in k$ such that x_j^i does not depend on $i \in u$, yet the value $p_i(x_j^i)$ is different for the two numbers $i \in u$, then $f(u) = \langle 1, 0, j \rangle$ for some such j ;
- if the previous items fail and there are $b \in 2$ and $j \in k$ such that the point $x_j^{i_0} \notin \text{dom}(p_{i_1-b})$ is Γ -connected with some point in $\text{dom}(p_{i_1-b}) \setminus \text{dom}(p_{i_0})$, then let $f(u) = \langle 2, b, j \rangle$ for some such b, j ;
- if all of the previous items fail then $f(u) = \text{OK}$.

Use the Ramsey property of m to find a set $a \subset m$ of cardinality n which is homogeneous for f . As in the proof of Theorem 3.3, one can argue that the sets $\text{dom}(p_i)$ for $i \in a$ form a Δ -system, $\bigcup_{i \in a} p_i$ is a function, and every Γ -edge in $\text{dom}(\bigcup_{i \in a} p_i)$ is a subset of $\text{dom}(p_i)$ for some $i \in a$. That edge will not be monochromatic since p_i is a Γ -coloring. In conclusion, $\bigcup_{i \in a} p_i$ is a common lower bound in $R(\Gamma)$ of the conditions p_i for $i \in a$. \square

4 A balanced coloring poset

Towards the proof of Theorem 1.4, I must find a balanced forcing which generates the desired model of ZF+DC over the choiceless Solovay model. I use this opportunity to isolate a rather general scheme for coloring ternary hypergraphs of the following form:

Definition 4.1. Let $\langle X, + \rangle$ be a standard Borel abelian group and Γ a finite union of slim linear ternary hypergraphs on X . Let $C = \bigcup_n C_n$ be a countable

set decomposed into infinitely many infinite sets. The coloring poset P consists of all p such that there is a countable Γ -closed set $\text{dom}(p) \subset X$ such that $p: \text{dom}(p) \rightarrow C$ is a Γ -coloring. The ordering is defined by $q \leq p$ if $p \subseteq q$ and for every $\text{dom}(p)$ -orbit $a \subset \text{dom}(q)$ there is a number $m \in \omega$ such that $q''a \subset \bigcup_{n \in m} C_m$.

For the last condition in the definition, note that the group $\text{dom}(p)$ acts on $X \setminus \text{dom}(p)$ by addition, and $\text{dom}(q) \setminus \text{dom}(p)$ is a set invariant under this action. The choice of sets C_n for $n \in \omega$ is clearly immaterial; for convenience, below I will set $C = \text{HF} \setminus \{0\}$ and $C_n = \{c \in C: |c| = n + 1\}$.

Proposition 4.2. *P is an analytic, transitive, σ -closed relation.*

Proof. The analyticity of P is clear. For the transitivity, assume that $r \leq q \leq p$ are conditions in P , and argue that $r \leq p$ must hold. For this, suppose that $a \subset \text{dom}(r) \setminus \text{dom}(p)$ is a $\text{dom}(p)$ -orbit; I must show that there is a number $m \in \omega$ such that $q''a \subset \bigcup_{n \in m} C_m$. There are two cases. Either, a contains some element of $\text{dom}(q)$. In this case, $a \subset \text{dom}(q)$ holds, and the existence of the number m follows from $q \leq p$. Or, $a \cap \text{dom}(q) = \emptyset$. In this case, a is a subset of a single $\text{dom}(q)$ -orbit and the existence of the number m follows from $r \leq q$.

For the σ -closure, it is clear that if $\langle p_n: n \in \omega \rangle$ is a descending sequence of conditions, then $\bigcup_n p_n$ is its lower bound. \square

Proposition 4.3. *For every $p \in P$ and every $x \in X$ there is $q \leq p$ such that $x \in \text{dom}(q)$.*

Proof. First, choose $\text{dom}(q)$ to be an arbitrary countable Γ -closed set containing $\text{dom}(p)$ as a subset and x as an element. Let $q: \text{dom}(q) \rightarrow C$ be a function such that $q \upharpoonright \text{dom}(p) = p$ and $q \upharpoonright \text{dom}(q) \setminus \text{dom}(p)$ is an injection whose range is a subset of C_0 . Then q is a Γ -coloring: any hyperedge $e \subset \text{dom}(q)$ is either contained in $\text{dom}(p)$ and not monochromatic as $p \in P$, or it contains more than one element of $\text{dom}(q) \setminus \text{dom}(p)$ by the Γ -closure of $\text{dom}(p)$, and it is not monochromatic as $q \setminus p$ is an injection. Since $q \leq p$ holds, the proposition follows. \square

