A tiling property for amenable groups along Tempelman Følner sequences

Jenna Zomback

University of Illinois at Urbana-Champaign

joint work with Jon Boretsky

pmp actions of groups and ergodicity

- (X, μ) — a standard probability space, e.g., $[0, 1]$ with Lebesgue measure.

pmp actions of groups and ergodicity

- (X, μ) — a standard probability space, e.g., $[0, 1]$ with Lebesgue measure.
- Γ — a countable (discrete) group acting on X.

pmp actions of groups and ergodicity

- (X, μ) — a standard probability space, e.g., $[0, 1]$ with Lebesgue measure.
- Γ — a countable (discrete) group acting on X.
- The action $\Gamma \curvearrowright (X, \mu)$ is probability measure preserving (pmp) if each function $\gamma : X \to X$ is Borel, and $\mu(\gamma \cdot A) = \mu(A)$, for each $\gamma \in \Gamma$ and measurable $A \subseteq X$.

pmp actions of groups and ergodicity

- (X, μ) — a standard probability space, e.g., $[0, 1]$ with Lebesgue measure.
- Γ — a countable (discrete) group acting on X.
- The action $\Gamma \curvearrowright (X, \mu)$ is probability measure preserving (pmp) if each function $\gamma : X \to X$ is Borel, and $\mu(\gamma \cdot A) = \mu(A)$, for each $\gamma \in \Gamma$ and measurable $A \subseteq X$.
- $A \subseteq X$ is invariant if A is a union of orbits. Equivalently, $\Gamma \cdot A = A$.

pmp actions of groups and ergodicity

- (X, μ) — a standard probability space, e.g., $[0, 1]$ with Lebesgue measure.
- Γ — a countable (discrete) group acting on X.
- The action $\Gamma \curvearrowright (X, \mu)$ is probability measure preserving (pmp) if each function $\gamma : X \to X$ is Borel, and $\mu(\gamma \cdot A) = \mu(A)$, for each $\gamma \in \Gamma$ and measurable $A \subseteq X$.
- $A \subseteq X$ is invariant if A is a union of orbits. Equivalently, $\Gamma \cdot A = A$.
- The action $\Gamma \curvearrowright (X, \mu)$ is ergodic if every invariant measurable subset is null or conull.

pmp actions of groups and ergodicity

- (X, μ) — a standard probability space, e.g., $[0, 1]$ with Lebesgue measure.
- Γ — a countable (discrete) group acting on X.
- The action $\Gamma \curvearrowright (X, \mu)$ is probability measure preserving (pmp) if each function $\gamma : X \to X$ is Borel, and $\mu(\gamma \cdot A) = \mu(A)$, for each $\gamma \in \Gamma$ and measurable $A \subseteq X$.
- $A \subseteq X$ is invariant if A is a union of orbits. Equivalently, $\Gamma \cdot A = A$.
- The action $\Gamma \curvearrowright (X, \mu)$ is ergodic if every invariant measurable subset is null or conull.
- Dating back to Birkhoff, **pointwise ergodic theorems** for pmp actions $\Gamma \curvearrowright (X, \mu)$ are bridges between the global condition of ergodicity and the a.e. local combinatorics of the action.

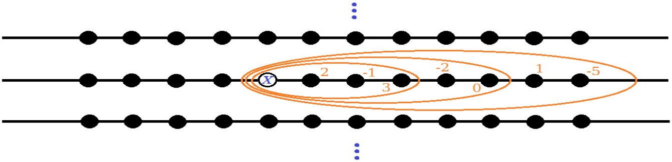
Pointwise ergodic theorem for $\mathbb{Z}$

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ *is ergodic if and only if for each* $f \in L^1(X, \mu)$ *and for a.e.* $x \in X$,

$$\lim_{n \to \infty} (\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.



