Isomorphism of compact structures

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Joint work with C. Rosendal

Compact structures

Question

What is the complexity of the topological group isomorphism relation between second countable t.d.l.c. groups?

- t.d.l.c — *totally disconnected, locally compact*
- complexity — *Borel reducibility*

Outcome of this project:
Obtain upper bounds on complexity for natural classes of mathematical objects by representing them in compact structures.

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Compact structures

- $\mathcal{L}$ — a countable, relational language
- $\alpha(R)$ — the arity of R

Definition

A (metrizable) compact structure is $\mathcal{M} = (M, (R^{\mathcal{M}})_{R\in\mathcal{L}})$ where

- M is compact Polish
- Each $R^{\mathcal{M}} \subseteq M^{\alpha(R)}$ is closed

A compact metric structure is $(M, d^{\mathcal{M}}, (R^{\mathcal{M}})_{R\in\mathcal{L}})$

Definition

A homeomorphic isomorphism is $f \colon \mathcal{M} \to \mathcal{N}$ where

- $f \colon M \to N$ is a homeomorphism
- $(x_1, \ldots, x_k) \in R^{\mathcal{M}} \iff (fx_1, \ldots, fx_k) \in R^{\mathcal{N}}$

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Compact structures

- $\mathcal{Q} = [0,1]^{\mathbb{N}}$ — Hilbert cube
- $K(X)$ — hyperspace of compact subsets of X

Definition

The space of compact $\mathcal{L}$-structures is

$$\mathfrak{K}_{\mathcal{L}} = \{(M, R^{\mathcal{M}})_{R\in\mathcal{L}} \mid R^{\mathcal{M}} \subseteq M^{\alpha(R)}\} \subseteq K(\mathcal{Q}) \times \prod_{R\in\mathcal{L}} K(\mathcal{Q}^{\alpha(R)})$$

Theorem (R.D. Anderson)

There is a homeomorphic embedding, $\iota\colon \mathcal{Q} \to \mathcal{Q}$, s.t. for compact $A, B \subseteq \iota[\mathcal{Q}]$, any homeo. $g\colon A \to B$ extends to $g' \in \mathrm{Homeo}(\mathcal{Q})$.

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Given compact $X, Y \subseteq \mathcal{Q}$:

$$\begin{aligned} & X \text{ and } Y \text{ are homeomorphic} \\ \iff & \iota[X] \text{ and } \iota[Y] \text{ are homeomorphic} \\ \iff & g\iota[X] = \iota[Y] \text{ for some } g \in \mathrm{Homeo}(\mathcal{Q}) \end{aligned}$$

Theorem (Kechris - Solecki)

The relation of homeomorphism between compact metric spaces is classifiable by a Polish group action (i.e., $\mathrm{Homeo}(\mathcal{Q}) \curvearrowright K(\mathcal{Q})$)

Any $f\colon \mathcal{Q} \to \mathcal{Q}$ induces

- $f^n\colon \mathcal{Q}^n \to \mathcal{Q}^n$
- $f^n_*\colon K(\mathcal{Q}^n) \to K(\mathcal{Q}^n)$
- $f_{\mathcal{L}}\colon K(\mathcal{Q}) \times \prod_{R\in\mathcal{L}} K(\mathcal{Q}^{\alpha(R)}) \to K(\mathcal{Q}) \times \prod_{R\in\mathcal{L}} K(\mathcal{Q}^{\alpha(R)})$

In particular, ι and each $g \in \mathrm{Homeo}(\mathcal{Q})$ induce $\iota_{\mathcal{L}}$ and $g_{\mathcal{L}}$.

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Properties of $\iota_{\mathcal{L}}$ and $g_{\mathcal{L}}$:

- $\iota_{\mathcal{L}}(\mathcal{M})$ is homeomorphically isomorphic to $\mathcal{M}$ but its domain is a subset of $\iota[\mathcal{Q}]$
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Proposition (Rosendal - Z.)

For any countable, relational $\mathcal{L}$, the relation of homeomorphic isomorphism between compact $\mathcal{L}$-structures is classifiable by a Polish group action (i.e., $\mathrm{Homeo}(\mathcal{Q}) \curvearrowright \mathfrak{K}_{\mathcal{L}}$ where $g \cdot \mathcal{M} = g_{\mathcal{L}}(\mathcal{M})$)

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Using compact structures to classify other structures

Theorem (Rosendal - Z.)

Topological group isomorphism between locally comp. and Roelcke precompact Polish groups is classifiable by a Polish group action

- (G, d) a Polish group, d left-invariant
- $\mathrm{Mult}_G = \{(g, h, k) \in G^3 \mid gh = k\}$
- $d_*(g, h) = \min\{d(g, h), \frac{1}{1+d(g,1)} + \frac{1}{1+d(h,1)}\}$
- $d_\wedge(g, h) = \inf_{k \in G} \max\{d(g, k), d(k^{-1}, h^{-1})\}$

Then:

- G locally comp. $\implies \overline{(G, d_*)}$ is one-point compactification
- G Roelcke precomp. $\implies \overline{(G, d_\wedge)}$ is Roelcke compactification

Assign $G \rightsquigarrow \mathcal{M}_G = (M_G, \overline{\mathrm{Mult}_G})$

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$$M_G \xrightarrow{\varphi} M_H$$

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$$
\begin{tikzcd}
M_G \arrow[rr, "\varphi"] & & M_H & \\
G \arrow[u, phantom, "\rotatebox{90}{$\subseteq$}"] & & & H \arrow[ul, phantom, "\rotatebox{135}{$\subseteq$}"]
\end{tikzcd}
$$

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\begin{tikzcd}
M_G \arrow[rr, "\varphi"] & & M_H & \\
G \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "\varphi \restriction G"] & G' \arrow[ur, phantom, sloped, "\subseteq"] & & H \arrow[ul, phantom, sloped, "\subseteq"]
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Using compact structures to classify other structures

Two observations:

- By Ferenczi - Louveau - Rosendal, the general relation of topological group isomorphism between Polish groups is a complete analytic equivalence relation

- The salient properties of $\mathcal{M}_G$:
  - The homeomorphism type of M_G is an invariant of the isomorphism type of G
  - M_G contains G as a comeagre subset
  - G structure is determined by any comeagre substructure

  (e.g., Polish metric spaces have the above—although their isometry relation has long been known to be classifiable be a group action by Gao - Kechris results)

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Non-Archimedean Polish groups

Suppose G is locally compact or Roelcke precompact and moreover non-Archimedean—admits a basis at 1_G of open subgroups.

- M_G is compact, non-empty, perfect, zero-dimensional
- By Brouwer's theorem M_G is homeomorphic to $2^{\mathbb{N}}$
- $G \rightsquigarrow (2^{\mathbb{N}}, \overline{\mathrm{Mult}_G})$
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- $\mathrm{Homeo}(2^{\mathbb{N}}) \leqslant S_\infty$
- Any action of a subgroup of S_∞ is classifiable by countable structures

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Isomorphism between non-Archimedean locally comp. and Roelcke precomp. Polish groups is classifiable by countable structures

- Every countable group is t.d.l.c. so isomorphism of second countable t.d.l.c. groups is Borel bi-reducible with a complete relation for S_∞ actions (e.g. graph isomorphism)
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Other notions of isomorphism

Upper bounds for ____ isomorphism between ____ structures

	Homeo.	Uniform	bi-Lipschitz	Isometric
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Thank you!