Proposition 4.4. *Let $\bar{p}: X \rightarrow C$ be a total Γ -coloring. Let $V[G_0]$ and $V[G_1]$ be generic extensions such that $V[G_0] \cap V[G_1] = V$. Let $p_0 \in P \cap V[G_0]$ and $p_1 \in P \cap V[G_1]$ be conditions such that $p_0 \leq \bar{p}$ and $p_1 \leq \bar{p}$. Then the conditions p_0, p_1 are compatible in P .*

Note that the analytic poset P is reinterpreted in every transitive model of set theory; in particular, the conditions p_0, p_1 are compatible in the poset P as reinterpreted in $V[G_0, G_1]$ or any larger model.

Proof. Work in the model $V[G_0, G_1]$.

Claim 4.5. *$p_0 \cup p_1$ is a function.*

Proof. This follows from the assumption that $V[G_0] \cap V[G_1] = V$: it then must be the case that $\text{dom}(p_0) \cap \text{dom}(p_1) \subset V$. The functions $p_0 \upharpoonright V$ and $p_1 \upharpoonright V$ are both the same, equal to \bar{p} . The claim follows. \square

Of course, this does not end the proof; I have to find a countable Γ -closed set $\text{dom}(q) \subset X$ which contains $\text{dom}(p_0 \cup p_1)$ and a condition $q \in P$ with this domain such that $q \leq p_0, p_1$ holds. To this end, for every $x \in X \setminus \text{dom}(p_0 \cup p_1)$ let $a_0(x) = \{x_0 \in \text{dom}(p_0) \setminus V : \exists x_1 \in \text{dom}(p_1) \setminus V \{x_0, x_1, x\} \in \Gamma\}$ and $a_1(x) = \{x_1 \in \text{dom}(p_1) \setminus V : \exists x_0 \in \text{dom}(p_0) \setminus V \{x_0, x_1, x\} \in \Gamma\}$.

Claim 4.6. *Let $x \in X \setminus \text{dom}(p_0 \cup p_1)$ be a point. Each of the sets $a_0(x), a_1(x)$ is a subset of a finite union of $X \cap V$ -orbits. In addition,*

1. *if b is a $\text{dom}(p_0)$ -orbit, then $\bigcup_{x \in b \setminus \text{dom}(p_1)} a_1(x)$ is a subset of a finite union of $X \cap V$ -orbits;*
2. *same with subscripts 0, 1 interchanged.*

Proof. Let g_0, g_1, g_2 be linear homomorphisms defining one of the slim linear components of the hypergraph Γ . For the first sentence of the claim, it will be enough to show that if $x_0, x'_0 \in \text{dom}(p_0) \setminus V$ and $x_1, x'_1 \in \text{dom}(p_1) \setminus V$ are points such that $g_0(x_0) + g_1(x_1) + g_2(x) = 0$ and $g_0(x'_0) + g_1(x'_1) + g_2(x) = 0$, then x_0, x'_0 are in the same $X \cap V$ -orbit, and so are x_1, x'_1 . To see this, subtract the latter equation from the former, and get $g_0(x_0 - x'_0) = -g_1(x_1 - x'_1)$. The left-hand side is in $V[G_0]$, the right-hand side is in $V[G_1]$, and since $V[G_0] \cap V[G_1] = V$, both are in V . Since V is closed under inverses of g_0 and g_1 , $x_0 = x'_0 \in X \cap V$ and $x_1 - x'_1 \in V$ as desired.

For (1) it will be enough to show that if $x_0, x'_0 \in \text{dom}(p_0) \setminus V$ and $x_1, x'_1 \in \text{dom}(p_1) \setminus V$ and $x, x' \in b$ are points such that $g_0(x_0) + g_1(x_1) + g_2(x) = 0$ and $g_0(x'_0) + g_1(x'_1) + g_2(x') = 0$, then x_1, x'_1 are in the same $X \cap V$ -orbit. Again, subtract the latter equation from the former, getting $g_0(x_0 - x'_0) - g_2(x - x') = -g_1(x_1 - x'_1)$. The left hand side is in $V[G_0]$ since $x - x'$ is; the right hand side is in $V[G_1]$. As before, this means that $-g_1(x_1 - x'_1) \in V$ and $x_1 - x'_1 \in V$ as desired. The proof of (2) is symmetric. \square