Pointwise ergodic theorems for pmp actions

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n \to \infty} (\text{average of } f \text{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Pointwise ergodic theorems for pmp actions

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n \to \infty} (\text{average of } f \text{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Group	**Pointwise ergodic theorem**
$\mathbb{Z}$	Birkhoff, 1931
Amenable (Tempelman)	various authors, 1967-1983
Amenable (tempered)	Lindenstrauss, 2001
Free groups	various authors, 1987-2013
Other nonamenable groups	Bowen–Nevo, 2013

Sufficient condition: tiling property

Definition (Tserunyan 2017)

We say that a countable group Γ has the tiling property along a sequence (F_n) of its finite subsets if:

Sufficient condition: tiling property

Definition (Tserunyan 2017)

We say that a countable group Γ has the tiling property along a sequence (F_n) of its finite subsets if:

- for any pmp action of Γ on a standard probability space (X, μ)

Sufficient condition: tiling property

Definition (Tserunyan 2017)

We say that a countable group Γ has the tiling property along a sequence (F_n) of its finite subsets if:

- for any pmp action of Γ on a standard probability space (X, μ)
- for any pointwise increasing sequence of functions $\ell_n : X \to \mathbb{N}$, $n \in \mathbb{N}$

Sufficient condition: tiling property

Definition (Tserunyan 2017)

We say that a countable group Γ has the tiling property along a sequence (F_n) of its finite subsets if:

- for any pmp action of Γ on a standard probability space (X, μ)
- for any pointwise increasing sequence of functions $\ell_n : X \to \mathbb{N}$, $n \in \mathbb{N}$
- for any $\epsilon > 0$

Sufficient condition: tiling property

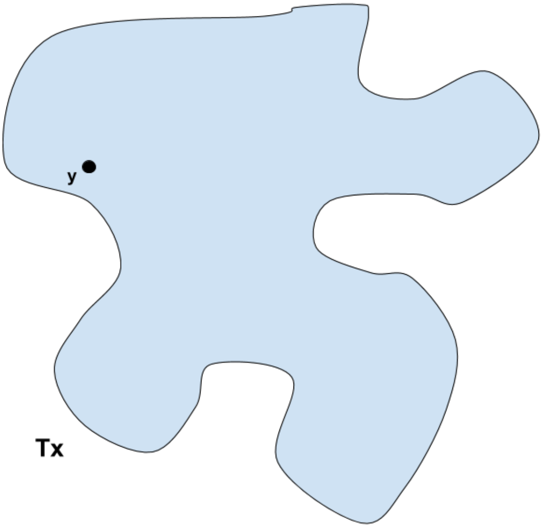
Definition (Tserunyan 2017)

We say that a countable group Γ has the tiling property along a sequence (F_n) of its finite subsets if:

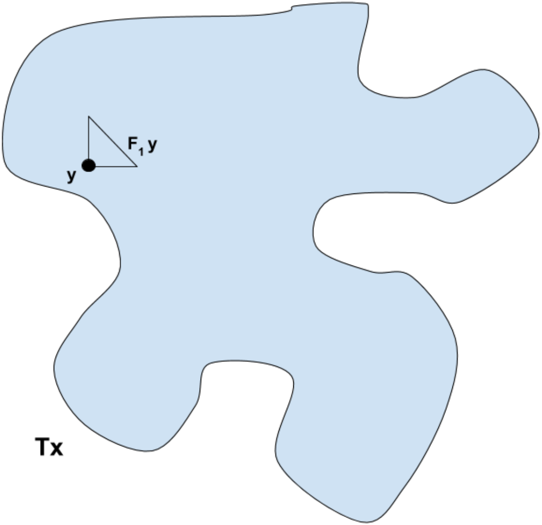
- for any pmp action of Γ on a standard probability space (X, μ)
- for any pointwise increasing sequence of functions $\ell_n : X \to \mathbb{N}$, $n \in \mathbb{N}$
- for any $\epsilon > 0$

there are arbitrarily large finite subsets $T \subseteq \Gamma$ such that for a set of points x of measure at least $(1-\epsilon)$, Tx can be covered up to ϵ fraction by disjoint sets of the form $F_{\ell_i(y)} \cdot y$.

