Isomorphism of compact structures

Joseph Zielinski

South Eastern Logic Symposium 2020 University of Florida Gainesville, Florida February 29, 2020

Joint work with C. Rosendal

Question

What is the complexity of the topological group isomorphism relation between second countable t.d.l.c. groups?

t.d.l.c — totally disconnected, locally compact
complexity — Borel reducibility

Outcome of this project:

Obtain upper bounds on complexity for natural classes of mathematical objects by representing them in compact structures.

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- \mathcal{L} a countable, relational language
- $\alpha(R)$ the arity of R

Definition

A (metrizable) compact structure is $\mathcal{M} = (M, (R^{\mathcal{M}})_{R \in \mathcal{L}})$ where

- M is compact Polish
- Each $R^{\mathcal{M}} \subseteq M^{\alpha(R)}$ is closed

A compact metric structure is $(M, d^{\mathcal{M}}, (R^{\mathcal{M}})_{R \in \mathcal{L}})$

Definition

A homeomorphic isomorphism is $f \colon \mathcal{M} \to \mathcal{N}$ where

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$$f: M \to N$$
 is a homeomorphism

 $(x_1, \dots, x_k) \in R^{\mathcal{M}} \iff (fx_1, \dots, fx_k) \in R^{\mathcal{N}}$

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$$\mathcal{Q} = [0,1]^{\mathbb{N}}$$
 — Hilbert cube

• K(X) — hyperspace of compact subsets of X

Definition

The space of compact \mathcal{L} -structures is

$$\mathfrak{K}_{\mathcal{L}} = \{ (M, R^{\mathcal{M}})_{R \in \mathcal{L}} \mid R^{\mathcal{M}} \subseteq M^{\alpha(R)} \} \subseteq K(\mathcal{Q}) \times \prod_{R \in \mathcal{L}} K(\mathcal{Q}^{\alpha(R)})$$

Theorem (R.D. Anderson)

There is a homeomorphic embedding, $\iota: \mathcal{Q} \to \mathcal{Q}$, s.t. for compact $A, B \subseteq \iota[\mathcal{Q}]$, any homeo. $g: A \to B$ extends to $g' \in \operatorname{Homeo}(\mathcal{Q})$.

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Given compact $X, Y \subseteq Q$: X and Y are homeomorphic $\iff \iota[X] \text{ and } \iota[Y]$ are homeomorphic $\iff g\iota[X] = \iota[Y]$ for some $g \in \text{Homeo}(Q)$

Theorem (Kechris - Solecki)

The relation of homeomorphism between compact metric spaces is classifiable by a Polish group action (i.e., $Homeo(Q) \curvearrowright K(Q)$)

Any $f: \mathcal{Q} \to \mathcal{Q}$ induces

- $\bullet f^n \colon \mathcal{Q}^n \to \mathcal{Q}^n$
- $\bullet f^n_* \colon K(\mathcal{Q}^n) \to K(\mathcal{Q}^n)$
- $f_{\mathcal{L}} \colon K(\mathcal{Q}) \times \prod_{R \in \mathcal{L}} K(\mathcal{Q}^{\alpha(R)}) \to K(\mathcal{Q}) \times \prod_{R \in \mathcal{L}} K(\mathcal{Q}^{\alpha(R)})$

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Properties of $\iota_{\mathcal{L}}$ and $g_{\mathcal{L}}$:

- *ι*_L(*M*) is homeomorphically isomorphic to *M* but its domain is a subset of *ι*[*Q*]
- Homeo(Q) acts continuously on $K(Q) \times \prod_{R \in \mathcal{L}} K(Q^{\alpha(R)})$ with invariant subspace $\mathfrak{K}_{\mathcal{L}}$

Proposition (Rosendal - Z.)

For any countable, relational \mathcal{L} , the relation of homeomorphic isomorphism between compact \mathcal{L} -structures is classifiable by a Polish group action (i.e., Homeo(\mathcal{Q}) $\curvearrowright \mathfrak{K}_{\mathcal{L}}$ where $g \cdot \mathcal{M} = g_{\mathcal{L}}(\mathcal{M})$) Properties of $\iota_{\mathcal{L}}$ and $g_{\mathcal{L}}$:

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Theorem (Rosendal - Z.)

Topological group isomorphism between locally comp. and Roelcke precompact Polish groups is classifiable by a Polish group action

• (G,d) a Polish group, d left-invariant

• $\operatorname{Mult}_G = \{(g, h, k) \in G^3 \mid gh = k\}$

$$d_*(g,h) = \min\{d(g,h), \frac{1}{1+d(g,1)} + \frac{1}{1+d(h,1)}\}$$

$$d_{\wedge}(g,h) = \inf_{k \in G} \max\{d(g,k), d(k^{-1}, h^{-1})\}$$

Then:

- G locally comp. $\implies \overline{(G, d_*)}$ is one-point compactification
- G Roelcke precomp. $\implies \overline{(G, d_{\wedge})}$ is Roelcke compactification

Assign $G \rightsquigarrow \mathcal{M}_G = (M_G, \overline{\operatorname{Mult}_G})$

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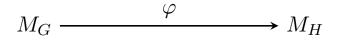
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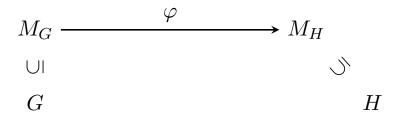
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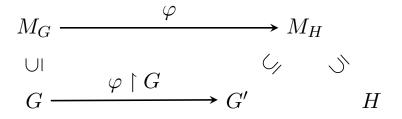
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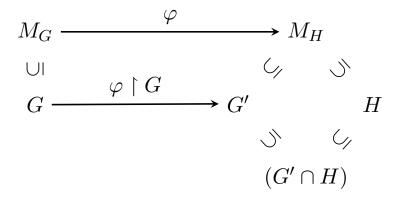
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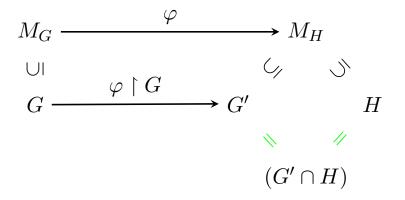








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Two observations:

 By Ferenczi - Louveau - Rosendal, the general relation of topological group isomorphism between Polish groups is a complete analytic equivalence relation

• The salient properties of \mathcal{M}_G :

- The homeomorphism type of M_G is an invariant of the isomorphism type of G
- M_G contains G as a comeagre subset
- *G* structure is determined by any comeagre substructure

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- \blacksquare M_G is compact, non-empty, perfect, zero-dimensional
- By Brouwer's theorem M_G is homeomorphic to $2^{\mathbb{N}}$
- $\blacksquare G \rightsquigarrow (2^{\mathbb{N}}, \overline{\mathrm{Mult}_G})$
- Group isomorphism is classified by the action $\operatorname{Homeo}(2^{\mathbb{N}}) \curvearrowright \{2^{\mathbb{N}}\} \times K((2^{\mathbb{N}})^3)$
- Homeo $(2^{\mathbb{N}}) \leqslant S_{\infty}$
- Any action of a subgroup of S_∞ is classifiable by countable structures

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Theorem (Rosendal - Z., Kechris - Nies - Tent)

Isomorphism between non-Archimedean locally comp. and Roelcke precomp. Polish groups is classifiable by countable structures

- Every countable group is t.d.l.c. so isomorphism of second countable t.d.l.c. groups is Borel bi-reducible with a complete relation for S_{∞} actions (e.g. graph isomorphism)
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Thank you!