Now, for each $x \in X \setminus \text{dom}(p_0 \cup p_1)$ let n_x be the smallest number $n \in \omega$ such that there are no points $x_0 \in \text{dom}(p_0) \setminus V$ and $x_1 \in \text{dom}(p_1) \setminus V$ such that $p_0(x_0) \in C_n, p_1(x_1) \in C_n$, and $\{x_0, x_1, x\} \in \Gamma$. The claim (together with $p_0, p_1 \leq \bar{p}$) shows that n_x is well-defined and in addition, for every $\text{dom}(p_0)$ -orbit b , the numbers n_x for $x \in b$ are bounded, and for every $\text{dom}(p_1)$ -orbit c , the numbers n_x for $x \in c$ are bounded. Now, let $\text{dom}(q) \subset X$ be any countable Γ -closed set containing $\text{dom}(p_0 \cup p_1)$, write $d = \text{dom}(q) \setminus \text{dom}(p_0 \cup p_1)$, and let $q: \text{dom}(q) \rightarrow C$ be any function such that $p_0 \cup p_1 \subset q$, for every $x \in d$ $q(x) \in C_{n_x}$, and $q \upharpoonright d$ is an injection. Such a function clearly exists since the sets C_n for every $n \in \omega$ are infinite. It will be enough to show that $q \in P$ and q is a common lower bound of p_0, p_1 .

First, I must argue that q is a Γ -coloring. Let $e \subset \text{dom}(q)$ be a Γ -hyperedge; I must argue that e is not monochromatic. The discussion splits into several possible configurations.

Case 1. $e \subset \text{dom}(p_0 \cup p_1)$. In this case, a counting argument shows that there is $i \in 2$ such that $\text{dom}(p_i)$ contains two points of e , and the Γ -closure of $\text{dom}(p_i)$ shows that $\text{dom}(p_i)$ contains all points of e . Then e cannot be monochromatic since p_i is a Γ -coloring.

Case 2. e contains exactly one point, say x , in the set d . In this case, the closure properties of the sets $\text{dom}(p_0)$ and $\text{dom}(p_1)$ show that there must be a point $x_0 \in \text{dom}(p_0) \setminus V$ and a point $x_1 \in \text{dom}(p_1) \setminus V$ such that $e = \{x_0, x_1, x\}$. Then, e is not monochromatic by the choice of the number n_x and the fact that $q(x) \in C_{n_x}$.

Case 3. e contains more than one point in the set d . In this case, e is not monochromatic since $q \upharpoonright d$ is an injection.

Finally, I must show that $q \leq p_0$; the proof of $q \leq p_1$ is symmetric. To this end, suppose that $b \subset \text{dom}(q) \setminus \text{dom}(p_0)$ is a $\text{dom}(p_0)$ -orbit; I must show that $q''b$ is a subset of the union of finitely many sets C_n for $n \in \omega$. Write $b_0 = b \cap \text{dom}(p_1)$ and $b_1 = b \setminus \text{dom}(p_1)$.

The first observation is that b_0 is a $\text{dom}(\bar{p})$ -orbit: if $x, x' \in b_0$ then $x - x' \in V[G_1]$ as $x, x' \in V[G_1]$, and $x - x' \in V[G_0]$ as b_0 is a subset of $\text{dom}(p_0)$ -orbit. Since $V[G_0] \cap V[G_1] = V$, it follows that $x - x' \in V$; as $x, x' \in b_0$ were arbitrary, it follows that b_0 is a $\text{dom}(\bar{p})$ -orbit. Since $p_1 \leq \bar{p}$ is assumed, it follows that $q''b_0$ is a subset of the union of finitely many sets C_n for $n \in \omega$.

The second observation is that the set $c = \bigcup_{x \in b_1} a_1(x) \subset \text{dom}(p_1)$ is a subset of a finite union of \bar{p} -orbits, and since $p_1 \leq \bar{p}$ is assumed, the set $q''c$ is covered by a finite union of the sets C_n for $n \in \omega$. Let $m \in \omega$ be some number such that $C_m \cap q''c = \emptyset$. It follows from the definitions that $n_x \leq m$ holds for every $x \in b_1$, therefore $q''b_1 \subset \bigcup_{n \leq m} C_n$ holds.