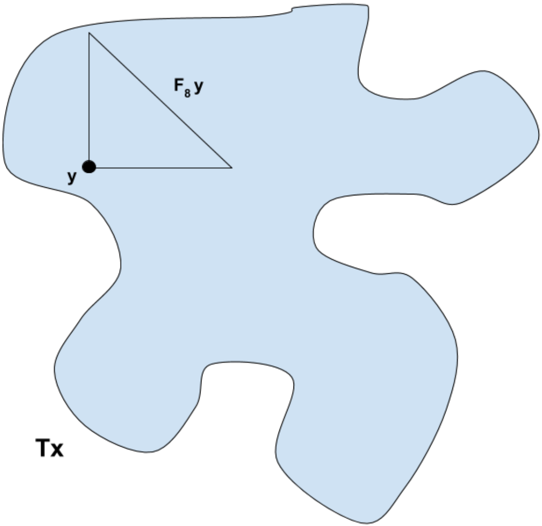
Tiling property



Tiling property



Tiling property



$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.

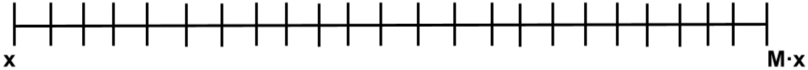
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



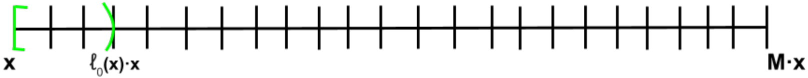
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



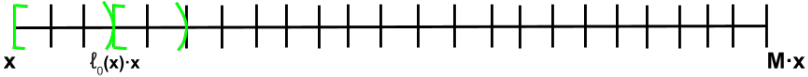
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



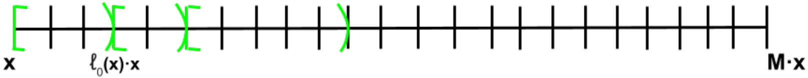
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



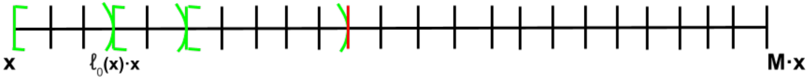
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



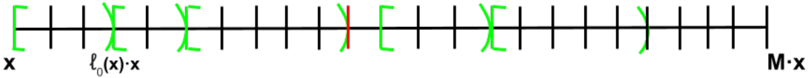
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



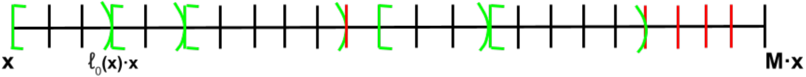
$\mathbb{Z}$ has the tiling property

Observation (Tserunyan 2017)

$\mathbb{Z}$ has the tiling property along the sequence $([0, n))_{n \in \mathbb{N}}$.

Proof Sketch:

- Let L be large enough so that for most $x \in X$, $\ell_0(x) < L$, and let M be large enough so that $\frac{L}{M}$ is small.
- We'll tile the interval $[0, M) \cdot x$ by placing tiles of the form $[y, \ell_0(y)) \cdot y$ when $\ell_0(y) < L$.



□

Tiling proof of the pointwise ergodic theorem for $\mathbb{Z}$

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n\to\infty} (\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Tiling proof of the pointwise ergodic theorem for $\mathbb{Z}$

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n \to \infty} (\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Proof Sketch (Tserunyan 2017):

Tiling proof of the pointwise ergodic theorem for $\mathbb{Z}$

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n\to\infty} (\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Proof Sketch (Tserunyan 2017):

- Assume WLOG $\int_X f \, d\mu = 0$ and assume for a contradiction that $\limsup(\text{average of } f \text{ over } F_n \cdot x) > \Delta > 0$.

Tiling proof of the pointwise ergodic theorem for $\mathbb{Z}$

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n \to \infty} (\text{average of } f \text{ over } F_n \cdot x) = \int_X f d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Proof Sketch (Tserunyan 2017):

- Assume WLOG $\int_X f \, d\mu = 0$ and assume for a contradiction that $\limsup(\text{average of } f \text{ over } F_n \cdot x) > \Delta > 0$.
- We use that $\limsup > \Delta$ and the tiling property to show that most of X can be covered by tiles with large ($> \Delta$) averages.