The two observations taken together show that $q''b$ is a subset of the union of finitely many sets C_n for $n \in \omega$ as desired. \square

Corollary 4.7. *The poset P is balanced, and in fact placid in the sense of [?, Definition 9.3.1].*

Proof. It will be enough, for every condition $p \in P$, to find a total coloring $\bar{p}: X \rightarrow \text{HF}$ such that the range of $\bar{p} \setminus p$ consists of sets of cardinality at most two. For then, in any generic extension collapsing the cardinality of the ground model continuum to \aleph_0 , it will be the case that $\bar{p} \in P$ and $\bar{p} \leq p$ holds. In addition, Proposition 4.4 then shows that \bar{p} will be exactly the balanced (or placid) virtual condition required in the definition of balance (or placidity).

Now, to obtain the extension of p to \bar{p} , use Corollary 2.11 to find a Γ -coloring $c: X \rightarrow \text{HF}$ and use Corollary 2.7 to find a $\Delta(\text{dom}(p))$ -coloring $d: \mathbb{R}^2 \setminus \text{dom}(p) \rightarrow \text{HF}$. Define the function \bar{p} by setting $\bar{p}(x) = p(x)$ if $x \in \text{dom}(p)$, and $\bar{p}(x) = \langle c(x), d(x) \rangle$ if $x \notin \text{dom}(p)$. This is literally an amalgamation of p with c and d in the sense of Definition 2.9, so Proposition 2.10 shows that \bar{p} is a Γ -coloring. By its definition, the range of $\bar{p} \setminus p$ consists of sets of cardinality at most two. This completes the proof. \square

5 The independence proofs

Finally, I am in the position to provide the proof of Theorem 1.4. Let κ be an inaccessible cardinal and let W be the choiceless Solovay model derived from κ . Let $G \subset P$ be a filter generic over W ; I will show that $W[G]$ is a model witnessing Theorem 1.4. There are several components of that proof.

Proposition 5.1. *In the model $W[G]$, there is no Vitali set.*

Proof. It will be enough to show that the poset P has Ramsey control as in [?, Section 11.6]. This means that in ZFC, for a condition $p \in P$ there is a triple $\langle R, \leq, \mathcal{C}, \bar{p} \rangle$ such that

1. R, \leq, \bar{p} are all definable from ordinal parameters and p ;
2. \mathcal{C} is a collection consisting of Ramsey centered subsets of R , each of which is definable from ordinal parameters and p , and $R = \bigcup \mathcal{C}$;
 tem \bar{p} is an R -name for a total Γ -coloring such that $R \Vdash \text{Coll}(\mathbb{R}, \aleph_0) \Vdash \bar{p} \leq \check{p}$.

The poset R is the finite support iteration of length ω_1 in which the α -th iterand (in the model $\mathbb{R}^2 \cap V[G_\beta: \beta \in \alpha]$) is the poset $R_{\alpha 0} \times R_{\alpha 1}$ where $R_{\alpha 0}$ is the finite condition poset coloring the hypergraph Γ from Theorem 3.3. To specify $R_{\alpha 1}$, use Example 2.6 to find a homomorphism $h_\alpha: X_\alpha \rightarrow Y_\alpha$, where $X_\alpha = X \setminus \bigcup_{\beta \in \alpha} \{V[G_\gamma: \gamma \in \beta]\}$ (if $\alpha = 0$) or $X \setminus \text{dom}(p)$ if $\alpha = 0$ and h_α is a homomorphism of the remainder graph $\Gamma_{X \setminus X_\alpha}$ to some locally finite graph Δ_α . Note that Example 2.6 provides $h_\alpha, Y_\alpha, \Delta_\alpha$ which are definable from p (if $\gamma = 0$) or from the sequence of intermediate generic extensions (if $\gamma > 0$). $R_{\alpha 1}$ will then be the finite condition poset adding a coloring of the graph Δ_α .

Each of these posets is suitably definably σ -Ramsey-centered, and c.c.c. This means that the finite support iteration R is c.c.c. and it is a union of Ramsey-centered pieces, each of which is definable from p and some ordinal parameters. This is proved as in [?, Proposition 11.6.3], noting that the Suslinity of the posets can be replaced with ordinal definability from the sequence of generic extensions.