Tiling proof of the pointwise ergodic theorem for $\mathbb{Z}$

Theorem (Pointwise ergodic for $\mathbb{Z}$, Birkhoff 1931)

A pmp action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n \to \infty} (\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f d\mu,$$

where $F_n := [0, n) \subseteq \mathbb{Z}$.

Proof Sketch (Tserunyan 2017):

- Assume WLOG $\int_X f \, d\mu = 0$ and assume for a contradiction that $\limsup(\text{average of } f \text{ over } F_n \cdot x) > \Delta > 0$.
- We use that $\limsup > \Delta$ and the tiling property to show that most of X can be covered by tiles with large ($> \Delta$) averages.
- Hence, $\int_X f \, d\mu$ is basically just an integral over the tiles (so $\int_X f \, d\mu > 0$). This contradicts $\int_X f \, d\mu = 0$. $\square$

Amenable groups and Følner sequences

- A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ is a Følner sequence if for all $\gamma \in \Gamma$,
$$\lim_n \frac{|F_n \triangle \gamma F_n|}{|F_n|} = 0.$$

Amenable groups and Følner sequences

- A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ is a Følner sequence if for all $\gamma \in \Gamma$,

$$\lim_n \frac{|F_n \triangle \gamma F_n|}{|F_n|} = 0.$$

- Γ is amenable if it admits a Følner sequence.

Amenable groups and Følner sequences

- A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ is a Følner sequence if for all $\gamma \in \Gamma$,
$$\lim_n \frac{|F_n \triangle \gamma F_n|}{|F_n|} = 0.$$
- Γ is amenable if it admits a Følner sequence.
- Given a Følner sequence (F_n), we say (F_n) is tempered if there is some natural number C such that for all n,
$$\left| \bigcup_{k<n} F_k^{-1} F_n \right| \leqslant C|F_n|.$$

Amenable groups and Følner sequences

- A sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of Γ is a Følner sequence if for all $\gamma \in \Gamma$,
$$\lim_n \frac{|F_n \triangle \gamma F_n|}{|F_n|} = 0.$$
- Γ is amenable if it admits a Følner sequence.
- Given a Følner sequence (F_n), we say (F_n) is tempered if there is some natural number C such that for all n,
$$\left|\bigcup_{k<n} F_k^{-1}F_n\right| \leqslant C|F_n|.$$
- We say a Følner sequence (F_n) is Tempelman if there is C such that for all n,
$$\left|\bigcup_{k\leqslant n} F_k^{-1}F_n\right| \leqslant C|F_n|.$$

Pointwise ergodic theorems for amenable groups

Theorem (Pointwise ergodic for amenable groups, Lindenstrauss 2001)

A pmp action $\Gamma \curvearrowright (X, \mu)$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

$$\lim_{n\to\infty} (\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where Γ is amenable, and F_n is a tempered Følner sequence.

Pointwise ergodic theorems for amenable groups

Theorem (Pointwise ergodic for amenable groups, Lindenstrauss 2001)

A pmp action $\Gamma \curvearrowright (X,\mu)$ *is ergodic if and only if for each* $f \in L^1(X,\mu)$ *and for a.e.* $x \in X$*,*

$$\lim_{n\to\infty}(\text{average of } f \text{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where Γ *is amenable, and* F_n *is a tempered Følner sequence.*

Theorem (Pointwise ergodic for amenable groups, Tempelman 1967)

A pmp action $\Gamma \curvearrowright (X,\mu)$ *is ergodic if and only if for each* $f \in L^1(X,\mu)$ *and for a.e.* $x \in X$*,*

$$\lim_{n\to\infty}(\text{average of } f \text{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where Γ *is amenable, and* F_n *is an increasing Tempelman Følner sequence.*

Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt:

Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt:

- Assume we have "liking" functions $\ell_i : X \to \mathbb{N}$ such that $\ell_i(x) < \ell_{i+1}(x)$ for all $x \in X$, $i \in \mathbb{N}$. We want to tile a large square in $\mathbb{Z}^2$ with good tiles (of the form $[0, \ell_i(x))^2 \cdot x$.

Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt:

- Assume we have "liking" functions $\ell_i : X \to \mathbb{N}$ such that $\ell_i(x) < \ell_{i+1}(x)$ for all $x \in X$, $i \in \mathbb{N}$. We want to tile a large square in $\mathbb{Z}^2$ with good tiles (of the form $[0, \ell_i(x))^2 \cdot x$.
- We can try the same thing as last time: find L such that most of X is such that $\ell_0(x) < L$, and find M such that $\frac{L}{M}$ is small.

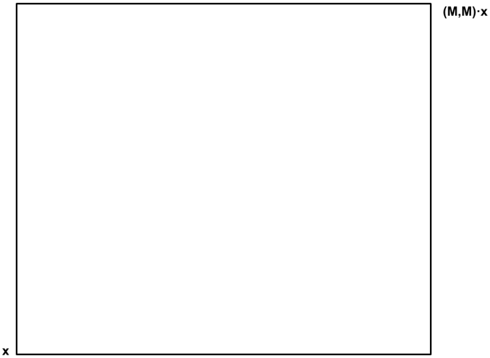
Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt:



Proof attempt for $\mathbb{Z}^2$

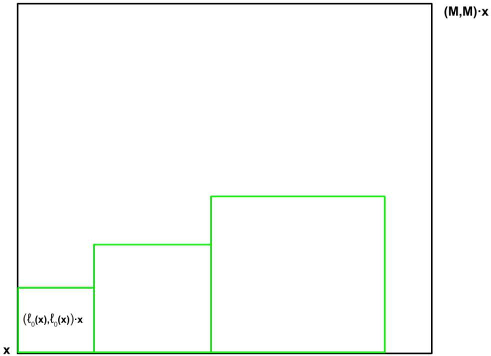
Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt:

Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

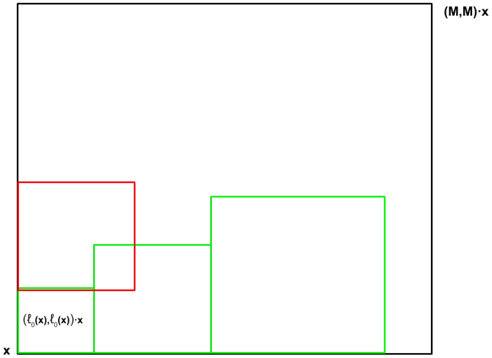
Proof attempt:



Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

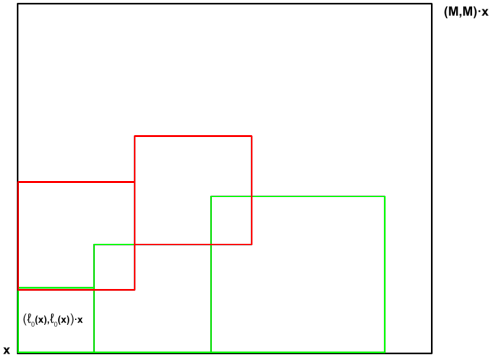
Proof attempt:



Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

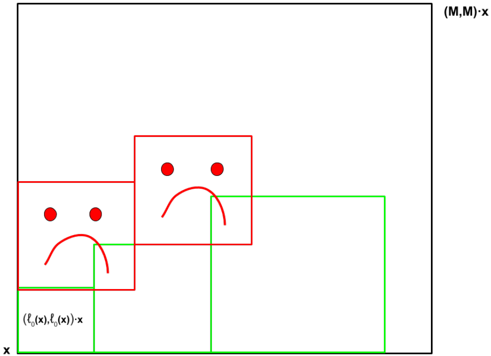
Proof attempt:



Proof attempt for $\mathbb{Z}^2$

Claim: $\mathbb{Z}^2$ has the tiling property along the sequence $[0, n)^2$.

Proof attempt:



Vitali covering lemma

- How large can the sad areas be?

Vitali covering lemma

- How large can the sad areas be?
- If $F_n x \cap F_m y \neq \emptyset$, then $x \in F_n^{-1} F_m y$.