At each stage of the iteration, $R_{\alpha+1}$ adds a Γ -coloring \dot{c}_α and $R_{\alpha 1}$ adds a coloring \dot{d}_α of the remainder graph which is the pullback of the generic Δ_α -coloring by the homomorphism h_α . In the end, let \bar{p} be the name for the amalgamation of the coherent sequence $\langle p, \dot{c}_\alpha, \dot{d}_\alpha: \alpha \in \omega_1 \rangle$ of colorings as in Definition 2.9. This will be a total Γ -coloring extending p by Proposition 2.10; $R \Vdash \text{Coll}(\mathbb{R}, \aleph_0) \Vdash \bar{p} \leq \check{p}$ follows from the definitions.

The proof of the proposition is now completed by a reference to [?, Theorem 11.6.5] applied to the Vitali equivalence relation viewed as a Borel graph. \square

Proposition 5.2. *In the model $W[G]$, the equivalence relation E_1 does not have a complete countable section.*

Proof. I will in fact show that the poset P is nested balanced [?, Definition 9.4.2]. Then, by [?, Theorem 9.4.4], in the P -extension of the Solovay model, the cardinality of the \mathbb{E}_1 -quotient space is not smaller than the cardinality of the orbit space of any Polish group action. In particular, there is no complete \mathbb{E}_1 section $S \subset (2^\omega)^\omega$, since then the map sending any \mathbb{E}_1 -class to its intersection with S would be an injection of the \mathbb{E}_1 -quotient space to the \mathbb{F}_2 -quotient space, which is an orbits space of an action of the full permutation group S_∞ .

To recall what nested balance means, a sequence $\langle M_i : i \in \omega \rangle$ of transitive models of ZFC is *coherent* [?, Section 4.2] if $M_0 \supseteq M_1 \supseteq \dots$ and for every ordinal α and every number $i \in \omega$, the sequence $\langle M_j \cap V_\alpha : j > i \rangle$ belongs to the model M_i . The sequence is *choice-coherent* [?, Section 4.3] if for every ordinal α there is a well-ordering \prec of V_α such that for every $i \in \omega$, $\prec \upharpoonright M_i \in M_i$ holds. For a choice coherent sequence $\langle M_i : i \in \omega \rangle$, write M_ω for the intersection model $\bigcap_i M_i$. It is a transitive model of ZFC.

To show that the poset P is nested balanced, it is enough, for a given choice-coherent sequence $\langle M_i : i \in \omega \rangle$ of transitive models of ZFC, to build a descending sequence of suitably balanced conditions. In the context of the poset P , it means building a total coloring $\bar{p} \in M_0$ such that for each $i \in \omega + 1$, the function $\bar{p}_i = \bar{p}_0 \upharpoonright M_i$ belongs to the model M_i , and for $i \leq j \leq \omega$, $\bar{p}_i \leq \bar{p}_j \leq p$ holds in any forcing extension which collapses $|M_0 \cap \mathbb{R}|$ to \aleph_0 .

To this end, let \prec be a well-ordering of the set of all partial maps from \mathbb{R}^2 to HF such that for each $i \leq \omega$, $\prec \upharpoonright M_i \in M_i$ holds. Let $I = \omega + 2$, and let \leq be the reverse of the usual ordinal ordering on I . For each $i \in I$, define the real closed field F_i : if $i \leq \omega$ then $F_i = \mathbb{R} \cap M_i$, and $F_{\omega+1} = \text{supp}(p)$. For each $i \in I$, define the colorings c_i and d_i : if $i \in \omega + 1$, then $c_i: F_i^2 \rightarrow \text{HF}$ is the \prec -least Γ -coloring in the model M_i , and $d_i: F_i^2 \setminus F_{i+1}^2 \rightarrow \text{HF}$ is the \prec -least Δ_i -coloring in the model M_i , where Δ_i is the remainder graph $\Delta(F_i^2, F_{i+1}^2)$. Also, $c_{\omega+1} = p$ and $d_{\omega+1} = 0$. It is not difficult to see that $\langle I, \leq, F_i, c_i, d_i : i \in I \rangle$ is a coherent sequence of colorings as in Definition 2.8. Let \bar{p} be the amalgamation of these colorings as in Definition 2.9. I claim that \bar{p} is as required.

First of all, \bar{p} is a Γ -coloring by Proposition 2.10. The definition of amalgamation, and the coherent choices of the colorings c_i and d_i , immediately imply that for each $i \in \omega + 1$, $\bar{p}_i = \bar{p} \upharpoonright M_i$ belongs to the model M_i . Finally, \bar{p} extends p , and by the definition of amalgamation, the range of $\bar{p} \setminus p$ consists of ordered pairs, i.e. of sets of cardinality at most two. It follows that for each $i \in j \in \omega + 1$, $\bar{p}_i \leq \bar{p}_j \leq p$ holds in any forcing extension in which $|F_i| = \aleph_0$. The proof is complete. \square

The proof of Theorem 1.6 is quite different. It closely follows [?, Section 11.8]. It uses an auxiliary hypergraph and a simple proper forcing notion of independent interest.

Definition 5.3. Let $n \geq 2$ be a number. Δ_n is the F_σ n -ary hypergraph on the space $Y_n = n^\omega$, where a Δ_n -hyperedge is a set $e \in [Y]^n$ such that there is a finite set $a \subset \omega$ and $z \in n^{\omega \setminus a}$ such that $e = \{y \in Y_n : y \upharpoonright \omega \setminus a = z \text{ and } y \upharpoonright a \text{ is constant}\}$.

The hypergraph Δ_n is unfortunately not actionable in the sense of [?, Definition 11.8.1], so the results of [?, Section 11.8] are not applicable verbatim and a restatement of the proof is necessary. There is a forcing notion associated with Δ_n . Using the Hales–Jewett theorem repeatedly, produce a sequence of consecutive nonempty finite intervals I_m of natural numbers for $m \in \omega$ such that for every partition of n^{I_m} into m pieces, one of the pieces contains a combinatorial line. Let ϕ_m be the submeasure on n^{I_m} defined by $\phi_m(m)$ = the smallest number of subsets of n^{I_m} , neither of which contains a combinatorial line, and together they cover b . Thus, $\phi_m(n^{I_m}) \geq m$ the poset R_n consists of sequences $r = \langle b_m : m \in \omega \rangle$ such that $b_m \subset n^{I_m}$ is a nonempty set and $\limsup_m \phi_m(b_m) = \infty$. The ordering on R_n is that of coordinatewise inclusion. The generic point of n^ω added by the poset R_n is the concatenation of the coordinatewise intersection of all conditions in the generic filter. The relevant properties of R_n are verified as in [?, Claim 11.8.4]:

Proposition 5.4. *The poset R_n is proper, bounding, and adds no independent reals. In addition, for every condition $r \in R_n$, in the R_n -extension below r there is a Δ_n -hyperedge whose elements are all R_n -generic and meet the condition r .*

The argument of [?, Section 11.8] can then be repeated to yield

Proposition 5.5. *In every compactly balanced extension of the choiceless Solovay model, the chromatic number of Δ_n is uncountable.*

Finally, I return to abelian groups and linear equations. Let $\langle X, + \rangle$ be a Polish abelian group, let $n \geq 3$ be a number, and let $g_i : X \rightarrow X$ for $i \in n$ be continuous homomorphisms such that $\sum_{i \in n} g_i = 0$. Write Γ for the hypergraph of arity n containing a tuple $\langle x_i : i \in n \rangle$ of pairwise distinct points if $\sum_{i \in n} g_i(x_i) = 0$. Suppose that open neighborhood of 0 in X contains a Γ -hyperedge consisting of nonzero points.

Let d be a left-invariant compatible metric on X . Since X is abelian, d is both-sided invariant, so complete. By recursion on $m \in \omega$, select nonzero pairwise distinct points $x_i(m) \in X$ for $i \in n$ so that

- $\sum_i g_i(x_i(m)) = 0$;
- $d(0, x_i(m)) < 2^{-m}$ times the minimum d -distance between distinct points of the form $x_j(k)$ for $k \in m$ and $j \in n$.

In addition, define the map $\pi_n : Y_n \rightarrow X$ by setting $\pi_n(y) = \sum_m x_{y_m}(m)$. The completeness and invariance of the metric d implies that the map π_n is well-defined and continuous. In addition, the continuity of the homomorphisms g_i for $i \in n$ show that if $\langle y_i : i \in n \rangle$ is a Δ_n -hyperedge, then $\langle \pi_n(y_i) : i \in n \rangle$ is a Γ -hyperedge; the map π_n is a homomorphism of Δ_n to Γ . In conjunction with Proposition 5.5 this concludes the proof of Theorem 1.6.

6 Problematic hypergraphs

There are algebraic hypergraphs for which the existence of colorings yields remarkable consequences in ZF+DC.