Vitali covering lemma

- How large can the sad areas be?
- If $F_n x \cap F_m y \neq \emptyset$, then $x \in F_n^{-1} F_m y$.
- Assuming $n \leqslant m$, this is bounded by $C|F_m|$, where C is the Tempelman constant.

Vitali covering lemma

- How large can the sad areas be?
- If $F_n x \cap F_m y \neq \emptyset$, then $x \in F_n^{-1} F_m y$.
- Assuming $n \leqslant m$, this is bounded by $C|F_m|$, where C is the Tempelman constant.

Lemma (Vitali covering lemma)

Given a function $\ell : X \to N$, a finite subset $S \subseteq X$, there exists a set K, which is a disjoint union of sets of the form $F_{\ell(x)} x$, $x \in S$, such that $|K| \geqslant \frac{1}{C} |S \cup K|$.

Proof sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof sketch:

- Find many (depending on C and ϵ) ranges of values such that for most of X, x likes values in each range.

Proof sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof sketch:

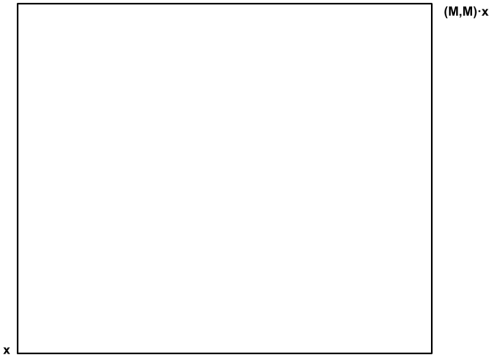
- Find many (depending on C and ϵ) ranges of values such that for most of X, x likes values in each range.
- Iterate the Vitali covering lemma, starting with the range of largest sizes.

Proof sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof sketch:

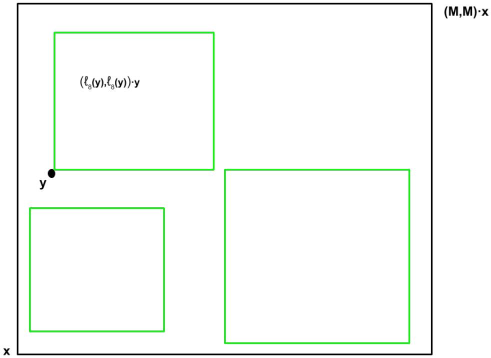


Proof sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof sketch:

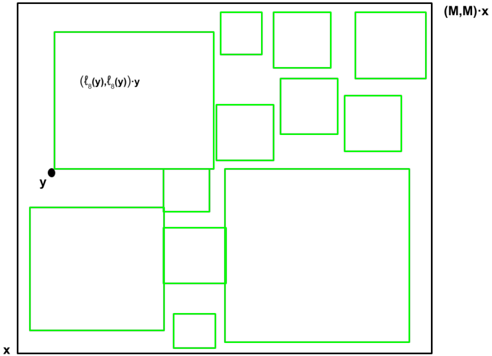


Proof sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof sketch:



Proof Sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Proof Sketch for Γ amenable and (F_n) Tempelman Følner

Theorem (Boretsky–Z. 2019)

If Γ is an amenable group with increasing Tempelman Følner sequence $(F_n)_{n\in\mathbb{N}}$, then Γ has the tiling property along (F_n).

Corollary (Less general version of Lindenstrauss' theorem)

A pmp action $\Gamma \curvearrowright (X,\mu)$ is ergodic if and only if for each $f \in L^1(X,\mu)$ and for a.e. $x \in X$,

$$\lim_{n\to\infty}(\textit{average of } f \textit{ over } F_n \cdot x) = \int_X f \, d\mu,$$

where Γ is amenable, and F_n is an increasing Tempelman Følner sequence.

Future work

Group	Pointwise ergodic theorem	Tiling property
$\mathbb{Z}$	Birkhoff, 1931	Tserunyan, 2017
Amenable (Tempelman)	various authors, 1967-1983	Boretsky–Z., 2019
Amenable (tempered)	Lindenstrauss, 2001	?
Amenable (nontempered)	?	?
Free groups	various authors, 1987-2013	?
Other nonamenable	Bowen–Nevo, 2013	?

Thank you!