Definition 6.1. Γ_R is the hypergraph of arity four on \mathbb{R}^2 consisting of all sets of the form $a \times b$ where both $a, b \subset \mathbb{R}^2$ are sets of cardinality two.

To tie this hypergraph in with the previous sections, note that it is a subgraph of the linear hypergraph on \mathbb{R}^2 given by the equation $x_0 - x_1 + x_2 - x_3 = 0$. In ZFC, it is a well-known result (a conjunction of [?] and [?, Theorem 2]) that existence of countable Γ_R -coloring is equivalent to the Continuum Hypothesis. In ZF, the situation is more nuanced.

Theorem 6.2. (ZF) *Suppose that Γ_R has countable chromatic number. Then*

1. *if there is an ω_1 -sequence of pairwise distinct reals, then there is such a sequence enumerating all reals;*
2. *there is a complete countable section for the \mathbb{E}_1 equivalence relation.*

The second item is the opposite of Theorem 1.4(3). While I do not know if the existence of a Vitali set follows from the countable chromatic number of Γ_R , the first item shows that any negative consistency result in this direction must use an inaccessible cardinal. It also shows that the results of [?, ?] require the inaccessible cardinal assumption as well.

Proof. Let $c: \mathbb{R}^2 \rightarrow \omega$ be a total Γ_R -coloring.

For (1), suppose that $s = \langle r_\alpha: \alpha \in \omega_1 \rangle$ be a sequence of pairwise distinct reals.

Claim 6.3. *The model $HOD_{s,c}$ contains all reals.*

Proof. Suppose towards a contradiction that it does not. Let $r \in \mathbb{R} \setminus HOD_{s,c}$ be any real. Use a counting argument to find distinct countable ordinals α, β such that the points $\langle r, r_\alpha \rangle, \langle r, r_\beta \rangle \in \mathbb{R}^2$ receive the same color in c , say $n \in \omega$. The real r cannot be unique with this property, since in such a case it would be definable and therefore an element of $HOD_{s,c}$. If r' is a different real with this property, then the set $\{r, r'\} \times \{r_\alpha, r_\beta\}$ is a monochromatic Γ_R -hyperedge with color n , a contradiction. \square

Now, work in the model $HOD_{s,c}$ and argue that the Continuum Hypothesis holds there as ZFC+countable chromatic number of Γ_R implies CH; this will conclude the proof. This is in fact a known result [?, Theorem 2]; for completeness, let me include it. Work in ZFC, suppose that $d: \mathbb{R}^2 \rightarrow \omega$ is a Γ_R -coloring, and assume towards a contradiction that the Continuum Hypothesis fails. Let M be an elementary submodel of cardinality \aleph_1 of a large structure containing the coloring d . Let $r_0 \in \mathbb{R} \setminus M$ be any point. By a counting argument, there must be a number $n \in \omega$ and distinct reals $s_0, s_1 \in \mathbb{R} \cap M$ such that

$d(s_0, r_0) = d(s_1, r_0) = n$. By elementarity of the model M , there has to be a point $r_1 \in \mathbb{R} \cap M$ satisfying the latter equation with r_0 replaced with r_1 . Then the set $\{s_0, s_1\} \times \{r_0, r_1\}$ is a Γ_R -hyperedge monochromatic in color n , contradicting the choice of the coloring d .

For (2), suppose that $x \in \mathbb{R}^\omega$ be a point, and for each $n \in \omega$ let M_{nx} be the class of all sets hereditarily ordinally definable from c and $x \upharpoonright M_{nx}$. These models clearly form a sequence decreasing with respect to inclusion. Thus, the ordinals $(\omega_1)^{M_{nx}}$ form a non-increasing sequence, which has to stabilize at some number $m \in \omega$. The same proof in (1) shows that the sets $\mathbb{R} \cap M_{nx}$ stabilize at that same number $m \in \omega$. If this stabilized set is uncountable, then there is a bijection between ω_1 and \mathbb{R} by (1), and the proof is complete.

Suppose then that for every sequence $x \in \mathbb{R}^\omega$, the sets $\mathbb{R} \cap M_{nx}$ for $n \in \omega$ stabilize to some countable set. Then, even the sets $[x]_{\mathbb{E}_1} \cap M_{nx}$ stabilize in some countable set, call it $a(x)$. Note that the set $a(x)$ depends only on the \mathbb{E}_1 -class of x . Thus, the set $\bigcup\{a(x) : x \in \mathbb{R}^\omega\}$ is a complete countable section for \mathbb{E}_1 . \square

Definition 6.4. Γ_T is the hypergraph of arity three on \mathbb{R}^2 consisting of those sets for which adding a single point results in a Γ_R -hyperedge.

Clearly, any Γ_T -coloring is a Γ_R -coloring; so, it is more difficult to find a Γ_T -coloring. Again, in ZFC, it is a well-known result ??? that existence of countable Γ_T -coloring is equivalent to the Continuum Hypothesis. Thus, in ZFC, hypergraphs Γ_R and Γ_T are conflated in this way, even though the construction of Γ_T -coloring is much more difficult. In ZF, the distinction between the two hypergraphs becomes immediately apparent:

Theorem 6.5. (ZF) *If Γ_T has countable chromatic number, then there is a total countable-to-one map from \mathbb{R} to ω_1 .*

In particular, in ZF, if Γ_T has countable chromatic number, then every equivalence relation on a Polish space has countable complete section. I do not know if a Vitali set must exist. However, balanced forcing cannot be applied to obtain any consistency result in this direction, since countable-to-one maps from \mathbb{R} to ω_1 do not exist in balanced extensions of the choiceless Solovay model.

Proof. Let $c: \mathbb{R}^2 \rightarrow \omega$ be a Γ_T coloring. For $x \in \mathbb{R}$ write M_x for the model of all sets hereditarily ordinally definable from x and c . Note that $c \upharpoonright M_x \in M_x$ holds. The argument divides into two cases.

Case 1. There is a real x such that $\mathbb{R} \cap M_x$ is uncountable. In this case, apply Theorem 6.2(1) to show that there is even a total injection from \mathbb{R} to ω_1 .

Case 2. Case 1 fails. Let $\pi: \mathbb{R} \rightarrow \omega_1$ be the map defined by $\pi(x) = \omega_1^{M_x}$. The case assumption shows that the range of this map is indeed a subset of ω_1 . We will show that π is in fact countable-to-one. Suppose towards contradiction that it is not, and let $\alpha \in \omega_1$ be an ordinal such that the set $\{x \in \mathbb{R} : \pi(x) = \alpha\}$ is uncountable. By the case assumption, there have to be points x_0, x_1 in this set such that $x_1 \notin M_{x_0}$. We will reach a contradiction by a split into subcases.

Case 1a. Suppose first that $x_0 \notin M_{x_1}$. Let L_0 be the line in \mathbb{R}^2 consisting of points whose 0-th coordinate is equal to x_0 and let L_1 be the line in \mathbb{R}^2 consisting of points whose 1-st coordinate is equal to x_1 . Let $n = c(x_0, x_1)$. Then $\langle x_0, x_1 \rangle$ is not the only point on L_0 which gets color n —otherwise x_1 would be definable from x_0 . Let $\langle x_0, x_2 \rangle \in L_0$ be a different point which gets color n . By the same argument, $\langle x_0, x_1 \rangle$ is not the only point on L_1 which gets color n —otherwise x_0 would be definable from x_1 . Let $\langle x_3, x_1 \rangle \in L_1$ be a different point which gets color n . Then $\{\langle x_0, x_1 \rangle, \langle x_0, x_2 \rangle, \langle x_3, x_1 \rangle\}$ is a monochromatic Γ_T -hyperedge of color n . A contradiction.

Case 1b. Assume now that $x_0 \in M_{x_1}$. The set $\mathbb{R} \cap M_{x_0}$ then belongs to M_{x_1} and must be uncountable there because the two models have the same ω_1 . By a counting argument in M_{x_1} , there must be distinct points $y_0, y_1 \in \mathbb{R} \cap M_{x_0}$ such that $\langle y_0, x_1 \rangle$ and $\langle y_1, x_1 \rangle$ get the same c -color, say n . Now, x_1 cannot be the only point such that $\langle y_1, x_1 \rangle$ gets the color n —otherwise x_1 would be definable from y_1 and then also from x_0 . So, pick a point $z \in \mathbb{R}$ such that $c(y_1, z) = n$ and note that the set $\{\langle y_0, x_1 \rangle, \langle y_1, x_1 \rangle, \langle y_1, z \rangle\}$ is a c -monochromatic Γ_T -hyperedge of color n . This is a final contradiction. \square